2D Systems Feedback Compensation: An Approach Based on Commutative Linear Transformations

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ABSTRACT

Algebraic properties of a pair of commutative matrices associated with an ideal in $\mathbf{R}[z_1, z_2]$ are exploited for characterizing the closed loop polynomial variety of a 2D system. Also algorithms are given to find under what constraints the closed loop variety can be assigned and to compute the MFD of a compensator.

1. INTRODUCTION

Let $N(z_1, z_2)$ and $D(z_1, z_2)$ be given matrices of dimension $p \times m$ and $m \times m$ respectively, with elements in the polynomial ring $\mathbf{R}[z_1, z_2]$. Our first concern in this paper is to analyze the structure of the complex variety of the polynomial

$$\Delta = \det(XN + YD), \tag{1.1}$$

where $X(z_1, z_2)$ and $Y(z_1, z_2)$ are arbitrary matrices of dimension $m \times p$ and $m \times m$ respectively, with elements in $\mathbf{R}[z_1, z_2]$. Particular attention is given to the case when N and D are right factor coprime, which has useful applications in 2D systems theory (i.e. state feedback design, observer synthesis, etc.).

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In the polynomial ring $\mathbf{R}[z]$, this problem and the more general one of solving the Bézout equation

$$XN + YD = C, \tag{1.2}$$

where C is a polynomial matrix of dimension $m \times m$, are well known, and there exists a satisfactory theory, which has a role in obtaining control algorithms for 1D systems using the polynomial matrix approach. Bézout equations in $\mathbf{R}[z_1, z_2]$ have been considered by several authors (see, for instance, [2, 3]) but at the moment the results available in the literature are not so strong as in $\mathbf{R}[z]$, and their application to control problems is sometimes questionable.

The way in which we will proceed is to introduce a Gröbner basis in the polynomial ideal $\mathfrak{J}(N, D)$ generated by the maximal order minors of

$$\begin{bmatrix} D \\ N \end{bmatrix}. \tag{1.3}$$

When N and D are coprime, we shall introduce a pair of commutative matrices (M_1, M_2) with the property that a polynomial p in $\mathbf{R}[z_1, z_2]$ is an element of $\mathfrak{J}(N, D)$ if and only if p is an annihilating polynomial of the pair (M_1, M_2).

The paper is divided into five sections. In Section 2 we define two commutative linear transformations on $\mathbf{R}[z_1, z_2]/\mathfrak{J}(N, D)$, and we investigate some properties of their matrix representations. In Section 3 internal and external representations of 2D systems are introduced, and the polynomial (1.1) is viewed as the characteristic polynomial of a feedback connection of 2D systems. In Sections 4 and 5 the assignability of the characteristic polynomial variety is related to the structure of matrix fraction descriptions (MFDs) representing the transfer matrix and the state equations of the closed loop system.

Throughout the paper we shall refer to right MFDs, right factors, and right coprimeness. Clearly all definitions and statements can be rephrased in terms of left MFDs in an obvious way.

2. LINEAR COMMUTATIVE TRANSFORMATIONS ON $\mathbf{R}[z_1, z_2]$

Let N be a $p \times m$ matrix and D a $m \times m$ matrix over $\mathbf{R}[z_1, z_2]$. By definition, N and D are right factor coprime (r.f.c.) if the relations

$$N = \overline{N}E, \qquad D = \overline{D}E,$$

where $\bar{N}, \bar{D}, E$ are matrices over $\mathbf{R}[z_1, z_2]$ with dimensions $p \times m$, $m \times m$, and $m \times m$ respectively, imply that $\det E$ is a nonzero constant. Denoting by $m_1, m_2, \dots, m_\mu$ the maximal order minors of (1.3), it is known that the following statements are equivalent:

(i) N and D are r.f.c.
(ii) $m_1, m_2, \dots, m_\mu$ are coprime polynomials.
(iii) The variety $\mathscr{V}(\mathfrak{J}(N, D))$ is a finite set.
(iv) The quotient ring $\mathbf{R}[z_1, z_2]/\mathfrak{J}(N, D)$ is a finite dimensional $\boldsymbol{R}$-vector space.

Coprimeness of N and D can be decided by computing a Gröbner basis $\mathscr{G} = (g_1, g_2, \dots, g_\nu)$ in the ideal $\mathfrak{J}(N, D)$, starting from the generators $m_1, m_2, \dots, m_\mu$. In fact the dimension over $\mathbf{R}$ of $\mathbf{R}[z_1, z_2]/\mathfrak{J}(N, D)$ is equal to the number of monic monomials $d_1, d_2, \dots$ that are not a multiple of the leading power products of any of the polynomials $g_1, g_2, \dots, g_\nu$ [4]. So N and D are r.f.c. if and only if $\{d_1, d_2, \dots\}$ is a finite set.

