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2-D Partial Fraction Expansions and Minimal Commutative Realizations

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2-D Partial Fraction Expansions and Minimal Commutative Realizations

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***Abstract*—This paper analyzes the class of rational functions in two variables which are realized by 2-D state-space models satisfying the commutativity assumption $A_1A_2 = A_2A_1$. A complete characterization of these transfer functions is given in terms of the existence of a suitable partial fraction expansion. It is shown that, under the commutativity assumption, minimal realizations are unique modulo algebraic equivalence, and that minimality is equivalent to local observability and reachability.**

I. INTRODUCTION

Consider a single-input single-output 2-D system (A_1, A_2, B, C) described by the following state equation:

$$x(h+1,k+1) = A_1x(h,k+1) + A_2x(h+1,k) + Bu(h+1,k+1)$$

$$y(h,k) = Cx(h,x) \tag{1}$$

where $x(h,k) \in \boldsymbol{R}^n$.

The 2-D transfer function $w(z_1, z_2)$ associated to the above system is easily derived and is expressed by a rational function of the following form:

$$w(z_1, z_2) = C(I - A_1z_1 - A_2z_2)^{-1}B. \tag{2}$$

It is well known [1]–[3] that as A_1, A_2, B, and C vary, $w(z_1, z_2)$ covers the class of causal rational functions in two variables, that is all rational functions whose denominators have a nonzero constant term. In other words, by updating (1), any causal rational function in two variables can be realized by a 2-D system. In general, it should be expected that any constraint we assume on the structure of the pair (A_1, A_2) translates into a restriction on the class of transfer functions which can be realized by (1). In this paper we shall concentrate our attention on pairs (A_1, A_2) of commutative matrices.

The main feature of the transfer functions obtainable from model (1) when A_1 and A_2 commute, is that their denominators factor completely in the complex field into linear factors [5]. As we shall see, this structural property is not sufficient to guarantee that a transfer function can be realized by a 2-D system with $A_1A_2 = A_2A_1$; it becomes sufficient if the commutativity assumption is weakened in the sense that A_1 and A_2 are assumed to be simultaneously triangularizable.

One of the main results of this paper is the complete characterization of the class of rational transfer functions which are realizable by 2-D systems (1) with $A_1A_2 = A_2A_1$. This is based on the existence of a partial fraction expansion of $w(z_1, z_2)$, whose structure is related to the spectral decomposition of the state space as a direct sum of (A_1, A_2)-invariant subspaces.

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The idea of considering commutative matrices was first investigated by Attasi [5], with reference to the special class of models given by the following updating equations:

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) \\ &\quad - A_1 A_2 x(h,k) + Bu(h,k) \\ y(h,k) &= Cx(h,k) \end{aligned}$$

with $A_1 A_2 = A_2 A_1$.

The transfer functions realizable by the Attasi's models are the so called "separable" rational functions, having the form $n(z_1,z_2)/p(z_1)q(z_2)$ where $n \in \mathbf{R}[z_1,z_2]$, $p \in \mathbf{R}[z_1]$, $q \in \mathbf{R}[z_2]$ and, conversely, any (causal) separable transfer function is realizable in the class of Attasi's models.

More recently the realization problem of separable transfer functions has been considered also in [6] and [7], dropping the commutativity assumption and adopting a 2-D state space model of Roesser's type

$$\begin{aligned} x(h+1,k+1) &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} x(h,k+1) \\ &\quad + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} x(h+1,k) \\ &\quad + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(h,k+1) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(h+1,k) \\ y(h,k) &= [C_1 \quad C_2] x(h,k) \end{aligned}$$

with $A_{12} = 0$. It turns out that also in this case the class of realizable transfer functions coincides with the set of separable rational functions.

However, since in this paper we are interested in 2-D models with *commuting* state updating matrices and with *first-order* recursive structure, the results presented here are quite different from those obtained in [6], [7] and [5] and, in particular, refer to transfer functions that *need not be separable*.

It is well known that the general problem of obtaining 2-D minimal realizations is still open, despite partial solutions have been obtained [5]–[9] by restricting the class of transfer functions to be realized and/or assuming state space models having particular structures.

As we shall see, differently from the general case, in a commutative context the solution of the minimal realization problem is quite similar to that obtained in the 1-D case.

By far, the most useful device for investigating the structure of the set of minimal 2-D commutative realizations is the Hankel matrix. In fact the existence of commutative realizations and their minimal dimension depend on the rank of a suitable Hankel matrix, whose elements are obtained from the power series expansion of the transfer function. Furthermore, Ho's linear algorithm for computing the matrices A_1, A_2, B, and C of a minimal realization applies.

