Dynamic Regulation of 2D Systems: A State-Space Approach

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Submitted by Jan C. Willems

ABSTRACT

The possibilities of modifying the dynamical behavior of 2D state-space models by output feedback compensation are investigated, and a complete characterization of the closed-loop polynomial varieties is given. It turns out that plant hidden modes and rank singularities of the transfer function are the unique constraints we have to cope with in the compensator synthesis. The proof of this result is based on algebraic manipulations of 2D MFDs and on a coprime realization algorithm.

1. INTRODUCTION

The first contributions [1–3] that discussed the problem of defining dynamical systems with input, output, and state functions depending on two independent variables appeared nearly 15 years ago. In principle, they were motivated by the necessity of investigating recursive structures for processing two-dimensional data.

This processing has essentially been performed for a long time using discrete filters given by ratios of polynomials in two indeterminates or by algorithms assigned via difference equations. Thus the idea of input-output description of systems by transfer functions in two indeterminates, as well as

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the design and analysis techniques based on the frequency response and on the two-dimensional z transform, has been well known for many years.

The new idea that originated research on 2D systems consisted in considering these algorithms (i.e., transfer functions and difference equations in two indeterminates) as external representations of dynamical systems and hence in introducing for such systems the concepts of state and its updating equations. It turns out that the models obtained in this way are suitable for providing state-space descriptions for a large class of processes which depend on two independent variables. Typically, they apply to two-dimensional data processing in various fields, as seismology, X-ray image enhancement, image deblurring, digital picture processing, etc. Also, 2D systems constitute a natural framework for modelling multivariable networks, large-scale systems obtained by interconnecting many subsystems, and, in general, physical processes where both space and time have to be taken into account.

In this paper we shall be concerned with the effects of output feedback compensators on 2D systems. We shall approach this subject from the point of view of classical system theory, by connecting the structural properties of the state-variable description with the possibility of assigning the closed-loop characteristic polynomial via output feedback.

The analysis will be developed on the basis of 2D polynomial matrix algebra. 2D matrix fraction descriptions (MFDs) provide a very convenient tool to investigate how input-output maps (characteristic of the classical methods in filter theory) are associated with internal representations (adopted in control problems) and to obtain the transfer matrices of compensators by solving Bézout polynomial equations in two variables.

A few observations might serve to motivate this detailed reexamination of feedback theory in the 2D context. Recently there has been increasing interest in studying 2D control problems, and mainly two different approaches have been pursued.

The first approach is essentially reductionist, in the sense that 2D systems are viewed as 1D systems over the ring of polynomials in one variable, while the second fully exploits the partial ordering of the 2D structure and data processing is not connected with any preferred direction.

In pursuing the first approach [4, 5], compensators have been introduced that preserve quarter-plane causality as well as compensators that do not. However, in the former case the feedback performance that can be obtained is not so good as for 2D compensators with unconstrained structure. Moreover, most results apply to Roesser models only.

Following the second approach, some authors [6] dealt with an input-output analysis of 2D systems, based on a factorization of the plant and compensator transfer matrices in two variables; others [7, 8] dealt with state-space models and 2D PBH controllability and reconstructibility criteria.

The unquestioned success of the input-output and the state-space compensation methods in 1D theory mainly relies on the canonical properties of minimal realizations, allowing for a polynomial-matrix (i.e. input-output) solution of control problems and for a subsequent synthesis of the compensator transfer matrix that does not introduce unwanted hidden modes in the feedback loop. However, since the equivalence between minimal and reachable and observable realizations no longer holds in the 2D case, the extension of classical techniques has presented a lot of difficulties.

One of our objectives in this paper is to formulate a realization procedure which leads to 2D systems free of hidden modes without pursuing the state-space minimization. The results are then applied to the analysis of closed-loop characteristic polynomials of 2D systems in state-space form. More specifically, we shall give necessary and sufficient conditions for the existence of a compensator that produces a closed-loop characteristic polynomial having a preassigned complex variety.

Finally, some algorithms are presented for deciding whether a given algebraic curve is assignable as the closed-loop characteristic variety of a 2D system and for computing the compensator transfer matrix which produces the desired variety.

2. PRELIMINARY NOTATION AND STATEMENT OF THE PROBLEM

A 2D system $\Sigma=(A_1, A_2, B_1, B_2, C, D)$ is a dynamical model [9]

$$
\begin{aligned}
x(h+1,k+1) &= A_1x(h,k+1)+A_2x(h+1,k)\\
&\quad + B_1u(h,k+1)+B_2u(h+1,k), \qquad (1)\\
y(h,k) &= Cx(h,k)+Du(h,k),
\end{aligned}
$$

where the *local state* x is an n-dimensional vector over the real field $\mathbf{R}$; input and output functions take values in $\mathbf{R}^m$ and $\mathbf{R}^p$; and A_1, A_2, B_1, B_2, C, and D are matrices of suitable dimensions with entries in $\mathbf{R}$. When $D=0$, Σ is called *strictly proper*.

Denoting by

$$
\mathfrak{X}_0=\sum_{i=-\infty}^{+\infty} x(-i,i)z_1^{-i}z_2^{i}
$$

the *global state* on the separation set

$$\mathfrak{C}_0 = \{(i,j) : i+j=0\},$$

and by

$$X(z_1,z_2) = \sum_{i+j\geqslant 0} x(i,j) z_1^i z_2^j,$$

$$U(z_1,z_2) = \sum_{i+j\geqslant 0} u(i,j) z_1^i z_2^j,$$

$$Y(z_1,z_2) = \sum_{i+j\geqslant 0} y(i,j) z_1^i z_2^j$$

the state, input, and output functions, one gets from (1)

$$(I - A_1 z_1 - A_2 z_2) X(z_1,z_2) - (B_1 z_1 + B_2 z_2) U(z_1,z_2) = \mathfrak{X}_0 \tag{2}$$

and

$$Y(z_1,z_2) = CX(z_1,z_2) + DU(z_1,z_2). \tag{3}$$

So, assuming zero initial conditions $\mathfrak{X}_0 = 0$, the rational transfer matrix

$$W(z_1,z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D \tag{4}$$

gives the input-output map

$$Y(z_1,z_2) = W(z_1,z_2) U(z_1,z_2).$$

The polynomial

$$\Delta(z_1,z_2) = \det(I - A_1 z_1 - A_2 z_2) \tag{5}$$

is called the characteristic polynomial of Σ.

