# Controllability and Reconstructibility Conditions for 2-D Systems

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***Abstract*—New necessary and sufficient conditions for local controllability and causal reconstructibility of multiinput multioutput 2-D systems are presented which are based on 2-D matrix polynomial equations. The conditions are computationally attractive: only unimodular operations in a Euclidean ring are required. In addition, when testing controllability and/or reconstructibility, one obtains a deadbeat regulator and/or an exact observer as byproducts.**

## I. Introduction

2-D signals and systems have been investigated in relation to several modern engineering fields such as 2-D digital filtering, digital picture processing, 2-D network realizability, seismic data processing, X-ray image and aerial photographs enhancement, image deblurring, etc. [1]–[3].

In most cases, 2-D systems result from a discretization procedure, both in space and in time, applied to spatially distributed continuous-time systems. This simplifies the evaluation of responses and makes it feasible to design computer implementable algorithms. In this context, it is quite natural to address 2-D state estimation and feedback control problems. In fact, the results easily apply to continuous distributed parameter systems modeled as 2-D systems. Moreover, as with 1-D systems, the output feedback compensator design reduces to independently synthesizing a state observer that processes measured data and a stabilizing state feedback law [4].

A special case of 2-D observers are the "exact observers," whose estimate error vanishes in a finite number of steps; these provide an exact state estimation in real time [5]. Dually, a special case of compensators are the deadbeat compensators, that drive to zero the state of the 2-D system in a finite number of steps. Using this kind of compensators leads to finite memory systems, in the sense that the initial conditions do not affect the dynamics after a finite interval and the impulse response exhibits FIR behavior.

In this note, we shall confine ourselves to exact observers and deadbeat controllers. In this, we are motivated not only by their intrinsic interest, but also by the fact that their synthesis is fully performed by using a 2-D polynomial matrix approach.

Moreover, a general theory of asymptotic observers and stabilizing compensators can be developed along similar lines, if only the ring of 2-D polynomials is substituted by the ring of 2-D stable rational functions [4].

The notions of local controllability and causal reconstructibility of 2-D systems have been introduced and discussed in [4] and [5], and play the same role as the notions of controllability and reconstructibility in the 1-D case.

The engineering significance of these concepts relies on the fact that they provide necessary and sufficient conditions for the existence of 2-D deadbeat compensators. More precisely, the synthesis of a state feedback deadbeat controller is made possible by local controllability, while causal reconstructibility allows one to build up exact observers [4]–[6]. From a computational point of view, the compensator synthesis essentially reduces obtaining the transfer matrices of the regulator and the observer by solving certain matrix polynomial equations and realizes these transfer matrices via 2-D state-space models.

The aim of this correspondence is to present a method for solving 2-D polynomial equations in a computationally attractive way and thereby, in fact, to derive corresponding necessary and sufficient conditions for controllability and reconstructibility of 2-D systems.

To test these conditions, we need just unimodular operations in a Euclidean ring. The method is applicable for generic multiinput multioutput 2-D systems. Moreover, when checking controllability, we compute also the transfer function of a deadbeat regulator. Similarly, when testing reconstructibility, we obtain an exact observer.

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## II. Definitions

Consider a 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ given by

$$x(h+1, k+1) = A_1 x(h, k+1) + A_2 x(h+1, k) + B_1 u(h, k+1) + B_2 u(h+1, k)$$

$$y(h, k) = Cx(h, k) + Du(h, k) \tag{1}$$

where the *local state* $x$ is an $n$ dimensional vector over the real field $R$, *input and output* $u$ and $y$ take values in $R^m$ and $R^p$, respectively, and $A_1, A_2, B_1, B_2, C, D$ are real matrices of suitable sizes [4].

Denote by

$$\mathfrak{X}_0 = \sum_{i=-\infty}^{+\infty} x(i, -i) z_1^i z_2^i \tag{2}$$

the *global state* on the *separation set*

$$S_0 = \{(i, j) : i + j = 0\}$$

and by

$$X(z_1, z_2) = \sum_{i+j \geq 0} x(i, j) z_1^i z_2^j$$

$$U(z_1, z_2) = \sum_{i+j \geq 0} u(i, j) z_1^i z_2^j$$

$$Y(z_1, z_2) = \sum_{i+j \geq 0} y(i, j) z_1^i z_2^j$$

the state, input, and output functions, respectively. Then one gets

$$Y(z_1, z_2) = [C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D] U(z_1, z_2) + C(I - A_1 z_1 - A_2 z_2)^{-1} \mathfrak{X}_0 \tag{3}$$

where the first term describes the input-output map of the system via the rational *transfer matrix*.

$$W(z_1, z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D \tag{4}$$

while the second stands for a free motion of $\Sigma$ from a nonzero global initial state $\mathfrak{X}_0$.