Finally, minimality of commutative realizations is proved to be equivalent to local observability and reachability, so that we can use reduction algorithms based on these properties to compute a minimal realization.

II. PARTIAL FRACTION EXPANSIONS AND COMMUTATIVE REALIZATIONS

The aim of this section is to obtain a complete characterization of 2-D transfer functions that admit a "commutative realization," i.e., are realized by 2-D systems (A_1, A_2, B, C) with $A_1 A_2 = A_2 A_1$. For this purpose, we summarize in the following two theorems the basic properties of commutative matrices [10] that are relevant to the solution of our problem.

Theorem 1: Let A_1 and A_2 be a pair of commutative matrices. Then A_1 and A_2 can be simultaneously reduced to triangular form by a similarity transformation over the complex field $\mathbf{C}$.

Theorem 2: Let (A_1, A_2) be a pair of commutative matrices. Then there exists a similarity transformation given by a complex matrix T that simultaneously reduces A_1 and A_2 to block diagonal forms

$$\begin{aligned} T^{-1}A_1T &= \operatorname{diag}\{A_{11}, A_{12}, \cdots, A_{1t}\} \\ T^{-1}A_2T &= \operatorname{diag}\{A_{21}, A_{22}, \cdots, A_{2t}\} \end{aligned} \tag{3}$$

where each submatrix A_{ij} has a unique eigenvalue α_{ij}, $i = 1,2$; $j = 1,2,\cdots,t$.

To prove Theorem 2, let us first analyze the structure exhibited by any matrix M that commutes with a Jordan form

$$J = \operatorname{diag}\{J_1(\lambda_1), J_2(\lambda_2), \cdots, J_q(\lambda_q)\}$$

$\lambda_i \neq \lambda_j$ for $i \neq j$. It turns out that M is block diagonal

$$M = \operatorname{diag}\{M_1, M_2, \cdots M_q\}$$

and M_i has the same dimension as $J_i(\lambda_i)$, $i = 1,2,\cdots q$.

Consider now a pair (A_1, A_2), with $A_1A_2 = A_2A_1$. Because of the above observation, there exists a matrix P that reduces A_1 to its Jordan form and A_2 to a conformably partitioned block diagonal matrix, i.e.,

$$\begin{aligned} P^{-1}A_1P &= \operatorname{diag}\{J_1(\lambda_1), J_2(\lambda_2), \cdots, J_q(\lambda_q)\} \\ P^{-1}A_2P &= \operatorname{diag}\{M_1, M_2, \cdots, M_q\}. \end{aligned}$$

Similarly, since M_i and $J_i(\lambda_i\}$ commute, there exists a matrix Q_i that reduces M_i to its Jordan form and $J_i(\lambda_i)$ to a conformably partitioned block diagonal matrix

$$\begin{aligned} Q_i^{-1}M_iQ_i &= \operatorname{diag}\{J_{i1}(\mu_1), J_{i2}(\mu_2), \cdots, J_{ir}(\mu_r)\} \\ Q_i^{-1}J_i(\lambda_i)Q_i &= \operatorname{diag}\{N_{i1}, N_{i2}, \cdots, N_{ir}\} \end{aligned}$$

where $N_{i1}, N_{i2}, \cdots, N_{ir}$ have the same eigenvalue λ_i.

Consequently, the product of the similarity transformations induced by P and

$$Q = \operatorname{diag}\{Q_1, Q_2, \cdots, Q_q\}$$

reduces simultaneously A_1 and A_2 to the block diagonal structure (3) with $T = PQ$.

The properties summarized by Theorems 1 and 2 can be exploited for analyzing the structure of the transfer function $w(z_1, z_2)$ of a 2-D system (A_1, A_2, B, C) with $A_1A_2 = A_2A_1$. Since $w(z_1, z_2)$ is invariant under algebraic equivalence and, by Theorem 1:

$$\det(I - A_1z_1 - A_2z_2) = \prod_{i=1,2,\cdots,n} (1 - \lambda_i z_1 - \mu_i z_2) \tag{4}$$

the denominator $d(z_1, z_2)$ factorizes into linear factors.