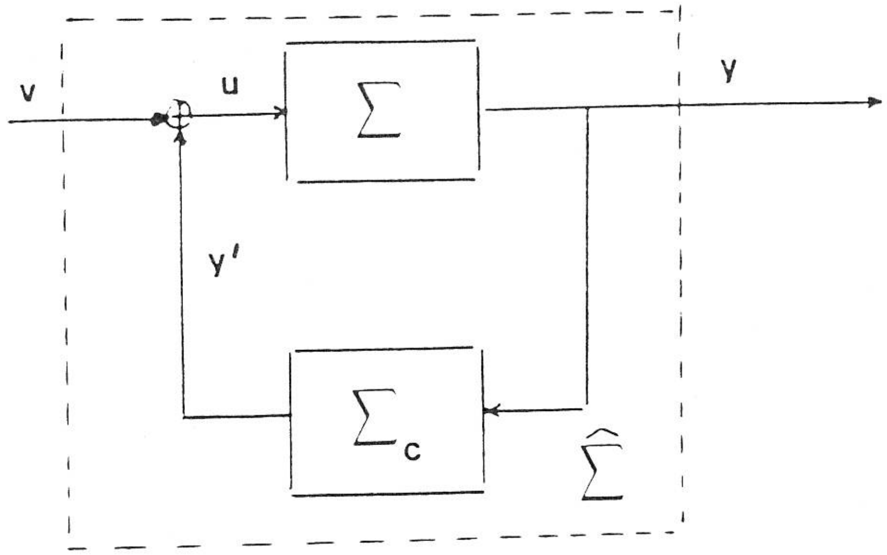

FIG. 1.

Suppose now that a 2D strictly proper *plant* $\Sigma=(A_1, A_2, B_1, B_2, C)$ has been given, and consider the feedback connection (see Figure 1) with a *compensator* $\Sigma_c=(F_1, F_2, G_1, G_2, H, J)$

$$\begin{aligned} x'(h+1,k+1) &= F_1x'(h,k+1)+F_2x'(h+1,k) \\ &\quad +G_1y(h,k+1)+G_2y(h+1,k), \\ y'(h,k) &= Hx'(h,k)+Jy(h,k), \\ u(h,k) &= -y'(h,k)+v(h,k), \end{aligned} \tag{6}$$

where $v(h,k)$ is the external input at (h,k).

The local state $x\oplus x'$ of the resulting closed-loop system $\hat{\Sigma}$ updates according to the following transition matrices:

$$\hat{A}_1=\begin{bmatrix} A_1-B_1JC & -B_1H \\ G_1C & F_1 \end{bmatrix}, \qquad \hat{A}_2=\begin{bmatrix} A_2-B_2JC & -B_2H \\ G_2C & F_2 \end{bmatrix}, \tag{7}$$

and the corresponding closed loop characteristic polynomial of $\hat{\Sigma}$,

$$\hat{\Delta}(z_1,z_2):=\det\left(I-\hat{A}_1z_1-\hat{A}_2z_2\right), \tag{8}$$

depends on the matrices of the compensator. We say that a polynomial $c(z_1, z_2)$ is *assignable* if it can be assumed as the closed-loop characteristic polynomial of the feedback connection of Σ and Σ_c, for a suitable compensator Σ_c.

Given Σ, the set of assignable polynomials is a proper subset of $\mathbf{R}[z_1, z_2]$. A first obvious constraint on assignable polynomials is that the constant term must be one. Depending on the structure of Σ, further constraints can arise, relative either to the plant transfer matrix or to the particular state-space model that realizes it. Referring to that, our objectives are the following:

(i) for a given plant, analyze the subset of assignable polynomials;

(ii) derive the conditions to be fulfilled in order that the subset above may include all polynomials in two variables with unit constant term;

(iii) given any specific $c(z_1, z_2)$ in $\mathbf{R}[z_1, z_2]$, decide about the assignability of its variety $\mathfrak{v}(c)$;

(iv) if $\mathfrak{v}(c)$ is assignable, give algorithms for realizing the compensator Σ_c.

The 2D matrix fraction description (MFD) approach provides the natural setting for studying these problems. In Section 3, elementary properties of MFDs will be briefly recalled and some new results will be presented to support the feedback analysis and the synthesis procedures of Sections 4–6.

3. SOME PROPERTIES OF 2D MFDs

Let $A(z_1, z_2)$ and $B(z_1, z_2)$ be matrices with entries in $\mathbf{R}[z_1, z_2]$, of dimensions $h \times h$ and $h \times k$ respectively, and assume $\det A(z_1, z_2) \neq 0$. Denote by $m_1, m_2, \dots, m_t$ the maximal-order minors of

$$\begin{bmatrix} A(z_1, z_2) & B(z_1, z_2) \end{bmatrix}, \tag{9}$$

and by $\mathfrak{J}(A, B) := (m_1, m_2, \dots, m_t)$ the ideal generated by $m_1, m_2, \dots, m_t$.

Clearly, the matrix (9) is full-rank except in the points of the complex variety

$$\mathfrak{v}(A, B) := \mathfrak{v}(\mathfrak{J}(A, B)),$$

where the maximal-order minors of (9) simultaneously vanish. When $\mathfrak{v}(A, B) = \varnothing$, A and B are called *left zero coprime* (l.z.c.). A necessary and sufficient condition for left zero coprimeness is that the Bézout equation

$$AX + BY = I_h \tag{10}$$

admits a 2D polynomial matrix solution in X and Y.

An $h \times h$ polynomial matrix $Q(z_1, z_2)$ is called a *common left divisor* of A and B if

$$A = Q\hat{A}, \qquad B = Q\hat{B}, \tag{11}$$

where $\hat{A}$ and $\hat{B}$ are polynomial matrices. A and B are *left factor coprime* (l.f.c.) if $\det Q$ is a nonzero constant for all Q satisfying (11).