Note that this set is empty if and only if the Gröbner basis $\mathscr{G}$ contains a nonzero constant polynomial. In this case $\mathscr{V}(\mathfrak{J}(N, D)) = \varnothing$ and in [4] an explicit construction of polynomials $q_1, q_2, \dots, q_\mu$ such that $1 = \sum_i q_i m_i$ is given.

Assume now that the monomials $d_1, d_2 \dots$ constitute a nonempty finite set $\{d_1, d_2, \dots, d_k\}$. Thus the $\mathbf{R}$-vector space $\mathbf{R}[z_1, z_2]/\mathfrak{J}(N, D)$ is finite dimensional and

$$d_1 + \mathfrak{J} := \bar{d}_1, \qquad d_2 + \mathfrak{J} := \bar{d}_2, \dots, \qquad d_k + \mathfrak{J} := \bar{d}_k$$

can be assumed as a basis in it.

Consider the following maps:

$$\mathfrak{z}_1 : \mathbf{R}[z_1, z_2]/\mathfrak{J} \to \mathbf{R}[z_1, z_2]/\mathfrak{J} : q + \mathfrak{J} \mapsto z_1 q + \mathfrak{J}, \tag{2.1}$$

$$\mathfrak{z}_2 : \mathbf{R}[z_1, z_2]/\mathfrak{J} \to \mathbf{R}[z_1, z_2]/\mathfrak{J} : q + \mathfrak{J} \mapsto z_2 q + \mathfrak{J}. \tag{2.2}$$

They are both well defined, commutative linear transformations on $\mathbf{R}[z_1, z_2]/\mathfrak{J}(N, D)$ and are represented by a pair of commutative matrices M_1, M_2 in $\mathbf{R}^{k \times k}$, once a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ in $\mathbf{R}^k$ has been associated with $\bar{d}_1, \bar{d}_2, \dots, \bar{d}_k$. Note that the smallest $\mathfrak{z}_1$- and $\mathfrak{z}_2$-invariant subspace generated by $\bar{d}_1 = \bar{1}$ is the whole space $\mathbf{R}[z_1, z_2]/\mathfrak{J}(N, D)$. Thus $M_1^i M_2^j \mathbf{v}_1$, $i, j \in \mathbf{N}$, generate $\mathbf{R}^k$.

The construction of M_1 and M_2 essentially requires one to express $\overline{z_1 d_i}$ and $\overline{z_2 d_i}$, $i = 1, 2, \ldots, k$, as linear combinations of $\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_k$. This can be accomplished by applying the *normal form algorithm* with respect to $\mathscr{G}$ [4].

The properties of $\mathfrak{J}(N, D)$, as well as those of its variety $\mathscr{V}(\mathfrak{J}(N, D))$, directly reflect into the structure of the pair M_1, M_2. Note first that the mapping

$$\mathbf{R} \to \mathbf{R}^{k \times k} : \alpha \mapsto \alpha I_k$$

is a monomorphism of $\mathbf{R}$ into $\mathbf{R}^{k \times k}$, so that the image set $\mathbf{R} I_k$ is a subfield of $\mathbf{R}^{k \times k}$ isomorphic to $\mathbf{R}$. Since the matrices M_1 and M_2 commute with each other and with every element αI_k, it follows that the mapping

$$p(z_1, z_2) = \sum_{ij} a_{ij} z_1^i z_2^i \mapsto \sum_{ij} a_{ij} M_1^i M_2^j := p(M_1, M_2)$$

is a homomorphism of $\mathbf{R}[z_1, z_2]$ into $\mathbf{R}^{k \times k}$. It is easy to see that the kernel of the homomorphism is the ideal $\mathfrak{J}(N, D)$, that is,

$$p(z_1, z_2) \in \mathfrak{J}(N, D) \quad \Leftrightarrow \quad p(M_1, M_2) = 0. \tag{2.3}$$

As far as $\mathscr{V}(\mathfrak{J}(N, D))$ is concerned, it is easily shown that for any p in $\mathbf{R}[z_1, z_2]$, $\mathscr{V}(p) \supset \mathscr{V}(\mathfrak{J}(N, D))$ if and only if $p(M_1, M_2)$ is a nilpotent matrix. In fact, by Hilbert's *Nullstellensatz*, $\mathscr{V}(p) \supset \mathscr{V}(\mathfrak{J}(N, D))$ implies $p^h \in \mathfrak{J}(N, D)$ for some h and hence $p^h(M_1, M_2) = 0$. Conversely, if $p^h(M_1, M_2) = 0$ for some h, then $p^h \in \mathfrak{J}(N, D)$ and $\mathscr{V}(p) \supset \mathscr{V}(\mathfrak{J}(N, D))$. Note that the dimension k of the matrices M_1 and M_2 provides an upper bound for h.

