LINEAR ALGORITHMS FOR COMPUTING CLOSED LOOP POLYNOMIALS OF 2D SYSTEMS

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ABSTRACT
In this paper some problems connected with the construction of 2D compensators and observers are analyzed. In particular we are concerned with algebraic criteria and linear algorithms for selecting 2D stable polynomials which can be realized as closed loop characteristic polynomials of a 2D system.

I. INTRODUCTION
Given a 2D system in state space form [1]

\[ x(h+1,k+1) = A_x x(h,k+1) + A_x x(h+1,k) + B u(h,k+1) + B u(h+1,k) \]
\[ y(h,k) = C x(h,k) \]

with m inputs, p outputs and n state variables, one of the fundamental control problems is to construct an asymptotic state observer and synthesize a state feedback law that provides a suitable dynamical behaviour.

The general structure of the solution to this problem comes out from the analysis of a Bezout 2D polynomial matrix equation and the state space realization of a matrix fraction description obtained from the solution of the Bezout equation.

More precisely, introducing the following matrices

\[ C(z_1,z_2) = \begin{bmatrix} 1-A_{11} z_1 & -A_{12} z_2 \\ -1 & -C \end{bmatrix} \]
\[ J(z_1,z_2) = \begin{bmatrix} I-A_{11} & -A_{12} z_2 \\ 1 & 1 \\ I & 1 \end{bmatrix} \]

which are the PBH test matrices for reconstructibility and controllability of (1), it has been proved [2,3] that

i) it is possible to compute a state observer whose estimation error e(h,k) converges to zero as h,k goes to infinity if and only if \( C(z_1,z_2) \) is full rank for all \( (z_1,z_2) \) in the closed polydisc

\[ \mathcal{P} = \{ (z_1,z_2) : |z_1| \leq 1, |z_2| \leq 1 \} \]

ii) given any polynomial \( q(z_1,z_2) \) in the ideal generated by the maximal order minors of \( C(z_1,z_2) \)
the equation

\[ P(z_1,z_2) C + Q(z_1,z_2) [I-A_{11} z_1 -A_{12} z_2] q(z_1,z_2) I \] (4)

is solvable. Moreover, if \( C(z_1,z_2) \) is full rank in \( \mathcal{P} \), we can select \( q(z_1,z_2) \) so that the intersection \( \mathcal{Y}(\mathcal{P}) \) is empty. In this case, any realization of

\[ \hat{\mathcal{Y}}(z_1,z_2) = \{ (z_1,z_2) (B z_1 + B z_2) q(z_1,z_2) \} \] (5)

satisfies the PBH controllability and reconstructibility criteria, provides an asymptotic observer.

iii) stabilizability by means of a dynamic state feedback compensator is equivalent to the full rank condition of \( \hat{\mathcal{Y}}(z_1,z_2) \) for all \( (z_1,z_2) \) in \( \mathcal{P} \).

iv) the equation

\[ (B z_1 + B z_2) N(z_1,z_2) \]

\[ + [I-A_{11} z_1 -A_{12} z_2] M(z_1,z_2) = I p(z_1,z_2) \]

is solvable for any polynomial \( p(z_1,z_2) \) in the ideal \( \mathcal{I} \) generated by the maximal order minors of \( \hat{\mathcal{Y}}(z_1,z_2) \). If \( \hat{\mathcal{Y}}(z_1,z_2) \) is full rank in \( \mathcal{P} \), we can choose \( p(z_1,z_2) \) that satisfies \( \mathcal{Y}(\mathcal{P}) \cap \mathcal{I} = \emptyset \). Given any state feedback compensator that realizes the matrix function

\[ N M^{-1} \]

and satisfies the PBH controllability and reconstructibility tests, the closed loop polynomial is \( p(z_1,z_2) \). So if \( p(z_1,z_2) \) has no zeros in \( \mathcal{P} \), then the above compensator makes the whole system stable.

Clearly, i), ii) and iii), iv) are relative to dual situations and the solutions are essentially the same. Hence in the sequel we only refer to the
synthesis of a stabilizing compensator with this
objective in mind, the following problems have to
be successively tackled:

a) check if the maximal order minors of \( R(z_1, z_2) \)
are devoid of common zeros in \( \mathcal{P} \).
b) if one has a positive answer for a), compute a
polynomial \( p(z_1, z_2) \) belonging to \( \mathcal{F} \) and having
no zeros in \( \mathcal{P} \).
c) solve the Bezout equation (5)
d) realize \( \mathbb{N}^{-1} \) by a 2D state space model that sat-
sifies the controllability and reconstructibili-
ty PRH tests.

In the sequel, we shall introduce an algorithm
which enables to solve a) and b) The reader is re-
ferred to [2] and [3] for a complete solution of
c) and d).

II. STABILIZABILITY CRITERION

Consider the ideal \( \mathcal{I} \) generated by the minors
of maximal order \( \mathfrak{M}(z_1, z_2), \mathfrak{M}_2(z_1, z_2), \ldots, \mathfrak{M}_s(z_1, z_2) \)
of the matrix (3) and compute their GCD (\( c(z_1, z_2) \)).