Theorem 2 allows us to get a more refined characterization of $w(z_1, z_2)$. In fact, by suitably partitioning B and C, we have

$$\begin{aligned} w(z_1,z_2) &= \sum_j C_j(I - A_{1j}z_1 - A_{2j}z_2)^{-1}B_j \\ &= \sum_j \frac{n_j(z_1,z_2)}{\det(I - A_{1j}z_1 - A_{2j}z_2)} \\ &= \sum_j \frac{n_j(z_1,z_2)}{(1 - \alpha_{1j}z_1 - \alpha_{2j}z_2)^{\nu_j}}. \qquad (5 \end{aligned}$$

Moreover, for any pair $(\alpha_{1j}, \alpha_{2j}) \neq (0,0)$ we have $\deg n_j < \nu_j = \dim A_{ij}$.

Comparing (4) and (5) we see that there are ν_j pairs (λ_j, μ_j) in (4) that assume the same value $(\alpha_{1j}, \alpha_{2j})$.

Remark: The last sum in (5) gives us the partial fraction expansion of the transfer function of a 2-D system that satisfies the commutativity assumption $A_1 A_2 = A_2 A_1$.

Now, it is known that, given a rational function in two variables, in general it does not admit a partial fraction expansion whose terms correspond to the irreducible factors of the denominator. So, the effect produced by the commutativity assumption on A_1 and A_2 on the transfer function $w(z_1, z_2)$ of (A_1, A_2, B, C) is that:

i) the irreducible factors of the denominator of $w(z_1, z_2)$ are polynomials of the first degree in z_1 and z_2 of the form $1 - \alpha_{1j} z_1 - \alpha_{2j} z_2$,

ii) there exists a partial fraction expansion of $w(z_1, z_2)$ with structure (5).

What makes the above conditions very significant from a system theoretic point of view, is that they are also sufficient for the existence of a commutative 2-D realization of $w(z_1, z_2)$. The proof of this fact is constructive and is a consequence of the following lemmas.

Lemma 1: The rational function $z_2^h/(1 - az_1)^{h+k}$ with $k > 0$, admits a commutative realization.

Proof: A commutative realization of dimension $h + k$ is given by

$$(h+1) \times (h+1)$$

$$A_1 = \left[\begin{array}{ccccc|ccc} a &&&&&&& \\ & a &&&&&& \\ && \cdot &&&&& \\ &&& a &&&& \\ &&&& a &&& \\ \hline &&&&& a && \\ &&&&&& \ddots & \\ &&&&&&& a \end{array}\right]$$

$$(k-1) \times (k-1)$$

$$A_2 = \left[\begin{array}{ccccc|ccc} 0 & -1 &&&&&& \\ & 0 & -1 &&&&& \\ && \cdot &&&&& \\ &&& 0 & -1 &&& \\ &&&& 0 &&& \\ \hline &&&&& 0 && \\ &&&&&& \ddots & \\ &&&&&&& 0 \end{array}\right] \qquad B = \left[\begin{array}{c} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \\ \hline 0 \\ \cdot \\ 0 \end{array}\right]$$

$$C = \left[\begin{array}{ccccc|ccc} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 \end{array}\right]. \tag{6}$$

Lemma 2: The rational function $z_1^s z_2^r/(1 - az_1)^{r+s+k}$ with $k > 0$, admits a commutative realization.

Proof: Assume first $a \neq 0$ and write z_1^s as a linear combination of powers of $(1 - az_1)$:

$$z_1^s = \gamma_0 + \gamma_1(1 - az_1) + \cdots + \gamma_s(1 - az_1)^s.$$

Then we have

$$\frac{z_1^s z_2^r}{(1 - az_1)^{k+r+s}} = \gamma_0 \frac{z_2^r}{(1 - az_1)^{k+r+s}} + \gamma_1 \frac{z_2^r}{(1 - az_1)^{k+r+s-1}} + \cdots + \gamma_s \frac{z_2^r}{(1 - az_1)^{k+r}} \tag{7}$$

so that, because of Lemma 1, (7) can be realized as a direct sum of 2-D systems with structure (6).