The causality of the system corresponds to the fact that the denominator of the entries of $W(z_1, z_2)$ has a nonzero constant term, so that $W(z_1, z_2)$ admits an expansion as power series in $z_1$ and $z_2$.

A system $\Sigma$ is *strictly proper* when $D = 0$ and is *finite memory*, if for any set of global initial conditions $\mathfrak{X}_0$, the free-state evolution goes to zero in a finite number of steps. Recall that for a finite memory system, (1) is the polynomial matrix $(I - A_1 z_1 - A_2 z_2)$ unimodular.

*Assumption:* We shall assume throughout this correspondence that the real matrix $A_2$ is invertible.

First, this is a generic case. If, all the same, $A_2$ is not invertible but $A_1$ is, we can simply interchange the roles of $z_1$ and $z_2$ in what follows. If even $A_1$ is not invertible, but the matrix $eA_1 + fA_2$ is for some $e, f$ in $R$, the substitution $z_1 = z_1$, $z_2 = ez_1 + fz_2$ will do the job. In such a way, most of the practical cases are covered.

In general, the characteristic polynomial of $\Sigma$ is

$$\det (I - A_1 z_1 - A_2 z_2) = (\det A_2) z_2^n + a_{n-1} z_2^{n-1} + \cdots + a_0 \tag{5}$$

where all the $a_{n-1} \cdots, a_0$ are polynomials in $z_1$.

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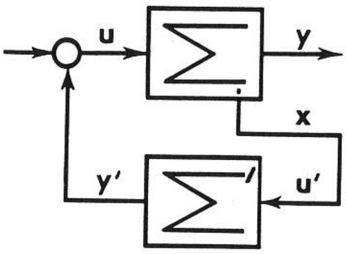


Fig. 1.

Throughout this correspondence, we use the standard notation: $R[z_1]$, $R(z_1)$, $R[z_1, z_2]$, and $R(z_1)$ $[z_2]$ stand for the ring of real polynomials in $z_1$, the ring of rational functions in $z_1$, the ring of real 2-D polynomials in $z_1$ and $z_2$, and the ring of polynomials in $z_2$ with coefficients in $R(z_1)$, respectively.

## III. CONTROLLABILITY AND REGULATORS

As in 1-D case, local controllability means the ability to drive any initial local state $x(0, 0)$ to zero in a finite number of steps by using an input function with support in the positive quarter plane. More precisely, we have the following definition.

*Definition 1:* A system $\Sigma$ is *locally controllable* if for any local-state $x(0, 0)$ there exists a polynomial input $U(z_1, z_2)$ such that the state evolution

$$X(z_1, z_2)=(I-A_1z_1-A_2z_2)^{-1}[x(0, 0)+(B_1z_1+B_2z_2)U(z_1, z_2)$$

is a polynomial vector.

The concept of local controllability has recently been investigated in [4]. From there, we quote also the theorem.

*Theorem 1:* The following facts are equivalent.

1) $\Sigma$ is locally controllable.

2) There exist polynomial matrices $M(z_1, z_2)$ and $N(z_1, z_2)$ such that the Bezout identity

$$(I-A_1z_1-A_2z_2)M(z_1, z_2)+(B_1z_1+B_2z_2)N(z_1, z_2)=I \tag{6}$$

holds.

3) For any $(z_1, z_2)$ in $C \times C$,

$$\text{rank}\ [I-A_1z_1-A_2z_2 \quad B_1z_1+B_2z_2]=n. \tag{7}$$

Now consider a *state feedback regulator* $\Sigma$, which is a 2-D dynamical system [4] governed by equations

$$x'(h+1, k+1)=A_1'x'(h, k+1)+A_2'x'(h+1, k) +B_1'u'(h, k+1)+B_2'u'(h+1, k)$$

$$y'(h, k)=Cx'(h, k)+Du'(h, k) \tag{8}$$

and connected with $\Sigma$ by a state feedback law (Fig. 1), so that

$$u'(h, k)=x(h, k)$$

$$u(h, k)=y'(h, k). \tag{9}$$

Denoting by $\mathfrak{X}_0'$ the global initial state of the regulator $\Sigma$, one gets from (8) and (9)