If A and B are not l.f.c., a greatest common left divisor (GCLD) can be extracted using either the primitive-factorization algorithm [10] or other procedures [11]. Left factor coprimeness is implied by, but does not imply, left zero coprimeness. In fact, l.f. coprimeness is equivalent to the finite cardinality of $\mathfrak{v}(A, B)$.

Let $W(z_1, z_2)$ be an $h \times k$ rational matrix in two variables, and suppose that the above polynomial matrices A and B satisfy

$$W = A^{-1}B. \tag{12}$$

Then $A^{-1}B$ is a left MFD of W. If further A and B are l.f.c., then $A^{-1}B$ is a left coprime MFD of W.

$W(z_1, z_2)$ is *proper* if any one of the following equivalent conditions holds:

(i) W admits a l.c. MFD $A^{-1}B$ with $A(0,0) = I$;
(ii) for any l.c. MFD $A^{-1}B = W$, $\det A(0,0) \neq 0$;
(iii) the entires of W are proper rational functions.

In the sequel, when dealing with proper left coprime MFDs, we shall assume $A(0,0) = I$.

Right MFDs can be introduced with the obvious changes. In particular, given a right MFD $W = CA^{-1}$, we denote by $\mathfrak{I}(C, A)$ the ideal generated by the maximal-order minors of $[A^T(z_1, z_2)\ \ C^T(z_1, z_2)]$.

The following theorem shows that the ideals generated by the maximal-order minors of a coprime MFD of $W(z_1, z_2)$ do not depend on the particular representation (left or right).

THEOREM 1. *Let* $N_R D_R^{-1} = D_L^{-1} N_L$ *be two coprime MFDs of W. Then* $\mathfrak{I}(N_R, D_R) = \mathfrak{I}(D_L, N_L)$.

The proof depends on two technical lemmas.

LEMMA 1 [10]. *Under the hypotheses of Theorem* 1, $\det D_L = \det D_R$. *Moreover, if C, A, and B are 2D polynomial matrices such that* $CA^{-1}B = W$, *then*

(i) $\det D_L \mid \det A$;

(ii) $\det D_L = \det A$ *if and only if* CA^{-1} *and* $A^{-1}B$ *are factor coprime MFDs.*

LEMMA 2. *Consider the polynomial matrix*

$$U = \begin{bmatrix} X & -YC \\ B & A \end{bmatrix}, \tag{13}$$

where X and A are square matrices and $\det A$ *is a nonzero polynomial. Then any right MFD* $N_R D_R^{-1}$ *of the rational matrix* $CA^{-1}B$ *satisfies the following equation*:

$$\det U = \frac{\det A}{\det D_R} \det(XD_R + YN_R). \tag{14}$$

The proof of Lemma 2 is an immediate consequence of the determinantal formula for block matrices.

Proof of Theorem 1. Putting $A = D_L$, $B = N_L$, $C = I$ in Lemma 2 and recalling Lemma 1, one gets

$$\det\begin{bmatrix} X & -Y \\ N_L & D_L \end{bmatrix} = \det(XD_R + YN_R). \tag{15}$$

Assume that $[X \quad -Y]$ is any permutation of the columns of $[I \ 0]$. Then, except for the sign, the right- and left-hand sides of (15) are maximal-order minors of $[N_L \ D_L]$ and $[D_R \ N_R]$ respectively. Moreover, as $[X \quad -Y]$ varies over the set of all permutations, we get a bijective correspondence between the maximal-order minors of $[N_L \ D_L]$ and $[D_R \ N_R]$. So $\mathfrak{I}(D_L, N_L) = \mathfrak{I}(N_R, D_R)$. ■

Consequently, there is no ambiguity in defining the *transfer-matrix ideal* $\mathfrak{I}(W)$ as the ideal of the maximal-order minors associated with an arbitrary right or left coprime MFD of W. The corresponding *transfer-matrix variety* $\mathfrak{v}(W) := \mathfrak{v}(\mathfrak{I}(W))$ is a (possibly empty) finite set, whose points are called the *rank singularities* of W.

REMARK. $\mathfrak{v}(W)$ is empty if and only if the factor coprime MFDs of $W(z_1, z_2)$ are zero coprime. This makes a substantial difference with respect to 1D transfer matrices, where zero coprimeness and factor coprimeness are equivalent concepts, and $\mathfrak{v}(W)$ is always empty. As we shall see, the existence of rank singularities plays an essential role in the closed-loop polynomial assignability problem.

THEOREM 2. *Assume that C, A, B, N_R, D_R are 2D polynomial matrices of suitable sizes with*

$$W(z_1, z_2) = CA^{-1}B = N_R D_R^{-1}$$

and that $N_R D_R^{-1}$ is a r.c. MFD. Then

$$\mathfrak{v}(A, B) \cup \mathfrak{v}(C, A) = \mathfrak{v}(W) \cup \mathfrak{v}(h)$$

where

$$h = \det A / \det D_R$$

is a 2D polynomial (by Theorem 1).

In proving Theorem 2, we need the following Lemma 3, which provides some additional properties of the matrix U introduced in Lemma 2.

LEMMA 3. *Let $(z_1^0, z_2^0) \in \mathbb{C} \times \mathbb{C}$. The matrix $U(z_1^0, z_2^0)$ is singular for any X and Y if and only if at least one of the matrices $[A \;\; B]$ and $[A^T \;\; C^T]$ is singular when evaluated at (z_1^0, z_2^0).*

Proof. Since all matrices are evaluated at the same point (z_1^0, z_2^0), in the notation for the matrices (z_1^0, z_2^0) will be omitted. The "if" part of the lemma is trivial. For the converse, assume that $[A \;\; B]$ and $[A^T \;\; C^T]$ are both full-rank. Then matrices M, N, P, Q of suitable dimension exist such that

$$AM + BN = I, \qquad QA + PC = I. \tag{16}$$

Moreover, by choosing a basis in the orthogonal complement of the row span of $[A \;\; B]$, we obtain a full-rank matrix $\left[\begin{smallmatrix} G \\ F \end{smallmatrix}\right]$ and matrices R, S satisfying

$$AG + BF = 0, \qquad RF + SG = I. \tag{17}$$

Letting

$$V = \begin{bmatrix} F & N \\ G & M \end{bmatrix}, \qquad X = R - SQB, \qquad Y = SP$$

and recalling (16) and (17), we obtain that in

$$UV = \begin{bmatrix} XF - YCG & XN - YCM \\ 0 & I \end{bmatrix} \tag{18}$$

the block $XF - YCG$ is the identity matrix. This proves that U is nonsingular for some X and Y. ∎