Let now $a = 0$ and consider the 1-D realizations of z_1^r and z_2^s given by

$$(r+1) \times (r+1)$$

$$F_1 = \begin{bmatrix} 0 & 1 &&& \\ & 0 & 1 && \\ && 0 && \\ &&& \cdot & 1 \\ &&&& 0 \end{bmatrix} \qquad G_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 1 \end{bmatrix}$$

$$H_1 = [0 \quad 0 \quad 0 \quad \cdots \quad 1].$$

and

$$(s+1) \times (s+1)$$

$$F_2 = \begin{bmatrix} 0 & 1 &&& \\ & 0 & 1 && \\ && 0 && \\ &&& \cdot & 1 \\ &&&& 0 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 1 \end{bmatrix}$$

$$H_2 = [0 \quad 0 \quad 0 \quad \cdots \quad 1].$$

A 2-D commutative realization of $z_1^r z_2^s$ is given by

$$\begin{aligned} A_1 &= F_1 \otimes I_{s+1} \\ A_2 &= I_{r+1} \otimes F_2 \\ B &= G_1 \otimes G_2 \\ C &= H_1 \otimes H_2 \binom{r+s}{s}^{-1}. \end{aligned} \tag{8}$$

In fact, the coefficient of $z_1^i z_2^j$ in $C(I - A_1 z_1 - A_2 z_2)^{-1} B$ is given by $\binom{i+j}{i} C A_1^i A_2^j B$ and, using (8), we have

$$\begin{aligned} CA_1^i A_2^j B &= (H_1 \otimes H_2)(F_1 \otimes I)^i (I \otimes F_2)^j (G_1 \otimes G_2) \\ &= (H_1 \otimes H_2)(F_1^i \otimes I)(I \otimes F_2^j)(G_1 \otimes G_2) \\ &= (H_1 \otimes H_2)(F_1^i \otimes F_2^j)(G_1 \otimes G_2) \\ &= [(H_1 F_1^i) \otimes (H_2 F_2^j)](G_1 \otimes G_2) = H_1 F_1^i G_1 \otimes H_2 F_2^i G_2 \\ &= (H_1 F_1^i G_1)(H_2 F_2^j G_2) = 1 \end{aligned}$$

if $(i, j) = (r, s)$ and 0 otherwise.

The commutativity of the matrices A_1 and A_2 defined by (8) is easily proven as a consequence of the following identities:

$$\begin{aligned} (F_1 \otimes I)(I \otimes F_2) &= (F_1 I) \otimes (I F_2) = (I F_1) \otimes (F_2 I) \\ &= (I \otimes F_2)(F_1 \otimes I). \end{aligned}$$

Lemma 3: The transfer function $p(z_1, z_2)/(1 - az_1 - bz_2)^{h+k}$, $h = \deg p$ and $k > 0$, admits a commutative realization.

Proof: If $ab = 0$, the proof is a direct consequence of Lemma 2, once $p(z_1, z_2)$ has been expressed as a linear combination of monomials in z_1 and z_2. Assume now $ab \neq 0$ and introduce the change of variables given by

$$w_1 = az_1 + bz_2 \qquad w_2 = z_2. \tag{9}$$

Thus (6) becomes

$$p'(w_1, w_2)/(1 - w_1)^{h+k}, \deg p' = h \tag{10}$$

and, by the first part of the proof, (10) admits a realization (F_1, F_2, B, C) with $F_1 F_2 = F_2 F_1$. It is easy to check that the commutative 2-D system (A_1, A_2, B, C) with:

$$A_1 = aF_1 \qquad A_2 = F_2 + bF_1$$

realizes the transfer function.

Now, combining the above lemmas with the condition (5) on the structure of the transfer function, we obtain the following result.

Theorem 3: A 2-D transfer function $w(z_1, z_2) \in \boldsymbol{R}(z_1, z_2)$ is realizable by a commutative 2-D system if and only if it admits a partial fraction expansion of the following form:

$$w(z_1, z_2) = n_0(z_1, z_2) + \sum_{j=1,\cdots,t} n_j(z_1, z_2)/(1 - a_{1j}z_1 - a_{2j}z_2)^{\nu_j} \tag{11}$$

with $n_j(z_1, z_2) \in \mathbb{C}[z_1, z_2]$, $\deg n_j > \nu_j$, $1 \leqslant j \leqslant t$ and $n_0(z_1, z_2) \in \boldsymbol{R}[z_1, z_2]$.

The rational functions with structure (11) constitute a proper subset of the set of rational functions whose denominators can be factored into a product of first order polynomials with nonzero constant terms.

Actually, given a polynomial $d(z_1, z_2)$ that factors into linear terms

$$d(z_1, z_2) = \prod_{i=1,2\cdots h} (1 - \lambda_i z_1 - \mu_i z_2) \tag{12}$$

and a positive integer k, the set of polynomials

$$N = \{\, n(z_1, z_2) : \deg n < k,\ n/d \text{ has structure (11)} \}$$

is a subspace of the space $\boldsymbol{R}_k[z_1, z_2]$ of polynomials with degree less than k. Note that N is a proper subspace of $\boldsymbol{R}_k[z_1, z_2]$ unless in (12) all factors coincide (i.e., $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$ for all i, j) and $k \leqslant h$.