Further properties of $\mathscr{V}(\mathfrak{J}(N, D))$ are obtained if we refer to the spectral structure of M_1 and M_2 [5]. For, as a corollary of the theorem on common eigenvectors for commutative matrices [6], we have that $(\alpha_1, \alpha_2) \in \mathscr{V}(\mathfrak{J}(N, D))$ if and only if M_1 and M_2 have a common eigenvector $\mathbf{v}$ and

$$M_1 \mathbf{v} = \alpha_1 \mathbf{v}, \qquad M_2 \mathbf{v} = \alpha_2 \mathbf{v}.$$

On the other hand, in view of the Frobenius theorem on simultaneous triangularization of commutative matrices [6], the variety $\mathscr{V}(\mathfrak{J}(N, D))$ can be characterized in the following way. Let $T_1 = [t_{ij}^{(1)}]$ and $T_2 = [t_{ij}^{(2)}]$ be triangular matrices such that $M_1 = P^{-1} T_1 P$ and $M_2 = P^{-1} T_2 P$ for some invertible matrix P in $\mathbf{C}^{k\times k}$. Then (α_1, α_2) belongs to $\mathscr{V}(\mathfrak{J}(N, D))$ if and only if there exists an integer i such that

$$t_{ii}^{(1)} = \alpha_1, \qquad t_{ii}^{(2)} = \alpha_2.$$

A natural question arises as to what extent the condition

$$p(R_1, R_2) = 0 \quad \Leftrightarrow \quad p(z_1, z_2) \in \mathfrak{J}(N, D) \tag{2.3'}$$

determines the structure of the pair of commutative matrices R_1 and R_2. The corresponding question for $\mathbf{R}[z]$ requires one to investigate the structure of matrices R that satisfy the condition

$$p(R) = 0 \quad \Leftrightarrow \quad p(z) \in \mathfrak{J} \tag{2.4}$$

for a given ideal $\mathfrak{J} \subset \mathbf{R}[z]$. In this case the answer is very simple: denoting by k the dimension of $\mathbf{R}[z]/\mathfrak{J}$, we have that

(a) matrices R have dimension greater than or equal to k;
(b) matrices R with dimension $k \times k$ are cyclic and similar to each other, and their minimum polynomial is the monic generator of $\mathfrak{J}$.

In the case of the ideal $\mathfrak{J}(N, D) \subset \mathbf{R}[z_1, z_2]$, point (a) naturally extends, as shown in the following theorem.

THEOREM 2.1. *Let $\mathfrak{J}(N, D) \subset \mathbf{R}[z_1, z_2]$ and let* $\dim \mathbf{R}[z_1, z_2]/\mathfrak{J}(N, D) = k > 0$. *Assume that R_1 and R_2 are commutative matrices satisfying* (2.3′). *Then R_1 and R_2 have dimension greater than or equal to k.*

Proof. To prove the theorem, it is enough to show that, for any pair of commutative matrices U and V in $\mathbf{R}^{n\times n}$, the subspace spanned by U^pV^q, $p, q = 0, 1, \ldots,$ has dimension not greater than n.

This is obviously true if U and V are simultaneously diagonalizable. If not, by a theorem of T. Motzkin and O. Taussky [7] there exist two sequences of matrices $\{U_h\}, \{V_h\}$ such that

$$U = \lim_{h\to\infty} U_h, \qquad V = \lim_{h\to\infty} V_h,$$

and for any h, U_h and V_h are simultaneously diagonalizable commutative matrices of dimension $n \times n$.