Denote by \( \mathcal{F} \) the ideal generated by the co-
prime polynomials \( m_1(z_1, z_2), m_2(z_1, z_2), \ldots, m_s(z_1, z_2) \),
where \( m_i(z_1, z_2) = \frac{m_i(z_1, z_2)}{c(z_1, z_2)}, i = 1, 2, \ldots, s \).

Since \( \mathcal{I} \cap \mathcal{F} = \mathcal{I} \cap \mathcal{F} \cup \mathcal{I} \cap \mathcal{F} \), for
testing stabilizability it is enough to check sepa-
ately

\[ \mathcal{F}(c) \cap \mathcal{F} = \mathcal{O} \] (7)

and

\[ \mathcal{F} \cap \mathcal{F} = \mathcal{O} \] (8)

As far as (1) is concerned, we can use standard
tests for 2D polynomial stability [5]. In order to
test if (8) is satisfied, we shall introduce a lin-
ear algorithm that does not require an explicit
computation of \( \mathcal{F}(c) \).

Let \( G = (g_1, g_2, \ldots, g_s) \) be a Gröbner basis in
\( \mathcal{F} \). Since \( \mathcal{F}(c) \) is a finite set, the quotient
ring \( R(z_1, z_2) / \mathcal{F} \) is a finite dimensional \( \mathbb{R} \)-vector
space and its dimension is equal to the number of
monic monomials \( d_1, d_2, \ldots, d_k \) that are not mul-
tiples of the leading power products of any of the
polynomials \( g_1, g_2, \ldots, g_s \) [5]. Note that this set
is empty if and only if the Gröbner basis \( G \) con-
sists of a non zero constant. In this case \( \mathcal{F}(c) = \mathcal{O} \)
and (8) is obviously true.

Assume now \( k > 0 \). Thus

\[ d_1, d_2, \ldots, d_k \]

can be assumed as a basis in \( R(z_1, z_2) / \mathcal{F} \).

Consider the following maps

\[ \mathcal{F}_1: R(z_1, z_2) / \mathcal{F} \rightarrow R(z_1, z_2) / \mathcal{F} \]

(9)

\[ \mathcal{F}_2: R(z_1, z_2) / \mathcal{F} \rightarrow R(z_1, z_2) / \mathcal{F} \]

(10)

They are both well defined, commutative linear
transformations on \( R(z_1, z_2) / \mathcal{F} \) and are represented
by a pair of commutative matrices \( M_1, M_2 \) in \( \mathbb{R}^{k \times k} \),
once a basis \( v_1, v_2, \ldots, v_k \) in \( \mathbb{R}^k \) has been asso-
ciated with \( \mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_k \). Note that the smallest \( \mathcal{F} \)
and \( \mathcal{F} \)-invariant subspace generated by \( \mathfrak{M} \) is the
whole space \( R(z_1, z_2) / \mathcal{F} \). Thus \( M_1, M_2 \), \( i \in \mathbb{N} \), generate \( \mathbb{R}^k \).

The construction of \( M_1 \) and \( M_2 \) essentially re-
quires to express \( z \mathfrak{M}_i \) and \( z \mathfrak{M}_j \), \( i = 1, 2, \ldots, k \), as
linear combinations of \( \mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_k \). This can be ac-
complished by applying the normal form algorithm
with respect to \( G \) [5].

The properties of \( \mathcal{F} \), as well as those of its
variety \( \mathcal{F}(c) \), directly reflect into the structure of
the pair \( M_1, M_2 \). Note first that the mapping

\[ R \rightarrow \mathbb{R}^{k \times k} : a \rightarrow a_k \]

is a monomorphism of \( R \) into \( \mathbb{R}^{k \times k} \), so that the image
set \( \mathbb{R} \mathfrak{M}_k \) is a subfield of \( \mathbb{R}^{k \times k} \) isomorphic to \( R \). Since
the matrices \( M_1 \) and \( M_2 \) commute each other and
with every element of \( \mathbb{R} \mathfrak{M}_k \), it follows that the mapping

\[ p(z_1, z_2) = \sum_{i=0}^{\infty} a_i z_1^i z_2^i \rightarrow p(M_1, M_2) \]

is a homomorphism of \( R(z_1, z_2) \) into \( \mathbb{R}^{k \times k} \). It is easy
to see that the kernel of the homomorphism is the
ideal \( \mathcal{F} \) that is

\[ p(z_1, z_2) \in \mathcal{F} \] is in \( p(M_1, M_2) = 0 \] (11)

As a corollary of the theorem on common eigenvectors
for commutative matrices [6], we have that

\[ (a, \beta) \in \mathcal{F}(c) \] if and only if \( M_1 \) and \( M_2 \) have a com-
mon eigenvector \( v \) and

\[ M_1 v = \alpha v \quad M_2 v = \beta v \]

On the other hand, basing on the Frobenius
theorem on simultaneous diagonalization of commu-
tative matrices, the variety \( \mathcal{F}(c) \) can be charac-
terized in the following way. Let \( T_1 = [t_{11}] \) and
\( T_2 = [t_{22}] \) be triangular matrices such that \( M_1 =
T_1^{-1} P T_1 \) and \( M_2 = T_2^{-1} P T_2 \) for some inverte-
tible matrix \( P \) in \( \mathbb{R}^{k \times k} \).