$$U(z_1, z_2)=W'(z_1, z_2)X(z_1, z_2)+C'(I-A_1'z_1-A_2'z_2)^{-1}\mathfrak{X}_0' \tag{10}$$

where the first term

$$W'(z_1, z_2)=C'(I-A_1'z_1-A_2'z_2)^{-1}(B_1'z_1+B_2'z_2)+D' \tag{11}$$

is the rational transfer matrix of the regulator.

In the sequel, we shall concentrate our interest in deadbeat regulation structures, by requiring that the feedback connection of Fig. 1 be a finite memory system. This amounts to the assumption that the free motion of $\Sigma$ and $\Sigma'$ converges to zero in a finite number of steps, independently on the initial conditions $\mathfrak{X}_0$ and $\mathfrak{X}_0'$.

The existence of a deadbeat regulator and local controllability are strongly connected.

*Theorem 2 [4]:* For a strictly proper system $\Sigma$, there exists a deadbeat regulator $\Sigma'$ if and only if $\Sigma$ is locally controllable.

Moreover, the deadbeat regulator is any locally controllable and causally reconstructible realization of the transfer matrix

$$W'(z_1, z_2)=-N(z_1, z_2)M(z_1, z_2)^{-1} \tag{12}$$

where the polynomial matrices $M(z_1, z_2)$ and $N(z_1, z_2)$ are given by (6).

From a glance at Theorems 1 and 2, it is clear how important the role of (6) is. $\Sigma$ is locally controllable and/or admits a deadbeat regulator iff (6) is solvable. Moreover, its solutions directly provide the transfer matrix of the deadbeat regulator (12). It is the aim of our correspondence to study this equation in detail.

Quite recently, a considerable deal of work has been done in attempting to extend some solution methods of 1-D matrix polynomial equations to the 2-D case [9]–[12].

The consideration of 2-D equations has opened up a new set of questions we shall explore in the sequel. However, they have many properties similar to 1-D equations; for example, as in 1-D case, linearity of (6) allows us to get the general solution from a particular one.

*Lemma 1:* Let $F(z_1, z_2)$ and $G(z_1, z_2)$ be right coprime matrices such that

$$G(z_1, z_2)F(z_1, z_2)^{-1}=(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2) \tag{13}$$

and let $M'(z_1, z_2)$ and $N'(z_1, z_2)$ be any (particular) solution of (6). Then the general solution of (6) is of the form

$$M(z_1, z_2)=M'(z_1, z_2)+G(z_1, z_2)T(z_1, z_2)$$

$$N(z_1, z_2)=N'(z_1, z_2)-F(z_1, z_2)T(z_1, z_2) \tag{14}$$

where $T(z_1, z_2)$ is an arbitrary polynomial matrix of suitable size.

*Proof:* The proof is same as in the corresponding 1-D case. See also [12].

Under the Assumption, also minimal solution resembles the 1-D case.

*Lemma 2:* Let $F(z_1, z_2)$ be given by (13) with $A_2$ invertible and let (6) be solvable. Then it possesses a unique solution such that the rational matrix

$$F(z_1, z_2)^{-1}N(z_1, z_2) \tag{15}$$

is strictly proper in $z_2$.

*Proof:* To get the minimum solution (15), we need to just take any particular $N'(z_1, z_2)$ and divide it from the left by $F(z_1, z_2)$ in (14) like in the 1-D case [5]. This, however, is not generally possible in the 2-D case as the ring $R[z_1, z_2]$ is not Euclidean.

Fortunately, we can perform it whenever the Assumption is satisfied: so, for any $N'(z_1, z_2)$

$$F(z_1, z_2)^{-1}N'(z_1, z_2)=(\text{adj}\ F)N'/\det F=E/\det F$$

where $E = (\text{adj}\ F)N'$ is a matrix with polynomial entries $e_{12}$.

Now $\det F$ divides $\det (I - A_1z_1 - A_2z_2)$ (see [7]) and the $z_2$ leading coefficient of $\det (I - A_1z_1 - A_2z_2)$ is a nonzero constant.