Proof of Theorem 2. By Lemma 3, we have

$$(z_1, z_2) \in \mathfrak{v}(A, B) \cup \mathfrak{v}(C, A) \Leftrightarrow \det U(z_1, z_2) = 0 \qquad \forall X, Y, \tag{19}$$

and, applying Lemma 2,

$$\det U(z_1, z_2) = 0 \quad \Leftrightarrow \quad h \det(XD_R + YN_R)(z_1, z_2) = 0. \tag{20}$$

Next observe that

$$(z_1, z_2) \in \mathfrak{v}(W) \Leftrightarrow \det(XD_R + YN_R)(z_1, z_2) = 0 \qquad \forall X, Y. \tag{21}$$

This follows directly from the equivalence of the statements below:

(i) $(z_1^0, z_2^0) \notin \mathfrak{v}(W)$;

(ii) $\begin{bmatrix} D_R(z_1, z_2) \\ N_R(z_1, z_2) \end{bmatrix}$ is full-rank at (z_1^0, z_2^0);

(iii) there exist constant matrices X^0, Y^0 such that

$$X^0 D_R(z_1^0, z_2^0) + Y^0 N_R(z_1^0, z_2^0) = I; \tag{22}$$

(iv) there exist polynomial matrices X and Y such that $X(z_1^0, z_2^0) = X^0$, $Y(z_1^0, z_2^0) = Y^0$, and (22) holds.

Finally, using (19), (20), and (21), one gets that (z_1^0, z_2^0) is in $\mathfrak{v}(A, B) \cup \mathfrak{v}(C, A)$ if and only if (z_1^0, z_2^0) belongs to $\mathfrak{v}(h) \cup \mathfrak{v}(W)$. ■

4. COPRIME REALIZATIONS

As we shall see in greater detail in the next section, the compensator synthesis is performed in two steps. The first one consists in solving a 2D Bézout equation, whose coefficients are determined by the plant transfer matrix and by some requirements on the structure of the characteristic polynomial of the closed-loop system. The solution provides us with an input-output representation of the compensator, and the second step calls for a state-space realization of it.

A problem which naturally arises in connection with the realization procedure is how to avoid the inclusion of unwanted "hidden modes" in the closed-loop polynomial. In order to introduce a concrete definition of the concept of "hidden modes" in 2D state-space models, we consider two complex varieties, associated with the polynomial matrices of the PBH controllability and reconstructibility criteria, and establish some connections between these varieties and the rank singularities of the transfer matrix. Interestingly, a 2D realization of $W(z_1, z_2)$ is free of hidden modes if and only if the join of the above varieties coincide with $\mathfrak{v}(W)$. So the natural question arises whether such a realization does exist and how may it be computed. The realization algorithm, presented at the end of this section, gives a positive answer to this question and provides a constructive realization procedure.

In designing state feedback laws and observers of a 2D system $\Sigma = (A_1, A_2, B_1, B_2, C)$, the following two matrices have proved to be of remarkable importance [7]:

$$\mathfrak{R} = \begin{bmatrix} I - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{bmatrix}, \tag{23}$$

$$\mathfrak{O} = \begin{bmatrix} I - A_1 z_1 - A_2 z_2 \\ C \end{bmatrix}. \tag{24}$$

In fact, the controllability and reconstructibility properties of Σ can be translated into terms of rank conditions on $\mathfrak{R}$ and $\mathfrak{O}$, which therefore will be called PBH controllability and PBH reconstructibility matrices.

Denote for short by $\mathfrak{v}(\mathfrak{R})$ and $\mathfrak{v}(\mathfrak{O})$ the complex varieties $\mathfrak{v}(I - A_1 z_1 - A_2 z_2, B_1 z_1 + B_2 z_2)$ and $\mathfrak{v}(C, I - A_1 z_1 - A_2 z_2)$, and assume that $N_R D_R^{-1}$ is any r.c. MFD of the system matrix. Then Theorem 2 can be easily rephrased in terms of $\mathfrak{R}$ and $\mathfrak{O}$ and

$$h = \frac{\det(I - A_1 z_1 - A_2 z_2)}{\det D_R}, \tag{25}$$

giving

$$\mathfrak{v}(\mathfrak{R}) \cup \mathfrak{v}(\mathfrak{O}) = \mathfrak{v}(h) \cup \mathfrak{v}(W). \tag{26}$$

Of course, if we assume that h is a nonzero constant, the finite cardinality of the right-hand side in (26) implies the factor coprimeness of $C(I - A_1 z_1 - A_2 z_2)^{-1}$ and $(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2)$. Viceversa, if h is a nonconstant polynomial, $\mathfrak{R}$ and/or $\mathfrak{O}$ are not full-rank along the algebraic curves associated with the irreducible factors of h. In this case, the uncontrollable and the unreconstructible modes (collectively, hidden modes) refer to the irreducible factors of h that appear as common factors of the maximal-order minors of $\mathfrak{R}$ and $\mathfrak{O}$ respectively.

By definition, a realization Σ of $W(z_1, z_2)$ is coprime if Σ is free of hidden modes. As a matter of fact, there are many equivalent definitions of coprime realizations. These are summarized in the following corollary, whose proof is a straightforward consequence of (26).

Corollary. *Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a realization of $W(z_1, z_2)$, and assume that $N_R D_R^{-1}$ is a r.c. MFD of $W(z_1, z_2)$. Then the following statements are equivalent:*

(i) $\det D_R = \det(I - A_1 z_1 - A_2 z_2)$;

(ii) $C(I - A_1 z_1 - A_2 z_2)^{-1}$ *and* $(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2)$ *are right and left f.c. MFDs respectively;*

(iii) $\mathfrak{v}(\mathfrak{R}) \cup \mathfrak{v}(\mathfrak{L}) = \mathfrak{v}(W)$;

(iv) Σ *is a coprime realization.*

Remark. Coprime realizations are not necessarily minimal, since their local state space need not have minimal dimension. For instance, the coprime realization

$$A_1 = A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = [0 \quad 1], \qquad D = 0$$

of the transfer function $(z_1 + z_2)/(1 - z_1 - z_2)$ is nonminimal. Even more, it is easy to show that whenever a transfer matrix admits a coprime realization, then it admits coprime realizations of arbitrary large dimension.