To prove this, let us first consider the simplest case, given by a transfer function of the following form:

$$w(z_1, z_2) = n(z_1, z_2)/(1 - a_{11}z_1 - a_{21}z_2)^{\nu_1}(1 - a_{12}z_1 - a_{22}z_2)^{\nu_2} \tag{13}$$

where $\deg n < \nu_1 + \nu_2$, and assume that $w(z_1, z_2)$ admits a partial fraction expansion:

$$w(z_1, z_2) = \frac{n_1(z_1, z_2)}{(1 - a_{11}z_1 - a_{21}z_2)^{\nu_1}} + \frac{n_2(z_1, z_2)}{(1 - a_{12}z_1 - a_{22}z_2)^{\nu_2}} \tag{14}$$

with $\deg n_1 < \nu_1$ and $\deg n_2 < \nu_2$. Equating (13) and (14) we obtain

$$n_1(z_1, z_2)(1 - a_{12}z_1 - a_{22}z_2)^{\nu_2} + n_2(z_1, z_2)(1 - a_{11}z_1 - a_{22}z_2)^{\nu_1} = n(z_1, z_2). \tag{15}$$

Letting $n_1(z_1, z_2)$ and $n_2(z_1, z_2)$ vary over $C_{\nu_1}[z_1, z_2]$ and $C_{\nu_2}[z_1, z_2]$, the polynomials $n(z_1, z_2)$ given by (15) span a proper subspace of $C_{\nu_1+\nu_2}[z_1, z_2]$ of dimension not greater than $\nu_1(\nu_1 + 1)/2 + \nu_2(\nu_2 + 1)/2$.

In particular, if $1 - a_{11}z_1 - a_{21}z_2$ and $1 - a_{12}z_1 - a_{22}z_2$ vanish on (z_{10}, z_{20}), then $n(z_{10}, z_{20}) = 0$ for any pair of polynomials (n_1, n_2).

This implies that if (13) is realizable by a commutative 2-D system, the variety of the numerator $n(z_1, z_2)$ includes the common zero of the linear factors in the denominator.

Similar arguments can be developed in the general case.

Remark 1: The property expressed in Theorem 1, i.e., that any pair of commutative matrices is simultaneously diagonalizable, provides a necessary commutativity condition.

However there are pairs of (lower) triangular matrices that do not commute. This suggests that simultaneously triangularizable matrices allow to realize a class of transfer functions which is wider than (11). In fact it has been proved in [4] that 2-D systems with (lower) triangular matrices realize any transfer function whose denominator factors into linear terms, as in (12).

Remark 2: The matrices A_1 and A_2 of the Roesser's model with $A_{12} = 0$ considered in [6], [7] can be simultaneously reduced to lower triangular form by a similarity transformation preserving the Roesser's structure. It is interesting to note that the class of separable transfer functions realized in this way is properly included in the class of transfer functions whose denominators factor into linear terms.

III. HANKEL MATRICES AND MINIMAL COMMUTATIVE REALIZATIONS

The results of the previous section, that provide a necessary and sufficient condition for the existence of commutative realizations, are based on the preliminary factorization (12) of a 2-D polynomial into linear factors.

In this section we shall introduce a different approach to commutative realizations, that exploits the representation theory of separable rational functions. As we shall see, many important realization problems can be formulated in terms of formal power series expansions of these functions and then solved by linear methods.

Let

$$\sum_{i,j} w_{ij} z_1^i z_2^j \tag{16}$$

be the power series expansion of a rational transfer function $w(z_1, z_2)$ and introduce the series

$$\sum_{i,j} w'_{ij} z_1 z_2 = w'(z_1, z_2) \tag{17}$$

where

$$w'_{ij} = \binom{i+j}{j}^{-1} w_{ij}. \tag{18}$$

Assume that $w(z_1, z_2)$ has a commutative realization (A_1, A_2, B, C). Then from

$$w(z_1, z_2) = C(I - A_1z_1 - A_2z_2)^{-1}B = \sum_{ij} \binom{i+j}{i} CA_1^iA_2^jBz_1^iz_2^j \tag{19}$$

we have

$$w'(z_1, z_2) = \sum_{ij} CA_1^iA_2^jBz_1^iz_2^j = C(I - A_1z_1)^{-1}(I - A_2z_2)^{-1}B \tag{20}$$

so that $w'(z_1, z_2)$ is the power series expansion of a separable rational function. The quadruplet A_1, A_2, B, and C, $A_1A_2 = A_2A_1$, is called a *representation* of $w'(z_1, z_2)$.