Consider $n+1$ distinct monic monomials $f_1, f_2, \dots, f_{n+1}$ in $\mathbf{R}[z_1, z_2]$. Since U_h and V_h are simultaneously diagonalizable, there exists a real valued vector $\mathbf{a}_h = (a_{1h} \ a_{2h} \ \cdots \ a_{n+1h})$ with dimension $n+1$, such that

$$0 = a_{1h} f_1(U_h, V_h) + a_{2h} f_2(U_h, V_h) + \cdots + a_{n+1,h} f_{n+1}(U_h, V_h) \quad (2.5)$$

and

$$1 = \sum_{i=1}^{n+1} |a_{ih}|. \quad (2.6)$$

Since $\mathbf{a}_1, \mathbf{a}_2, \dots$ is a compact sequence, by the Bolzano-Weierstrass property a converging subsequence $\mathbf{a}_{s_1}, \mathbf{a}_{s_2}, \dots$ can be extracted from it. Letting

$$\mathbf{b} = \begin{pmatrix} b_1 & b_2 & \cdots & b_{n+1} \end{pmatrix} = \lim_{j \uparrow \infty} \mathbf{a}_{s_i},$$

(2.6) implies

$$1 = \sum_{i=1}^{n+1} |b_i|.$$

Recalling (2.5), we have

$$\left\| \sum_{i=1}^{n+1} b_i f_i(U,V) \right\| = \left\| \sum_{i=1}^{n+1} \left(b_i - a_{s_j 1}\right) f_i(U,V) + a_{s_j i}\left[f_i(U,V) - f_i\left(U_{s_j}, V_{s_j}\right)\right] \right\|$$

$$\leqslant \sum_{i=1}^{n+1} |b_i - a_{s_j i}| \| f_i(U,V) \|$$

$$+ \sum_{i=1}^{n+1} |a_{s_j i}| \left\| f_i(U,V) - f_i\left(U_{s_j}, V_{s_j}\right) \right\|$$

Since, as j increases, $\mathbf{a}_{s_j}$ converges to $\mathbf{b}$ and $f_i(U_{s_j}, V_{s_j})$ converge to $f_i(U,V)$, $i = 1, 2, \dots, n+1$, we obtain

$$\sum_{i=1}^{n+1} b_i f_i(U,V) = 0.$$

Therefore, the space spanned by the matrices $U^p V^q$, $p, q = 0, 1, 2, \dots$, has dimension not greater than n. ■

The following theorem provides an extension of point (b) to pairs of commutative matrices with dimension $k \times k$.

THEOREM 2.2. *Let R_1 and R_2 be commutative matrices with dimension $k \times k$, satisfying the condition (2.3′). Furthermore, assume that there exists a nonzero vector* w *in* $\mathbf{R}^k$ *such that the smallest R_1- and R_2-invariant subspace of $\mathbf{R}^k$ which contains* w *is the whole space $\mathbf{R}^k$. Then R_1 and R_2 are simultaneously reducible by similarity to the matrices M_1 and M_2 associated with the maps $\mathfrak{z}_1$ and $\mathfrak{z}_2$ in* (2.1) *and* (2.2).

Proof. Consider the monomials $z_1 d_i$ and $z_2 d_i$, $i = 1,2,\ldots,k$. Then in the $\mathbf{R}$-linear span of $d_1, d_2, \ldots, d_k$ the polynomials

$$p_i = \sum_{j=1}^{k} \gamma_{ji} d_j = z_1 d_i \bmod \mathfrak{J}(N, D), \qquad i = 1,2,\ldots,k, \tag{2.7}$$

and

$$q_i = \sum_{j=1}^{k} \eta_{ji} d_j = z_2 d_i \bmod \mathfrak{J}(N, D), \qquad i = 1,2,\ldots,k, \tag{2.8}$$

are uniquely defined and can be computed using the normal form algorithm [4].

Introduce the $k \times k$ matrices

$$T = \left[d_1(M_1, M_2)\mathbf{v}_1 \quad d_2(M_1, M_2)\mathbf{v}_1 \quad \cdots \quad d_k(M_1, M_2)\mathbf{v}_1 \right], \tag{2.9}$$

$$P = \left[d_1(R_1, R_2)\mathbf{w} \quad d_2(R_1, R_2)\mathbf{w} \quad \cdots \quad d_k(R_1, R_2)\mathbf{w} \right]. \tag{2.10}$$

Both matrices are invertible. In fact let

$$0 = \sum_i \alpha_i d_i(M_1, M_2)\mathbf{v}_1$$

be any linear combination of the columns of T. This implies

$$0 = \sum_i \alpha_i d_i(M_1, M_2)(M_1^r M_2^s \mathbf{v}_1), \qquad r, s = 0, 1, \ldots,$$

and, since span$\{ M_1^r M_2^s \mathrm{v}_1,\ r, s = 0,1,\dots \} = \mathbf{R}^k$, we have

$$0 = \sum_i \alpha_i d_i(M_1, M_2).$$

Hence, recalling the condition (2.3), we obtain

$$\sum_i \alpha_i d_i(M_1, M_2) \in \mathfrak{I}(N, D),$$

which shows that the scalars $\alpha_1, \alpha_2, \dots, \alpha_k$ are all zero. The same argument applies to the columns of P.