Then \( (a, \beta) \) belongs to \( \mathcal{F}(c) \) if and only if
By Lemma 2 the measure of the interval
\[ \mathcal{L} \left( \lambda \in [0,1] : |t_{rr}^{(2)} + \lambda (t_{rr}^{(1)} - t_{rr}^{(1)})| < 1 \right) \]
satisfies the inequalities
\[ \text{meas} \left( \mathcal{L} \right) < \frac{2}{p-1} < \frac{1}{K} \]
Since the minimum distance between \( \lambda_1 \) and \( \lambda_2 \), \( i=j \), is \( 1/K \), each interval contains at most one of the \( \lambda_i \).

Hence at least one \( \lambda_1 \), say \( \lambda_1 \), falls out of \( \mathcal{L} \) and the spectrum of \( P \)
\[ \lambda_i = \left( \begin{array}{cc} 2 \alpha & \beta \\ \gamma & \delta \end{array} \right) \]
satisfies the conditions of Section 2.

Theorem 1 and the above observations make possible to set a procedure for checking (2) using the following sequence of steps:

STEP 1: Compute the matrices \( M \) and \( M^2 \).

STEP 2: Compute a positive integer \( \rho \) such that \( \rho \) does not intersect the spectrum of \( M \) and \( M^2 \).

STEP 3: Compute \( \rho \) and the integer \( \gamma \).

STEP 4: Solve the Lyapunov equations:
\[ P_i - P_i \lambda_i + I = 0, \ldots, k \]
The system is stabilizing if and only if at least one of the above equations admits a positive definite solution.

3. COMPUTATION OF A STABLE 2D POLYNOMIAL

Assume that the procedure of Section 2 has been successful, which means that the system is stabilizable.

The aim of this section is to solve the problem mentioned at point (b) in the Introduction, that is, the computation of a stable 2D polynomial in \( \mathcal{F} \). As we shall see, the techniques of the previous section constitute the basic tools for obtaining this goal.

Let \( P_i \) be one of the matrices considered at step 4 in section 2, whose spectrum lies outside the unit disc
\[ P_i = \lambda_i \begin{pmatrix} 1 & \gamma \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \]
Denote by \( h(z_1, z_2) \in \mathcal{F} \) the characteristic polynomial of \( P_i \) and introduce the polynomial
\[ h(z_1, z_2) = z_1^\gamma (z_1^\alpha + (1-\lambda_i) z_1^{\gamma \alpha}) \]
Recalling (11), it is easy to see that \( h(z_1, z_2) \in \mathcal{F} \).

We have now to prove that \( h(z_1, z_2) \) defined in (16) is 2D-stable. For, factorize \( P_i \) as a product of prime factors:
\[ P_i = \prod_{r=1}^{R} \left( \begin{array}{cc} \rho_r & \gamma_r \\ \alpha_r & 1 \end{array} \right) \]
and notice that \( |\gamma_r| > 1, r=1,2,\ldots,k \). Each factor in the corresponding factorization of \( h(z_1, z_2) \)
\[ h(z_1, z_2) = \prod_{r=1}^{R} (z_1 - \frac{\gamma_r}{\rho_r}) \]
(14) turns out a stable 2D polynomial. In fact assume that \( (z_1, z_2) \) is a zero of the \( r \)-th factor in (14)
\[ \lambda_r^\gamma \gamma_r = (1-\gamma_r^\alpha) \]
Then
\[ \lambda_r^\gamma \gamma_r > 1 \]
proves that \( z_1^\gamma z_2^\alpha \notin \mathcal{F} \), since for any \( (z_1, z_2) \) in \( \mathcal{F} \) we have
\[ z_1^\gamma z_2^\alpha \leq 1 \]

Once \( h(z_1, z_2) \) has been obtained, a stable 2D polynomial in \( \mathcal{F} \), is given by
\[ \hat{p}(z_1, z_2) = c_{\lambda_1} c_{\alpha_1} h(z_1, z_2) \]
Note that by assumption \( c_{\lambda_1} c_{\alpha_1} \) which is the a.c. d. of the maximal order minors in (3), \( c_{\lambda_1} c_{\alpha_1} \)
satisfies (7) and hence is a stable 2D polynomial.

REMARK: If the entries of \( A_1, A_2, B_1, \) and \( B_2 \) belong to a subfield \( \mathbb{K} \) of the real field, the algorithm given above provides a stable polynomial \( h(z_1, z_2) \) in \( \mathcal{F} \cap \mathbb{K}[z_1, z_2], \) i.e., a polynomial having coefficients in \( \mathbb{K} \).

4. REFERENCES