Vice versa, assume $w'(z_1, z_2)$ to be the expansion of a separable rational function. Then $w'(z_1, z_2)$ can be represented as in (20),(see [11]), with $A_1A_2 = A_2A_1$ and we go back to (19) following the previous steps in the reverse order.

The above discussion shows that the set of commutative realizations of $w(z_1, z_2)$ coincides with the set of representations of $w'(z_1, z_2)$.

In particular, $w(z_1, z_2)$ admits a commutative realization if and only if $w'(z_1, z_2)$ is the expansion of a separable transfer function. Moreover, minimal commutative realizations of $w(z_1, z_2)$ coincide with minimal representations of $w'(z_1, z_2)$. Consequently the algorithms for testing separability of $w'(z_1, z_2)$ and computing minimal representations apply directly to solve the problem of constructing minimal commutative realizations of $w(z_1, z_2)$.

The basic tool for checking the separability property of $w'(z_1, z_2)$ and for obtaining its minimal representation is provided by Hankel matrices. The Hankel matrix $\boldsymbol{H}(w')$ of a power series w' is an infinite matrix, whose rows and columns are indexed by the monomials $z_1^i z_2^j$. The matrix element indexed by $(z_1^i z_2^j, z_1^h z_2^k)$ is the coefficient $w'_{i+h,\, j+k}$ of the monomial $z_1^{i+h} z_2^{j+k}$.

The following facts are relevant in connecting the properties of $\boldsymbol{H}(w')$ with the representations of w' [11]:

i) w' is the expansion of a separable rational function if and only if $\operatorname{rank} \boldsymbol{H}(w') < \infty$;

ii) $\operatorname{rank} \boldsymbol{H}(w')$ gives the dimension of minimal representations (20) of w';

iii) all minimal representations of w' are algebraically equivalent.

As a consequence of iii), the minimal commutative realizations of $w(z_1, z_2)$ are essentially unique, modulo a change of basis in the local state space.

Since *noncommutative* minimal realizations of $w(z_1, z_2)$ need not be algebraically equivalent [1], the nature of solutions to the problem of realizing $w(z_1, z_2)$ by state-space models (1) of minimal dimension essentially depends on the preliminary assumption of commutativity of A_1 and A_2.

The finite rank condition i) on $\boldsymbol{H}(w')$ gives us a different way to prove that the structure of the denominator of a transfer function does not contain enough information to conclude about the existence of commutative realizations. This is illustrated by the following rational function:

$$w(z_1, z_2) = 1/(1 - z_1)(1 - z_1 - z_2). \tag{21}$$

Its power series expansion in a neighborhood of the origin is

$$w = \sum_{i,j} \binom{i+j+1}{j+1} z_1^i z_2^j.$$

So, by (18), we have

$$w' = \sum_{i,j} (i+j+1)/(j+1)\, z_1^i z_2^j.$$

In the associated Hankel matrix

$$\boldsymbol{H}(w') = \begin{array}{c|c:c:c:}
 & (1) & (z_1)(z_2) & (z_1^2)(z_1 z_2)(z_2^2) \\ \hline
(1) & H_{00} & H_{01} & H_{11} \\ \hdashline
(z_1) & & & \\
 & H_{10} & H_{11} & H_{12} \\
(z_2) & & & \\ \hdashline
(z_1^2) & & & \\
(z_1 z_2) & H_{20} & H_{21} & H_{22} \\
(z_2^2) & & &
\end{array}$$

the nth diagonal block matrix is

$$H_{nn} = (2n+1) \cdot \begin{bmatrix} 1 & 1/2 & 1/3 & \cdots & 1/(n+1) \\ 1/2 & 1/3 & 1/4 & \cdots & 1/(n+2) \\ & \cdots & & & \\ & \cdots & & & \\ 1/(n+1) & 1/(n+2) & & \cdots & 1/(2n+1) \end{bmatrix}$$

Now notice that $H_{nn}/(2n+1)$, $n = 0, 1, 2, \cdots$ are the $(n+1) \times (n+1)$ submatrices appearing in the upper left-hand corner of the Hankel matrix associated with the nonrational power series $-\log(1-x) = \Sigma_n x^n/n$.