From (2.7), (2.8), and (2.3) it follows that

$$\left[M_1 d_i(M_1, M_2) - \sum_j \gamma_{ji} d_j(M_1, M_2) \right] \mathrm{v}_1 = 0,$$

$$\left[M_2 d_i(M_1, M_2) - \sum_j \eta_{ji} d_j(M_1, M_2) \right] \mathrm{v}_1 = 0,$$

so that

$$T^{-1} M_1 T = [\gamma_{ji}] \quad \text{and} \quad T^{-1} M_2 T = [\eta_{ji}].$$

By the same arguments one obtains

$$P^{-1} R_1 P = [\gamma_{ji}] \quad \text{and} \quad P^{-1} R_2 P = [\eta_{ji}].$$

We therefore have that R_1 and R_2 are simultaneously reducible to M_1 and M_2 by the similarity transformation induced by PT^{-1}. ∎

REMARK. When we consider an ideal $\mathfrak{I}$ in $\mathbf{R}[z]$, the $k \times k$ matrices R which satisfy (2.4) turn out to be cyclic. So in this case we do not need to assume cyclicity to prove that these matrices are related each other by similarity transformations.

On the other side, as we have seen, the cyclicity assumption for R_1 and R_2 or, equivalently, the existence of w is necessary in the case of an ideal in $\mathbf{R}[z_1, z_2]$, as we cannot exclude the existence of $k \times k$ commutative matrices R_1 and R_2 which satisfy (2.3′) and cannot be reduced to M_1 and M_2 by similarity transformations. To see this, consider for instance the polynomial ideal $\mathfrak{I}$ generated by $z_1^2, z_1 z_2, z_2^2$. The quotient space $\mathbf{R}[z_1, z_2]/\mathfrak{I}$ has dimension 3, and using the normal form algorithm, we obtain

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.11}$$

The commutative pair given by

$$R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.12}$$

although it satisfies (2.3′), cannot be reduced by similarity to (2.11). To see that, it is enough to assume $\mathbf{v}_1 = [1\ \ 0\ \ 0]'$, so that $\mathbf{v}_1, M_1\mathbf{v}_1, M_2\mathbf{v}_1$ span $\mathbf{R}^3$, while there isn't any vector $\mathbf{w}$ such that $R_1^r R_2^s \mathbf{w}$, $r, s = 0, 1, \dots,$ span $\mathbf{R}^3$.

3. ASSIGNABILITY OF THE CLOSED LOOP POLYNOMIAL VARIETY

Any $p \times m$ rational matrix in two variables $W(z_1, z_2)$ can be expressed in terms of right or left MFDs as

$$W = N_R D_R^{-1} = D_L^{-1} N_L. \tag{3.1}$$

We say that $N_R D_R^{-1}$ ($D_L^{-1} N_L$) is right (left) factor coprime if N_R and D_R (N_L and D_L) are r.f.c. (l.f.c.). According to the procedure introduced in Section 1, each MFD in (3.1) is associated with the ideal generated by the maximal order minors in the matrices $[N_R'\ D_R']$ or $[N_L\ D_L]$ and with the varieties $\mathscr{V}(\mathfrak{I}(N_R, D_R))$ or $\mathscr{V}(\mathfrak{I}(N_L, D_L))$.

If we restrict to (right or left) factor coprime MFDs of $W(z_1, z_2)$ it can be proved that the corresponding ideals and varieties are independent of the MFD. In other terms, given any l.c. MFD $D_L^{-1} N_L$ and any r.c. MFD $N_R D_R^{-1}$ of W, we have [8]

$$\mathfrak{I}(N_R, D_R) = \mathfrak{I}(N_L, D_L)$$

and, obviously

$$\mathscr{V}(\mathfrak{J}(N_R, D_R)) = \mathscr{V}(\mathfrak{J}(N_L, D_L)). \tag{3.2}$$

The points of (3.2) only depend on $W(z_1, z_2)$ and are therefore called *rank singularities* of W. The finite set (3.2) will be denoted by $\mathscr{V}(W)$.