The question of the existence of coprime realizations for any proper transfer matrix is positively answered by a corollary of Theorem 3, which provides also an explicit realization procedure.

Theorem 3. *Let $N_R D_R^{-1}$ be a right MFD of the transfer matrix $W(z_1, z_2)$ satisfying $D_R(0,0) = I$. Then there exists a 2D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ that realizes W and satisfies the following conditions:*

(i) $\mathfrak{R}(z_1, z_2)$ *is full rank in* $\mathbb{C} \times \mathbb{C}$;

(ii) $\det(I - A_1 z_1 - A_2 z_2) = \det D_R$.

Proof. There is no restriction in assuming $W(z_1, z_2)$ strictly proper, so that $N_R(0,0) = 0$. Denote by k_i, $i = 1, 2, \ldots, m$, the column degree of the ith column of

$$\begin{bmatrix} N_R \\ D_R \end{bmatrix},$$

that is the degree of the maximal-order polynomial in the ith column. We can write

$$D_R = I_m - D_{HT}\Psi$$

where

$$\Psi^T(z_1, z_2) = \left[\begin{array}{cccccccc|ccc|cccccccc} z_2^{k_1} & z_1 z_2^{k_1-1} & \cdots & z_1^{k_1} & z_2^{k_1-1} & \cdots & z_1 & z_2 & 0 & \cdots & 0 & & & & \bigcirc & & & & \\ & & & & & & & & & \ddots & & & & & & & & & \\ & & & \bigcirc & & & & & 0 & \cdots & 0 & z_2^{k_m} & z_1 z_2^{k_m-1} & \cdots & z_1^{k_m} z_2^{k_m-1} & \cdots & z_1 & z_2 & \end{array}\right],$$

$$D_{HT} = \begin{bmatrix} D_{11} & \cdots & D_{1m} \\ \cdots & \cdots & \cdots \\ D_{m1} & \cdots & D_{mm} \end{bmatrix}, \qquad N_{HT} = \begin{bmatrix} N_{11} & \cdots & N_{1m} \\ \cdots & \cdots & \cdots \\ N_{p1} & \cdots & N_{pm} \end{bmatrix},$$

$$D_{HT} = \begin{bmatrix} D_{11} & \cdots & D_{1m} \\ \cdots & \cdots & \cdots \\ D_{m1} & \cdots & D_{mm} \end{bmatrix}, \qquad N_{HT} = \begin{bmatrix} N_{11} & \cdots & N_{1m} \\ \cdots & \cdots & \cdots \\ N_{p1} & \cdots & N_{pm} \end{bmatrix},$$

and D_{ij} and N_{ij} are row vectors whose elements are the coefficients of the (i, j)-indexed polynomial in $-D_R + I_m$ and in N_R.

Introduce now the following matrices:

$$A_{10}^{(h)} = \left[\begin{array}{cccccc} & M_h & & & & \\ & & M_{h-1} & & & \\ & & & & \ddots & \\ & & & & & M_2 \\ \hline 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{array}\right], \qquad B_1^{(h)} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 1 \end{array}\right],$$

$$A_{20}^{(h)} = \left[\begin{array}{cccccc} & N_h & & & & \\ & & N_{h-1} & & & \\ & & & & \ddots & \\ & & & & & N_2 \\ \hline 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{array}\right], \qquad B_2^{(h)} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \end{array}\right],$$

with

$$M_j = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ & & I_j & & \end{bmatrix}, \qquad N_j = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ & & O_j & & \end{bmatrix},$$

and define

$$A_{10} = \operatorname{diag}\left\{ A_{10}^{(k_1)}, A_{10}^{(k_2)}, \dots, A_{10}^{(k_m)} \right\},$$

$$A_{20} = \operatorname{diag}\left\{ A_{20}^{(k_1)}, A_{20}^{(k_2)}, \dots, A_{20}^{(k_m)} \right\},$$

$$B_1 = \operatorname{diag}\left\{ B_1^{(k_1)}, B_1^{(k_2)}, \dots, B_1^{(k_m)} \right\},$$

$$B_2 = \operatorname{diag}\left\{ B_2^{(k_1)}, B_2^{(k_2)}, \dots, B_2^{(k_m)} \right\}.$$

It is a matter of simple computation to show that

$$(I - A_{10}z_1 - A_{20}z_2)^{-1}(B_1z_1 + B_2z_2) = \Psi.$$

Assuming now

$$\mathfrak{A}_0 := A_{10}z_1 + A_{20}z_2, \qquad \mathfrak{B} := B_1z_1 + B_2z_2, \qquad \mathfrak{A} := \mathfrak{A}_0 + \mathfrak{B}D_{HT},$$

we have

$$\begin{aligned}(I - \mathfrak{A})^{-1}\mathfrak{B} &= (I - \mathfrak{A}_0 - \mathfrak{B}D_{HT})^{-1}\mathfrak{B} \\ &= \left\{\left[I - \mathfrak{B}D_{HT}(I - \mathfrak{A}_0)^{-1}\right](I - \mathfrak{A}_0)\right\}^{-1}\mathfrak{B} \\ &= (I - \mathfrak{A}_0)^{-1}\left[I - \mathfrak{B}D_{HT}(I - \mathfrak{A}_0)^{-1}\right]^{-1}\mathfrak{B} \\ &= (I - \mathfrak{A}_0)^{-1}\mathfrak{B}\left[I - D_{HT}(I - \mathfrak{A}_0)^{-1}\mathfrak{B}\right]^{-1} \\ &= \Psi(z_1, z_2)(I - D_{HT}\Psi)^{-1} = \Psi D_R^{-1}.\end{aligned}$$

Since

$$\begin{aligned}N_R D_R^{-1} = N_{HT}\Psi D_R^{-1} &= N_{HT}(I - \mathfrak{A})^{-1}\mathfrak{B} \\ &= N_{HT}\left[I - (A_{10} + B_1D_{HT})z_1 - (A_{20} + B_2D_{HT})z_2\right]^{-1}(B_1z_1 + B_2z_2),\end{aligned}$$

the matrices $A_1 := A_{10} + B_1D_{HT}$, $A_2 := A_{20} + B_2D_{HT}$, B_1, B_2, C furnish a realization of $N_R D_R^{-1}$.