So the $z_2$ leading coefficient of $\det F$ is a nonzero constant, too, and we can divide in (15)

$$e_{ij}/\det F=p_{ij}+h_{ij}/\det F \tag{17}$$

to get a polynomial part $p_{ij}$ and a strictly proper (in $z_2$) part $h_{ij}/\det F$. As a result,

$$F(z_1, z_2)^{-1}N'(z_1, z_2)=P(z_1, z_2)+H(z_1, z_2) \tag{18}$$

where $P = [p_{ij}]$ is a polynomial matrix while $H = [h_{ij}/\det F]$ is a strictly proper (in $z_2$) rational matrix. Finally, (18) yields

$$N'(z_1, z_2)=F(z_1, z_2)P(z_1, z_2)+F(z_1, z_2)H(z_1, z_2) \tag{19}$$

so that we get the desired minimum solution

$$\begin{aligned} M(z_1, z_2)&=M'(z_1, z_2)+G(z_1, z_2)P(z_1, z_2)\\ N(z_1, z_2)&=F(z_1, z_2)H(z_1, z_2) \end{aligned} \tag{20}$$

from (14) by taking $T = P$. Indeed,

$$F(z_1, z_2)^{-1}N(z_1, z_2)=H(z_1, z_2)$$

is strictly proper in $z_2$ and this solution is unique as is the division (17).

To proceed, we can use the same plan of attack as in [9] for the scalar case: we will work in $R(z_1)\,[z_2]$—the "nearest greater" Euclidean ring to $R[z_1, z_2]$. The following lemma is a standard Euclidean ring result.

*Lemma 3:* Let

$$(I-A_1z_1-A_2z_2)M''(z_1, z_2)+(B_1z_1+B_2z_2)N''(z_1, z_2)=I \tag{21}$$

be solvable for $M''(z_1, z_2)$, $N''(z_1, z_2)$ with entries in $R(z_1)\,[z_2]$.

Then it possesses a unique solution such that

$$F(z_1, z_2)^{-1}N''(z_1, z_2) \tag{22}$$

is strictly proper in $z_2$.

*Proof:* If $F(z_1, z_2)$, $G(z_2, z_2)$ are right coprime in $R[z_1, z_2]$, they are right coprime in $R(z_1)\,[z_2]$ as well. We can, therefore, repeat the proof of Lemma 2 even without the assumption, since the ring $R(z_1)\,[z_2]$ is Euclidean.

Now we are prepared to state our main result.

*Theorem 3:* Let $A_2$ be an invertible matrix. Then (6) is solvable (in $R[z_1, z_2]$) if and only if:

1) it is solvable in $R(z_1)$, $[z_2]$ and

2) its $R(z_1)\,[z_2]$ solution $M''(z_1, z_2)$, $N''(z_1, z_2)$ given by (21) and (22) satisfy

$$M''(z_1, z_2),\ N''(z_1, z_2) \in R[z_1, z_2] \tag{23}$$

*Proof:* Clearly 1) and 2) imply solvability in $R[z_1, z_2]$. On the other hand, the solvability of (6) in $R[z_1, z_2]$ implies 1) as $R[z_1, z_2] \subset R(z_1)\,[z_2]$. Moreover, it implies also 2) since $N(z_1, z_2)$ satisfying (15) in Lemma 2 satisfies (22) in Lemma 3 as well.

In a system theoretic version, the main result reads as follows.

*Theorem 4:* For a strictly proper system $\Sigma$ with $A_2$ invertible, the following facts are equivalent.

1) $\Sigma$ is locally controllable.

2) $\Sigma$ admits a deadbeat regulator.

3) There exist matrices $F(z_1, z_2)$ and $G(z_1, z_2)$ (generally with entries in $R(z_1)\,[z_2]$) satisfying both (21) and (22). At the same time, these matrices are polynomial

$$F(z_1, z_2),\ G(z_1, z_2) \in R[z_1, z_2].$$

*Proof:* This is an easy consequence of Theorems 1, 2, and 3.

To check the controllability and/or to find the deadbeat regulator, any Euclidean ring algorithm (in $R(z_1)\,[z_2]$) can be applied to solve (6). We recommend the following procedure.