Polynomial matrices with elements in $\mathbf{R}[z_1, z_2]$ constitute a fundamental tool for studying the dynamics of 2D systems. We recall that a finite dimensional discrete linear 2D system in state space form $\Sigma = (A_1, A_2, B_1, B_2, C, L)$ is defined by the following equations:

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h, k+1) + A_2 x(h+1, k) \\ &\quad + B_1 u(h, k+1) + B_2 u(h+1, k), \\ y(h, k) &= Cx(h, k) + Lu(h, k), \end{aligned}$$

where A_1 and A_2, B_1 and B_2, C, and L are matrices of respective dimensions $n \times n$, $n \times m$, $p \times n$, and $p \times m$, and where $u(h, k)$, $x(h, k)$, $y(h, k)$ are vectors of respective dimensions m, n, p.

The input/output behavior of Σ is given by its *transfer matrix*

$$W_\Sigma(z_1, z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + L, \tag{3.3}$$

which is a $p \times m$ proper rational matrix in two variables. The polynomial $\det(I - A_1 z_1 - A_2 z_2)$ is called the *characteristic polynomial* of Σ.

Consider now the feedback connection of Σ and $\tilde{\Sigma} = (F_1, F_2, G_1, G_2, H, J)$, where F_1 and F_2, G_1 and G_2, H, J are matrices of respective dimensions $\tilde{n} \times \tilde{n}$, $\tilde{n} \times p$, $m \times \tilde{n}$, $m \times p$; and assume that Σ is strictly proper, i.e. $L = 0$. The closed loop characteristic polynomial of the feedback connection can be expressed in terms of suitable MFDs of the transfer matrices W_Σ and $W_{\tilde{\Sigma}}$ [2]. In fact, let

$$ND^{-1} = W_\Sigma, \qquad X^{-1}Y = W_{\tilde{\Sigma}}$$

be MFDs such that

$$\det D = \det(I - A_1 z_1 - A_2 z_2), \tag{3.4a}$$

$$\det X = \det(I - F_1 z_1 - F_2 z_2). \tag{3.4b}$$

Then the closed loop characteristic polynomial is given by

$$\Delta(z_1, z_2) = \det(XD + YN). \tag{3.5}$$

From a system theoretic point of view, and in particular in the framework of stability analysis, it is quite important to investigate to what extent the variety $\mathscr{V}(\Delta)$ can be modified by varying X and Y over the polynomial matrices in two variables, with the constraint $\det X(0,0) \neq 0$ (which corresponds to assuming that $X^{-1}Y$ is a proper rational matrix). If D and N are not coprime, a greatest common right divisor $E(z_1, z_2)$ can be extracted, and after substituting $N_R E$ for N and $D_R E$ for D, (3.5) becomes

$$\Delta(z_1, z_2) = \det E \det(XD_R + YN_R). \tag{3.6}$$

Thus the closed loop polynomial Δ is the product of $\det E$, which is an invariant factor under feedback connection, and a polynomial $\Delta_R = \det(XD_R + YN_R)$, whose variety is completely characterized by the following theorem.

THEOREM 3.1. *Let $N_R D_R^{-1}$ be a r.c. MFD of W_Σ, and let*

$$\Delta_R = \det(XD_R + YN_R) \tag{3.7}$$

Then, for any X and Y, $\mathscr{V}(\Delta_R) \supseteq \mathscr{V}(W_\Sigma)$ and, conversely, for any algebraic curve $\mathscr{C}$ that includes $\mathscr{V}(W_\Sigma)$, there exist X and Y such that $\mathscr{V}(\Delta_R) = \mathscr{C}$.

Proof. The inclusion $\mathscr{V}(\Delta_R) \supseteq \mathscr{V}(W_\Sigma)$ *is a direct consequence of the Binet-Cauchy formula.*

For the converse, let q be a polynomial such that $\mathscr{V}(q) = \mathscr{C}$. By Hilbert's *Nullstellensatz*, q^r belongs to $\mathfrak{I}(W_\Sigma)$ for some integer r.