The PBH controllability matrix is full-rank in $\mathbb{C} \times \mathbb{C}$. In fact

$$\begin{aligned}\operatorname{rank}[I - \mathfrak{A} \,\vdots\, \mathfrak{B}] &= \operatorname{rank}[I - \mathfrak{A}_0 - \mathfrak{B} D_{HT} \,\vdots\, \mathfrak{B}] = \operatorname{rank}[I - \mathfrak{A}_0 \,\vdots\, \mathfrak{B}] \\ &= \operatorname{rank}\Big[\operatorname{diag}\{ I - A_{10}^{(k_i)} z_1 - A_{20}^{(k_i)} z_2,\ i = 1,2,\ldots,m \} \\ &\qquad \times \operatorname{diag}\{ B_1^{(k_i)} z_1 + B_2^{(k_i)} z_2,\ i = 1,2,\ldots,m \}\Big]\end{aligned}$$

is full for every (z_1, z_2), since the matrices

$$\left[I - A_{10}^{(k_i)} z_1 - A_{20}^{(k_i)} z_2 \quad B_1^{(k_i)} z_1 + B_2^{(k_i)} z_2 \right], \qquad i = 1,2,\ldots,m,$$

have full rank for every (z_1, z_2).

It remains to prove that $\det(I - A_1 z_1 - A_2 z_2) = \det D_R$. This follows from the identities

$$\det D_R = \det(I - D_{HT}\Psi) = \det(I - \Psi D_{HT})$$

and

$$\begin{aligned}\det(I - \mathfrak{A}) &= \det(I - \mathfrak{A}_0 - \mathfrak{B} D_{HT}) \\ &= \det(I - \mathfrak{A}_0)\det\left[I - (I - \mathfrak{A}_0)^{-1}\mathfrak{B} D_{HT}\right] = \det(I - \Psi D_{HT}). \ \blacksquare\end{aligned}$$

COROLLARY. *If $N_R D_R^{-1}$ is a r.c. MFD of $W(z_1, z_2)$, the system Σ given in Theorem 3 is a coprime realization of $W(z_1, z_2)$.*

5. ASSIGNABILITY OF THE CLOSED-LOOP CHARACTERISTIC POLYNOMIAL

At the end of Section 2 we posed the problems (i)–(iv) relative to the system of Figure 1, obtained by interconnecting a strictly proper plant $\Sigma = (A_1, A_2, B_1, B_2, C)$ and a compensator $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$. Our aim now is to give a solution to these problems.

Let $W(z_1, z_2)$ and $W_c(z_1, z_2)$ be the transfer matrices of Σ and Σ_c respectively, and consider two MFDs PQ^{-1} and $X^{-1}Y$ satisfying

$$W(z_1, z_2) = PQ^{-1}, \qquad \det Q = \det(I - A_1 z_1 - A_2 z_2), \tag{27}$$

$$W_c(z_1, z_2) = X^{-1}Y, \qquad \det X = \det(I - F_1 z_1 - F_2 z_2). \tag{28}$$

Then the closed-loop characteristic polynomial (8) is given by

$$\hat{\Delta}(z_1, z_2) = \det(XQ + YP) \tag{29}$$

On the other hand, by Theorem 3 any left MFD $X^{-1}Y$ with $X(0,0) = I$ admits a realization $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$ that satisfies the condition

$$\det X(z_1, z_2) = \det(I - F_1 z_1 - F_2 z_2).$$

So, as (X, Y) varies over the set of polynomial matrix pairs with $X(0,0) = I$, (29) produces all assignable closed-loop polynomials for the given plant Σ.

Let E be a GCRD of P and Q. Then

$$P = N_R E, \qquad Q = D_R E, \tag{30}$$

and $N_R D_R^{-1}$ is a r.c. MFD of W. As a consequence of (25) and (27), we have

$$h(z_1, z_2) = \frac{\det(I - A_1 z_1 - A_2 z_2)}{\det D_R} = \det E, \tag{31}$$

and (29) becomes

$$\hat{\Delta}(z_1, z_2) = h \det(XD_R + YN_R). \tag{32}$$

The above formula clearly shows that $h(z_1, z_2)$, which represents the hidden modes of Σ, is an invariant factor of $\hat{\Delta}(z_1, z_2)$ with respect to feedback compensation. In other words, as far as fixed modes are concerned, 2D systems behave exactly in the same way as 1D systems do. However, a deep difference between 2D and 1D systems comes out when we consider the factor $\det(XD_R + YN_R)$. In fact, as we established in the proof of Theorem 2, this factor must vanish for any choice of X and Y on the set $\mathfrak{v}(W)$ of rank singularities. Such a restriction does not exist in the 1D case, where the solvability of the Bézout equation $XD_R + YN_R = I$ and hence the complete assignability of the polynomial $\det(XD_R + YN_R)$ are consequences of the coprimeness of N_R and D_R.

The next theorem shows how the conditions that $\hat{\Delta}$ vanishes on $\mathfrak{v}(h)$ and $\mathfrak{v}(W)$ and $\hat{\Delta}(0,0) = 1$ represent the only constraints imposed on the closed-loop polynomial variety by the structure of the plant.