Letting n go to infinity on both sides of $\operatorname{rank} \boldsymbol{H}(w') > \operatorname{rank} H_{nn}$, we obtain $\operatorname{rank} \boldsymbol{H}(w') = \infty$. This implies that (21) cannot be realized using commutative matrices A_1 and A_2, despite the denominator of (21) is a product of linear factors.

Note that the same conclusion can be drawn on the basis of Theorem 3. In fact there are no A and B such that the equality $(1-z_1)^{-1}(1-z_1-z_2)^{-1} = A(1-z_1)^{-1} + B(1-z_1-z_2)^{-1}$ holds.

For the explicit computation of a minimal representation of w', and hence of a minimal realization of w, we can resort to a modified Ho's algorithm [2]. As an alternative procedure, we can introduce the infinite matrices $\boldsymbol{O}$ and $\boldsymbol{R}$ defined by

$$\boldsymbol{O} = \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1 A_2 \\ CA_2^2 \\ \cdots \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} B & A_1 B & A_2 B & A_1^2 B & A_1 A_2 B & A_2^2 B \cdots \end{bmatrix}. \tag{22}$$

Because of the commutativity of A_1 and A_2, the columns of $\boldsymbol{R}$ span the local reachability subspace and the rows of $\boldsymbol{O}$ span the orthogonal complement of the unobservable subspace of (A_1, A_2, B, C). In fact the columns of the local reachability matrix of (A_1, A_2, B, C) have the structure [1], [2]

$$\left(A_1^i \sqcup\!\sqcup\, {}^j A_2 \right) B, \qquad i, j = 0, 1, 2 \cdots. \tag{23}$$

where the matrices $A_i^{-i} \sqcup\!\sqcup\, {}^i A_2$ are inductively defined as

$$A_1^r \sqcup\!\sqcup\, {}^s A_2 = A_1^r; \qquad A_1^0 \sqcup\!\sqcup\, {}^s A_2 = A_2^s$$

$$A_1^0 \sqcup\!\sqcup\, {}^s A_2^s = A_1 \left(A_1^{r-1} \sqcup\!\sqcup\, {}^s A_2 \right) + A_2 \left(A_1^r \sqcup\!\sqcup\, {}^{s-1} A_2 \right).$$

By the commutativity assumption, we have

$$\left(A_1^i \sqcup\!\sqcup\, {}^j A_2 \right) B = \binom{i+j}{j} A_1^i A_2^j B$$

which shows that the columns of $\boldsymbol{R}$ coincides with the columns of the reachability matrix, except for a nonzero multiplicative scalar. A similar argument applies to the rows of $\boldsymbol{O}$.

Since the Markov parameters are given by

$$w'_{ij} = CA_1^i A_2^j B$$

we have

$$\boldsymbol{H}(w') = \boldsymbol{OR} \tag{24}$$

and the same arguments used in the 1-D case show the equivalence between minimality and local reachability and observability in the commutative 2-D case. So, if we start with a nonminimal commutative realization (A_1, A_2, B, C), we can use linear algorithms [13] to eliminate the unreachable and the unobservable parts and thus obtain a minimal commutative realization.

Finally, it is worthwhile to observe that the set of commutative realizations of a given transfer function is properly included in the set of its (commutative and noncommutative) realizations. So, in general, it is possible to have minimal realizations with dimension smaller than minimal commutative realizations. In some cases, as shown by the following example, the difference is quite considerable.

Example: Consider the polynomial transfer function

$$w(z_1, z_2) = \sum_{i=1,2,\cdots m} \sum_{j=1,2,\cdots n} w_{ij} z_1^i z_2^j, \qquad w_{mn} \neq 0. \quad (25)$$

The dimension of its minimal *commutative* realizations is given by rank $H(w')$, where $w'_{ij} = \binom{i+j}{j}^{-1} z_1^i z_2^j$. Taking into account the positions occupied by the coefficients of the maximum degree monomial in $H(w')$, it is easy to see that rank $H(w') = (m+1)(n+1)$. Thus the Ho's algorithm directly provides a minimal commutative realization of dimension $(m+1)(n+1)$.