Let $M_i(z_1, z_2)$, $i = 1, 2, \ldots, \mu$, be the submatrices of maximal order in $[N_R', D_R']$, and denote by $m_i(z_1, z_2)$, $i = 1, 2, \ldots, \mu$, the corresponding maximal order minors. Then there exist polynomials $s_1, s_2, \ldots, s_\mu$, such that

$$q^r = \sum_i m_i s_i$$

and constant matrices of suitable size X_i, Y_i, $i = 1, 2, \ldots,$ such that

$$M_i(z_1, z_2) = X_i D_R + Y_i N_R. \tag{3.8}$$

Premultiplying (3.8) by $\operatorname{adj} M_i\ s_i$ and summing over i, we obtain

$$\left[\sum_i s_i(\operatorname{adj} M_i)X_i\right]D_R + \left[\sum_i s_i(\operatorname{adj} M_i)Y_i\right]N_R$$

$$= \sum_i s_i(\operatorname{adj} M_i)M_i = \sum_i s_i m_i I_m = q^r I_m.$$

So, if we choose in (3.7) $X = \Sigma_i s_i(\operatorname{adj} M_i)X_i$ and $Y = \Sigma_i s_i(\operatorname{adj} M_i)Y_i$, we obtain

$$\Delta_R = \det(q^r I_m) = q^{rm}.$$

This gives $\mathscr{V}(\Delta_R) = \mathscr{V}(q) = \mathscr{C}$. ∎

The previous theorem clarifies the role played by the set of rank singularities $\mathscr{V}(W_\Sigma)$ in the assignability of the closed loop variety $\mathscr{V}(\Delta)$. Actually, given a right MFD $ND^{-1} = W_\Sigma$, the algebraic curve $\det(XD + YN) = 0$ includes $\mathscr{V}(W_\Sigma)$ for any X and Y. Moreover, when ND^{-1} is factor coprime, the algebraic curve is freely assignable except that it must include $\mathscr{V}(W_\Sigma)$.

Remark. When we deal with transfer matrices in one indeterminate, the problem of assigning the closed loop characteristic polynomial is much simpler. In fact factor coprimeness in $\mathbf{R}[z]$ implies that the Bézout equation (1.2) is solvable for any C and in particular for $C = I_m$, so that $\mathscr{V}(W_\Sigma)$ is empty.

4. ALGORITHMS FOR COMPUTING $X^{-1}Y$

Theorem 3.1 and the factorization (3.6) give us a complete characterization of the closed loop polynomial variety in the sense that $\mathscr{V}(\Delta)$ can be arbitrarily assigned except that it must include $\mathscr{V}(W)$ and $\mathscr{V}(\det E)$. In other words, $\mathscr{V}(W)$ and $\mathscr{V}(\det E)$ represent the whole set of points of $\mathscr{V}(\Delta)$ which are not affected by feedback compensation.

The aim of this section is to provide some algorithms for effectively computing a MFD $X^{-1}Y$ of a feedback compensator $\tilde{\Sigma}$. More precisely, given a pair of polynomial matrices $N(z_1, z_2)$ and $D(z_1, z_2)$ of dimensions

$p \times m$ and $m \times m$ respectively and an algebraic curve $\mathscr{C} \subset \mathbf{C} \times \mathbf{C}$ associated to the equation $p(z_1, z_2) = 0$, the algorithms we shall introduce solve the following problems:

(i) Decide if $\mathscr{C}$ is assignable, i.e. decide if there exist X and Y such that

$$\mathscr{V}(\Delta) = \mathscr{V}(\det(XD + YN)) = \mathscr{C}. \tag{4.1}$$

(ii) Whenever $\mathscr{C}$ is assignable, compute a pair of matrices X and Y which solve (4.1).

In order to answer question (i), we have to decide whether $\mathscr{C}$ includes $\mathscr{V}(\det E)$ and $\mathscr{V}(W)$. This will be done in three steps.

Step 1 [*Computation of* $\det E$]. An obvious way to determine $\det E$ is to express the transfer matrix ND^{-1} as a right coprime MFD $N_R D_R^{-1}$ using either the primitive factorization algorithm [9] or the Lai and Chen algorithm [10]. This gives

$$\det E = \frac{\det D}{\det D_R} \tag{4.2}$$

An alternative approach is required to evaluate the maximal order minors $q_1, q_2, \ldots, q_\mu$ of the polynomial matrix $[D' \;\; N']$ and, using linear operations, to compute $\det E$ as their greatest common divisor.

Note that, by equating the maximal order minors on both sides of

$$\begin{bmatrix} D \\ N \end{bmatrix} = E \begin{bmatrix} D_R \\ N_R \end{bmatrix}, \tag{4.3}$$

one gets

$$q_i = m_i \det E, \qquad i = 1, 2, \ldots, \mu, \tag{4.4}$$

which gives a set of generators $m_1, m_2, \ldots, m_\mu$ for the ideal $\mathfrak{I}(N_R, D_R) = \mathfrak{I}(W)$.