### *Algorithm*

*Step 1)* Using elementary column operations (in $R(z_1)\,[z_2]$) perform the reductions

$$\left[\begin{array}{c|c} I-A_1z_1-A_2z_2 & B_1z_1+B_2z_2\\ \hline I & 0\\ 0 & I \end{array}\right] \rightarrow \left[\begin{array}{c|c} I & 0\\ \hline X & U\\ Y & V \end{array}\right]$$

such that $U$, $V \in R[z_1, z_2]$. If this is not possible STOP; in fact, (6) is not solvable even in $R(z_1)\,[z_2]$.

*Step 2)* Perform the division (in $R(z_1)\,[z_2]$)

$$V^{-1}Y=P+H \tag{25}$$

to get a "polynomial" (in $R(z_1)\,[z_2]$) part $P$ and a strictly proper part $H$.

*Step 3)* Set $F(z_1, z_2) = X - UP$ and $G(z_1, z_2) = Y - VP$. If either $F \notin R[z_1, z_2]$ or $G \notin R[z_1, z_2]$ then STOP; in fact, (6) has no solution.

This procedure is numerically efficient as it needs only unimodular operations in a Euclidean ring. When comparing it to the method employed in [4], [5], which is based on checking the rank condition (7) and consists in computing the zeros of 2-D polynomials, our method is more attractive.

*Example:* Consider the system $\Sigma$ given by

$$A_1=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\quad A_2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\quad B_1=\begin{bmatrix}0\\ 0\end{bmatrix}\quad B_2=\begin{bmatrix}1\\ 0\end{bmatrix}.$$

Note that the $A_2$ invertibility assumption is satisfied.

*Step 1)* By elementary column operations, reduce the matrix

$$\begin{bmatrix} I-A_1z_1-A_2z_2 & B_1z_1+B_2z_2\\ I & 0\\ 0 & I \end{bmatrix}=\begin{bmatrix} 1 & -(z_1+z_2) & z_2\\ -z_2 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

to the following structure

$$\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 1 & z_1+z_2 & -z_2\\ z_2 & 1+z_2(z_1+z_2) & -z_2^2\\ z_1+z_2 & (z_1+z_2)^2 & 1+z_2(z_1+z_2) \end{bmatrix}=\begin{bmatrix} I & O\\ X & U\\ Y & V \end{bmatrix}.$$

*Step 2)* In the ring $R(z_1)\,[z_2]$, divide $Y$ on the left by $V$

$$V^{-1}Y=[0\ \ -1]+\begin{bmatrix}\dfrac{z_1+z_2}{1-z_1z_2-z_2^2} & \dfrac{1+z_1^2-z_1z_2}{1-z_1z_2-z_2^2}\end{bmatrix}=P+H.$$

So one gets

$$P=[0\ \ -1].$$

*Step 3)* Compute the matrices

$$M'=X-UP=\begin{bmatrix}1 & z_1\\ z_2 & 1+z_1z_2\end{bmatrix}$$

$$N'=Y-VP=[z_1+z_2\ \ 1+z_1^2+z_1z_2].$$

Since the entries of $M'$ and $N'$ are polynomials, we obtained a solution in $R[z_1, z_2]$ of the Bezout equation (6). By (12), the transfer matrix of a deadbeat controller is

$$W'=-N'M'^{-1}=[-z_1\ \ -1]. \tag{26}$$

The locally controllable and causally reconstructible realization $\Sigma'$ *of* (26), *given by*

$$A_1'=A_2'=[0]\quad C'=[1]\quad B_1'=[-1\ \ 0]\quad B_2'=[0\ \ 0]\quad D=[0\ \ -1]$$

is a deadbeat controller for $\Sigma$.

In fact, it is easy to check that the state transition matrices $F_1$ and $F_2$ of the whole system of Fig. 1 are

$$F_1=\begin{bmatrix}A_1+B_1D' & B_1C'\\ B_1' & A_1'\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}$$

$$F_2=\begin{bmatrix}A_2+B_2D & B_2C'\\ B_2' & A_2'\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

and det $(I - F_1z_1 - F_2z_2) = 1$. This confirms that the whole system is finite memory.

*Remark:* Local controllability is stronger than global controllability. In fact, by linearity and shift invariance, zero controllability of local states implies zero controllability of global states, but the viceversa is not true, as shown by the following counterexample. Consequently, globally controllable 2-D systems exist whose states cannot be driven to zero in a finite number of steps by means of a causal feedback law.