THEOREM 4. *Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a realization of the transfer matrix $W(z_1, z_2)$. For any compensator Σ_c, the closed-loop polynomial variety $\mathfrak{v}(\hat{\Delta})$ satisfies the inclusion*

$$\mathfrak{v}(\hat{\Delta}) \supseteq \mathfrak{v}(h) \cup \mathfrak{v}(W),$$

where h is given by (31) *and $\mathfrak{v}(W)$ is the set of rank singularities of W. Viceversa, given any algebraic curve $\mathfrak{C}$ that includes $\mathfrak{v}(h) \cup \mathfrak{v}(W)$ and excludes the origin, a compensator Σ_c exists such that $\mathfrak{v}(\hat{\Delta}) = \mathfrak{C}$.*

Proof. The first part of the theorem has already been proved. For the second, let M_i be the submatrices of maximal order in $[P^T \ Q^T]^T$ that correspond to the minors m_i, $i = 1, 2, \dots, t$. Then there exist constant matrices L_i and K_i that satisfy $M_i = L_i Q + K_i P$, $i = 1, 2, \dots, t$, and we have

$$m_i I = (\operatorname{adj} M_i) M_i = (\operatorname{adj} M_i) L_i Q + (\operatorname{adj} M_i) K_i P. \tag{33}$$

Consider a 2D polynomial c such that

$$\mathfrak{v}(c) = \mathfrak{C}, \qquad c(0,0) = 1.$$

The inclusion

$$\mathfrak{v}(c) \supseteq \mathfrak{v}(W) \cup \mathfrak{v}(h) = \mathfrak{v}(N_R, D_R) \cup \mathfrak{v}(E) = \mathfrak{v}(P, Q)$$

and Hilbert's *Nullstellensatz* imply

$$c^r = \sum_{i=1}^{t} m_i g_i \in \mathfrak{I}(P, Q) \tag{34}$$

for a suitable integer r and suitable polynomials g_i. Tying (33) and (34) together yields

$$c^r I = \left(\sum_{i=1}^{t} m_i g_i \right) I = XQ + YP \tag{35}$$

with

$$X = \sum_i g_i (\operatorname{adj} M_i) L_i, \qquad X(0,0) = I; \qquad Y = \sum_i g_i (\operatorname{adj} M_i) K_i.$$

By Theorem 3 we are able to construct a compensator $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$ that realizes $X^{-1}Y$ under the constraint $\det(I - F_1 z_1 - F_2 z_2) = \det X$. Thus the corresponding closed-loop polynomial is given by

$$\hat{\Delta}(z_1, z_2) = \det(XQ + YP) = c^{rm},$$

and $\mathfrak{C}$ is the variety of $\hat{\Delta}$.

When dealing with MISO and SIMO systems, an alternative characterization of the feedback action is available in terms of polynomial ideals, instead of polynomial varieties. Assignable polynomials of a strictly proper MISO system Σ are easily characterized as the elements with unit constant term in the ideal $h\mathfrak{J}(W)$. For, let q be the characteristic polynomial of Σ, and $[p_1, p_2, \dots, p_p]q^{-1}$ its transfer matrix; and consider any polynomial c in $(q, p_1, p_2, \dots, p_q) = h\mathfrak{J}(W)$ and satisfying $c(0,0) = 1$. Then there exist 2D polynomials $r, s_1, s_2, \dots, s_p$ such that

$$c = qr + \sum_i p_i s_i$$

and $r(0,0) = 1$. Clearly any 2D realization $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$ of $r^{-1}[s_1, s_2, \dots, s_p]^T$ that satisfies $r(z_1, z_2) = \det(I - F_1 z_1 - F_2 z_2)$ gives $c(z_1, z_2)$ as the closed-loop characteristic polynomial. The same property can be shown for SIMO systems, using dual reasoning.

We are now in the position for deriving a set of necessary and sufficient conditions for the complete assignability of the closed-loop characteristic polynomial or, equivalently, of its variety. These are a direct consequence of Theorem 4 and are summarized in the following corollary.

Corollary. *Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a strictly proper 2D system. The following are equivalent:*

(i) *the closed-loop characteristic polynomial is arbitrarily assignable;*

(ii) *there exists a compensator Σ_c (dead-beat controller) such that the closed-loop characteristic polynomial is $\hat{\Delta}(z_1, z_2) = 1$;*

(iii) *$\mathfrak{v}(\mathfrak{R}) = \mathfrak{v}(\mathfrak{D}) = \varnothing$ (i.e., the plant is PBH controllable and reconstructible);*

(iv) *the set of rank singularities of the plant is empty, and* $h(z_1, z_2) = 1$.

Proof. The equivalence (iii) $\Leftrightarrow$ (iv) is a consequence of (26). Furthermore, assuming $\mathfrak{C} = \varnothing$ in Theorem 4, the equivalence (ii) $\Leftrightarrow$ (iv) follows immediately.

In order to prove (ii) $\Rightarrow$ (i), let X and Y satisfy the Bézout equation (35) with $c(z_1, z_2) = 1$. Given any polynomial $q(z_1, z_2)$ and a matrix $M(z_1, z_2)$ with $\det M = q$, the pair $(\tilde{X}, \tilde{Y}) = (MX, MY)$ satisfies $q = \det(\tilde{X}Q + \tilde{Y}P)$. ■

In the above proof the closed-loop characteristic polynomial has been obtained by introducing hidden modes in the compensator. It turns out that in the compensator synthesis hidden modes could have been avoided, since the equation $\det(XQ + YP) = q$ admits left zero coprime solutions.

In fact, assume that the number of inputs m is not greater than the number of outputs p, and consider a solution $(\hat{X}, \hat{Y})$ of the equation $XQ + YP = I_m$. Then the general solution of the equation

$$XQ + YP = M \qquad \text{with} \quad \det M = q$$

is given by

$$[X \quad Y] = M[\hat{X} \quad \hat{Y}] + T[S \quad -R] = [M \quad T]\begin{bmatrix} \hat{X} & \hat{Y} \\ S & -R \end{bmatrix}, \qquad (36)$$

where $R^{-1}S$ is a left zero coprime MFD of PQ^{-1}, T is an arbitrary polynomial matrix, and $\begin{bmatrix} \hat{X} & \hat{Y} \\ S & -R \end{bmatrix}$ is unimodular [9].

When choosing $T = [I_m \quad 0]$, the matrix $[X \,\vdots\, Y]$ in (36) is full-rank in $\mathbb{C} \times \mathbb{C}$.

The case $m > p$ can be solved in a similar way through a left MFD of the plant and a right MFD of the controller.