Step 2 [*Check the inclusion* $\mathscr{C} \supset \mathscr{V}(\det E)$]. Clearly $\mathscr{C} \supset \mathscr{V}(\det E)$ if and only if $\det E$ is a divisor of p^j for some positive integer j, and $\deg(\det E)$ provides an upper bound for the integer j.

Step 3 [*Check the inclusion* $\mathscr{C} \supset \mathscr{V}(W)$]. According to Section 2 and using the generators $m_1, m_2, \dots, m_\mu$ obtained in step 1, we associate a pair of commutative matrices M_1, M_2 with the ideal $\mathfrak{J}(W)$. Then the inclusion $\mathscr{C} \supset \mathscr{V}(W)$ is equivalent to the nilpotency of $p(M_1, M_2)$.

Suppose now that $\mathscr{C}$ is assignable. The computation of the matrices X and Y required at point (ii) can be performed in the following way.

Using steps 1 and 3 above, compute the polynomials $\det E, m_1, m_2, \dots, m_\mu$ and the matrices M_1, M_2. Let r be the smallest integer such that $p(M_1, M_2)^r = 0$. Then $p^r \in \mathfrak{J}(W)$ and the normal form algorithm gives the polynomials s_i satisfying

$$\sum_i s_i m_i = p^r. \tag{4.5}$$

Compute now the matrices $X = \Sigma_i s_i (\operatorname{adj} M_i) X_i$ and $Y = \Sigma_i s_i (\operatorname{adj} M_i) Y_i$, where X_i and Y_i are the constant matrices considered in the proof of Theorem 3.1. Then $\Delta = XD + YN$ provides a solution of the equation (4.1). In fact, according to the proof of Theorem 3.1 and recalling (4.4) and (4.5), we have

$$\det(XD + YN) = \det\left[I_m \sum_i s_i q_i\right] = (\det E)^m \left(\sum_i s_i m_i\right)^m = (\det E)^m p^{mr}.$$

Since $\mathscr{V}(\det E) \subset \mathscr{C} = \mathscr{V}(p)$, the variety of $\det(XD + YN)$ is the curve $\mathscr{C}$.

5. CONNECTIONS WITH THE STATE SPACE APPROACH

In Section 3 we have shown that when we tackle the problem of synthesizing a feedback compensator, we need to investigate the variety associated with the polynomial (1.1). The algorithms given in Section 4 refer to MFDs of the transfer matrices W_Σ and $W_{\tilde{\Sigma}}$ and make no explicit mention to the underlying 2D state space models of Σ and $\tilde{\Sigma}$.

The connection between the transfer matrix and the state variable approaches occurs through the possibility of expressing the transfer matrix W_Σ as a MFD that satisfies Equation (3.3) and, conversely, of realizing $X^{-1}Y$ by means of a 2D system $\tilde{\Sigma}$ that satisfies Equation (3.4b). The problem of constructing a right MFD ND^{-1} starting from the matrices of Σ, under the constraint given by Equation (3.4a), can be linearly solved using the method presented in [10]. Also, the problem of constructing a 2D system $\tilde{\Sigma} = (F_1, F_2, F_1, G_2, H, J)$ that realizes $X^{-1}Y$ under the constraint (3.4b) has an explicit solution that can be obtained from X and Y using linear algorithms [11].

As a final remark, it is interesting to point out that the problem of deciding if a given algebraic curve $\mathscr{C}$ is assignable is solvable in terms of linear operations utilizing directly the matrices of the state model of Σ.

In order to compute a set of generators $m_1 = \det D_R, m_2, \dots, m_\mu$ for the ideal $\mathfrak{J}(W)$, note that

$$W = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2)$$

$$= [C \operatorname{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2)][\det(I - A_1 z_1 - A_2 z_2) I_m]^{-1}$$

and apply the procedure of Section 4 to the right MFD ND^{-1} with

$$N = C \operatorname{adj}(I - A_1 z_1 A_2 z_2)(B_1 z_1 + B_2 z_2),$$

$$D = \det(I - A_1 z_1 - A_2 z_2) I_m.$$

This gives us the maximal order minors of $[N' \ \ D']$, and by eliminating their g.c.d. we obtain the maximal order minors $m_1 = \det D_R, m_2, \dots, m_\mu$ of $[N_R' \ \ D_R']$.

Finally, $\det E$ can be computed as $\det(I - A_1 z_1 - A_2 z_2)/m_1$.

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