*Example:* Consider the system $\Sigma$ given by the following matrices:

$$A_1 = B_2 = [1]$$

$$A_2 = B_1 = C = D = [0]$$

$\Sigma$ is globally controllable, since any initial global state $\mathcal{X}_0 = \Sigma x(h, -h)z_1^h z_2^{-h}$ is driven to zero in one step by the input function

$$u(h, k) = \begin{cases} -x(h-1, k+1) & \text{if } h+k=0 \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

However, (6) is not solvable in $R[z_1, z_2]$, so that $\Sigma$ is not locally controllable. In other words, $\Sigma$ is open, but not closed-loop controllable.

This is easy to explain. In fact, the input function (27) does not "causally" depend on the state; in particular, the single local-state $x(0, 0)$ is driven to zero by the input

$$u(h, k) = \begin{cases} -x(0, 0) & \text{if } (h, k) = (1, -1) \\ 0 & \text{otherwise} \end{cases}$$

where the "time points" (0, 0) and (1, −1) are not causally related.

Actually, input functions with support in the future of (0, 0) cannot control $x(0, 0)$ to zero in a finite number of steps. Therefore, a feedback controller that causally processes the states of $\Sigma$ is not a deadbeat controller.

### *Reconstructibility and Observers*

Analogously to 1-D case, causal reconstructibility means the possibility of determining the local-state $x(0, 0)$ when the input and output values are known on a finite set of points in the past.

In [5], the formal definition is given, and it is known that it is equivalent to a rank condition and to a Bezout identity "dual" to (5) and (6).

Furthermore, the causal reconstructibility property is equivalent to the existence of an exact observer that furnishes a state estimation whose estimate error vanishes after a finite number of steps, and the observer matrices are easily computed once this Bezout identity is solved.

So, in order to check the reconstructibility and/or to construct an exact observer, it suffices to solve a Bezout equation.

This equation can be solved in a way similar to the algorithm presented in the last section.

This procedure again depends on unimodular operations in the Euclidean ring $R(z_1)[z_2]$ and is, therefore, computationally attractive.

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# $n-D$ Polynomial Matrix Equations

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*Abstract*—**Linear matrix equations in the ring of polynomials in $n$ indeterminates ($n$-$D$) are studied. General- and minimum-degree solutions are discussed. Simple and constructive necessary and sufficient solvability conditions are derived. A new algorithm to solve the equations with general $n$-$D$ polynomial matrices is presented. It is based on elementary reductions in a greater ring of polynomials in one indeterminate, having as coefficients polynomial fractions in the other $n$-1 indeterminates, which makes the use of Euclidean division possible.**


## INTRODUCTION

When solving various control problems for standard systems, linear equations with polynomials in one indeterminate (1-$D$) are often employed. Facing the same problems for some nonstandard systems which are described by polynomials in more than one indeterminate ($n$-$D$) (such as delay-differential systems, multidimensional digital filters, systems depending on parameters, and systems described by partial differential equations), one naturally encounters similar equations in $n$-$D$ polynomials.

Although 1-$D$ polynomial equations are now, at least theoretically, well understood (see, e.g., [6]), their $n-D$ counterparts (especially the matrix ones) were not systematically studied until recently. Hence, let us first briefly review existing methods of solution of 1-$D$ polynomial equations in the light of their possible generalization for $n$-$D$ polynomial matrices.

The classical *indeterminate coefficients method* depends heavily on our ability to foretell the degrees of an expected solution. Such degree bounds have recently been found in [7], [8] for scalar 2-$D$ equations which makes it possible to apply this method. In contrast, no such estimates are available for the matrix equations so that attempts to employ the method [2] for them are not successful. Namely, whenever we fail to calculate a solution up to a certain degree, we can hardly infer whether the equation possesses a solution of a higher degree or if it is not solvable at all (see [7]).

The *polynomial reductions* via elementary operations do not work in general in the ring of $n$-$D$ polynomials as this ring is not Euclidean. Even endeavors to apply them, at least for zero coprime left-hand side [3], [4], fail as shown recently in [9]. On the other hand, an algorithm of reduction for scalar 2-$D$ equations [7], [8] makes the Euclidean division possible as it allows operations with polynomial fractions in one of the two indeterminates. This procedure has been successfully generalized for a special matrix equation in [10].

Finally, the methods based on certain state-space realizations [1], [12] could be essentially generalized also for $n$-$D$ matrices (recursively if $n \geqslant 3$). However, this has not yet been explicitly done up to now. Besides, the


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