On the other side, a minimal *noncommutative* realization of $w(z_1, z_2)$ is the following:

$$(m+1)\times(m+1)$$

$$A_1 = \left[\begin{array}{cccc|cccc} & 0 & & & & 0 & & \\ \hdashline w_{10} & w_{11} & \cdots & w_{1m} & 0 & 1 & & \\ w_{20} & w_{21} & \cdots & w_{2m} & & 0 & 1 & \\ \cdot & \cdot\cdot & \cdot & & & & & 1 \\ & \cdot & & & & & & 0 \\ w_{n0} & w_{n1} & \cdots & w_{nm} & & & & \end{array}\right]$$

$$n \times n$$

$$A_2 = \left[\begin{array}{cccc|c} 0 & & & & \\ 1 & 0 & & & \\ & & & & 0 \\ & & 1 & & 0 \\ \hdashline w_{01} & \cdots & w_{0m} & 0 & \\ 0 & & 0 & & \\ & & \cdot & & \end{array}\right] \qquad B = \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \\ \\ \end{array}\right]$$

$$C = \left[\begin{array}{cccc|cccc} w_{00} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{array}\right]$$

whose dimension is $m+n+1$. In [9], using a Roesser's type model, a minimal realization of dimension $m+n$ has been presented. The difference of one unit depends on the fact that the Roesser's state updating equation introduces a one step delay between input and state values.

IV. CONCLUDING REMARKS

In this paper we have investigated the class of 2-D transfer functions which are realized by 2-D systems (A_1, A_2, B, C) whose state updating equations have structure (1) and the pair (A_1, A_2) satisfies the commutativity assumption $A_1 A_2 = A_2 A_1$. A peculiar property of this class is that the minimal commutative realization of a transfer function is locally reachable and observable as well as essentially unique (modulo a similarity transformation). Thus in some sense the commutativity assumption leads to a class of 2-D systems whose behaviour reminds the dynamics of 1-D systems.

Moreover, the check for the existence of commutative realizations of a transfer function $w(z_1, z_2)$ and the procedure for obtaining the matrices characterizing a minimal realization are essentially based on the properties of an associated Hankel matrix.

There are further points of tangency between 1-D theory and the theory of 2-D commutative systems [14].

For instance, consider a 1-D system (A, B, C) where A is in Jordan from and the column vector B is conformably partitioned

$$A = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \cdots & \\ & & & J_t \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \cdots \\ B_t \end{bmatrix}.$$

Then the system is reachable if and only if $J_1, J_2, \cdots, J_t$ have no common eigenvalues and each pair (J_i, B_i) is reachable.

An analogous reachability condition holds for a 2-D commutative system (A_1, A_2, B, C), where A_1 and A_2 have been reduced to the block diagonal forms (3) and B has been conformably partitioned. In fact, referring to this structure, local reachability is equivalent to the following pair of conditions:

i) there are no repeated pairs of eigenvalues $(\alpha_{1i}, \alpha_{2i})$ relative to the pairs (A_{1i}, A_{2i}), $i = 1, 2, \cdots, t$;

ii) the subsystems (A_{1i}, A_{2i}, B_i) are locally reachable.

Also when we deal with modal analysis of 2-D systems the commutativity assumption plays an essential role and enables us to work out a long term behavior analysis which reminds very closely the well-known 1-D theory of dominant modes.

Let $(\alpha_{1j}, \alpha_{2j})$, $j = 1, 2, \cdots, t$, be the pairs of eigenvalues corresponding to the block diagonal form (3) of the commutative matrices A_1 and A_2. Assume that

$$|\alpha_{11}| > |\alpha_{1j}|, \quad j \neq 1; \qquad |\alpha_{21}| > |\alpha_{2j}|, \quad j \neq 1 \qquad (26)$$

and that the diagonal blocks A_{11} and A_{21} have dimension 1. Then the pair $(\alpha_{11}, \alpha_{21})$ of greatest magnitude plays the role of "dominant pair," since it largely determines the asymptotic dynamics of the free evolution corresponding to a single initial local state $x(0,0)$.

In fact, suppose that $\{e_{11}\}$, $\{e_{21}, e_{22}, \cdots, e_{2k}\}, \cdots, \{e_{t1}, e_{t2}, \cdots, e_{tk}\}$ are bases of the A_1, A_2-invariant subspaces corresponding to the block diagonal representations (3) and refer A_1 and A_2 to the basis $\{e_{11}; e_{21}, \cdots e_{2k}; \cdots; e_{t1}, \cdots e_{tk}\}$.

Then the value of the local state in (h,k) produced by $x(0,0) = \gamma e_{11} + \Sigma_{i=1,2\cdots t} \Sigma_j \gamma_{ij} e_{ij}$ can be expressed as

$$x(h,k) = \binom{h+k}{k} A_1^h A_2^k x(0,0)$$

and, as $h+k$ goes to infinity, it can be approximated by

$$x(h,k) = \binom{h+k}{k} \alpha_{11}^h \alpha_{21}^k \gamma e_{11}. \qquad (27)$$

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