REMARK 1. Equations (15) and (35) show that any 2D polynomial matrix $[D_L \quad N_L]$ with N_L and D_L left zero coprime can be row bordered up into a square unimodular matrix, i.e., there exist X and Y such that

$$\det\begin{bmatrix} X & Y \\ D_L & N_L \end{bmatrix} = 1. \qquad (36)$$

This is easily shown using the following procedure:

(i) Construct a zero coprime right MFD PQ^{-1} of $D_L^{-1}N_L$.

(ii) According to the corollary of Theorem 4, compute X and Y that satisfy the equation

$$1 = \det(XQ + YP).$$

Because of (15), X and Y satisfy (36).

Note that this constitutes a simple derivation of the Quillen-Suslin theorem for polynomial matrices in two indeterminates [12].

REMARK 2. For any choice of the compensator Σ_c, the set of rank singularities $\mathfrak{v}(W)$ is included in the closed-loop characteristic polynomial variety; actually it is the subset of this variety which is invariant under compensation.

Nevertheless, the set of rank singularities of the closed-loop transfer function does not necessarily include $\mathfrak{v}(W)$. This can be easily seen by taking, for instance

$$W(z_1, z_2) = \frac{z_1(1 - 2z_2 - z_1^2)}{1 + 2z_2}, \tag{37}$$

$$W_c(z_1, z_2) = \frac{z_1}{1 + 2z_2 + z_1^2}. \tag{38}$$

The closed-loop transfer function

$$W(z_1, z_2) = \frac{W}{1 + WW_c} = z_1 \tag{39}$$

is devoid of rank singularities, while $\mathfrak{v}(W) = \mathfrak{v}(W_c) = \{0, -\frac{1}{2}\}$.

Note that, whatever the realizations of $W(z_1, z_2)$ and $W_c(z_1, z_2)$ may be, the resulting closed-loop system is internally unstable. In fact, independently of the internal description Σ and Σ_c of W and W_c, the variety of the closed-loop characteristic polynomial $\hat{\Delta}(z_1, z_2)$ must include $\mathfrak{v}(W)$ [and $\mathfrak{v}(W_c)$], and $\hat{\Delta}(z_1, z_2)$ is a hidden mode of the closed-loop system.

6. ALGORITHMS FOR COMPENSATOR DESIGN

In this concluding section we shall outline some algorithms connected with the solution of the points (iii) and (iv) in Section 2.

The first problem which naturally arises is to decide whether a given algebraic curve $\mathfrak{C}$, described by a polynomial equation $c(z_1, z_2) = 0$, is assignable (i.e. can be viewed as the closed-loop polynomial variety of the system depicted in Figure 1). By Theorem 4, the procedure will consist in verifying if

$$(0,0) \notin \mathfrak{C}, \tag{40}$$

$$\mathfrak{v}(h) \subseteq \mathfrak{C}, \tag{41}$$

$$\mathfrak{v}(W) \subseteq \mathfrak{C}. \tag{42}$$

Checking (40) is trivial, and once the polynomial $h(z_1, z_2)$ has been computed, we can easily verify (41) using any linear algorithm to see if h divides $c^{\deg h}$. The condition (42) can be checked by first computing a set of generators of $\mathfrak{J}(W)$ and successively exploiting them for constructing a pair of commutative matrices M_1 and M_2 with the property

$$p(z_1, z_2) \in \mathfrak{J}(W) \quad \Leftrightarrow \quad p(M_1, M_2) = 0.$$

Thus $\mathfrak{v}(W) \subseteq \mathfrak{C}$ if and only if $c(M_1, M_2)$ is a nilpotent matrix. For the construction of M_1 and M_2 the reader is referred to [14].

So it remains to show how to compute the polynomial h and a set of generators for $\mathfrak{J}(W)$ starting from the system matrices A_1, A_2, B_1, B_2, C. For this, let

$$\left[C \operatorname{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2)\right]\left[I_m \det(I - A_1 z_1 - A_2 z_2)\right]^{-1} = \overline{N}\overline{D}^{-1}$$

be a MFD of the transfer matrix $W(z_1, z_2)$. The generator set can be obtained by evaluating the maximal-order minors $m_1, m_2, \dots, m_t$ in $[\overline{N}^T \ \overline{D}^T]$ and then eliminating their g.c.d. $d(z_1, z_2)$. Thus h is given by

$$h = \frac{\det(I - A_1 z_1 - A_2 z_2)}{\det D_R} = \frac{\det(I - A_1 z_1 - A_2 z_2)\, d(z_1, z_2)}{\det \overline{D}}.$$

Assume now that a variety $\mathfrak{C} = \mathfrak{v}(c)$ that fulfills the conditions (40) through (42) has been given, and suppose we want to synthesize a compensator Σ_c that produces a closed-loop polynomial $\hat{\Delta}$ whose variety is $\mathfrak{C}$. The procedure can be summarized as follows:

1. Evaluate a r.c. MFD $N_R D_R^{-1}$ of W. This can be performed by using the primitive factorization algorithm [10] or other algorithms that do not require primitive factorizations [11].
2. Compute the maximal-order minors $m_1, m_2, \dots, m_t$ of $[N_R^T \; D_R^T]$.
3. Compute an integer r and a Gröbner basis $g_1, g_2, \dots, g_w$ such that $c^r = \sum_i m_i g_i$. A technique for performing this step has been presented in [14].
4. Solve the Bézout equation $c^r I_m = XD_R + YN_R$ as in the proof of Theorem 4.
5. Exploit the realization algorithm of Theorem 3 for computing a coprime realization Σ_c of $X^{-1}Y$.

The correctness of the procedure is easily seen from the following chain of equalities:

$$\begin{aligned} \mathfrak{v}(\hat{\Delta}) &= \mathfrak{v}(h) \cup \mathfrak{v}(\det(XD_R + YN_R)) && \text{by (32)} \\ &= \mathfrak{v}(h) \cup \mathfrak{v}(c^r I_m) \\ &= \mathfrak{v}(h) \cup \mathfrak{v}(c) && \text{by (41)} \\ &= \mathfrak{v}(c) = \mathfrak{C}. \end{aligned}$$

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Received 16 June 1988; final manuscript accepted 3 November 1988