ALGEBRAIC ASPECTS OF 2D SINGULAR SYSTEMS

E. Fornasini and S. Zampieri
Dept. of Electr. and Inform. University of Padova
via Gradenigo 6/a, 35131 Padova, Italy, fax 39-49-8927099

Abstract The paper investigates the behaviour $B$ of a singular 2D system on a half plane. Some connections between the matrices appearing in the updating equations and the restrictions of $B$ to the separation sets are presented.

1 Introduction

Consider a 2D system given by the following equation

$$\overline{E}z(h+1,k+i) = \overline{A}z(h,k+i) + \overline{B}z(h+1,k)$$

where $\overline{E}, \overline{A}, \overline{B}$ are $q \times n$ matrices with entries in $\mathbb{R}$.

Clearly, if rank $\overline{E} = n$, (1) can be reduced to the equation of an unforced nonsingular 2D system [1], as follows

$$\overline{E}(h+1,k+i) = (\overline{E}^T \overline{E})^{-1} \overline{E}^T \overline{A}z(h,k+i) + (\overline{E}^T \overline{E})^{-1} \overline{E}^T \overline{B}z(h+1,k)$$

If rank $\overline{E} = r < n$, we are allowed to introduce two nonsingular matrices $Q \in \mathbb{R}^{n \times n}$ and $N \in \mathbb{R}^{n \times r}$, such that

$$Q \overline{E}N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

So, letting

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = N^{-1} \overline{z}, \quad Q \overline{A}N = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad Q \overline{B}N = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

equation (1) can be rewritten as follows

$$z_1(h+1,k+i) = A_{11}z_1(h,k+i) + A_{12}z_2(h,k+1) + B_{11}z_1(h,k+i) + B_{12}z_2(h,k+1) + B_{12}z_2(h+1,k)$$

$$0 = A_{21}z_1(h,k+1) + A_{22}z_2(h,k+1) + B_{21}z_1(h,k+i) + B_{22}z_2(h,k+1) + B_{22}z_2(h+1,k)$$

In the particular case when $A_{11}, B_{11}, A_{22}, B_{22}$ are simultaneously zero, $z_1$ can be viewed as an $n - r$ dimensional input and (5) provides the state updating equation of a nonsingular 2D system. More generally, however, $z_2$ is the direct sum of exogenous variables (i.e. inputs), and auxiliary variables that induce some dynamical constraints on the system trajectories, and (5) can be considered a singular 2D system, as studied in [2].

This paper constitutes a preliminary report on a research still in progress, concerning the analytical structure of the trajectories of system (5) in the half plane $\mathbb{H} = \{(h,k) : h + k \geq 0\}$. No "a priori" assumption is made on which components of $z_2$ can be given the role of exogenous variables. Following the philosophy that underlies the behavioural approach by J. Willems and P. Rocha [3-5], the nature of the input functions is determined "a posteriori", after establishing what variables are constrained by equations (5).

2 An algebraic approach via duality

All signals $\overline{x}$ that will be considered in this paper are sequences indexed on the half plane $\mathbb{H}$ and taking values in some finite dimensional $\mathbb{R}$-vector space $\mathbb{X} : \mathbb{H} \rightarrow \mathbb{R}^n : (h,k) \rightarrow \overline{x}(h,k)$. The single step updating structure (5) makes it convenient to introduce a partition of $\mathbb{H}$ into a countable family of separation sets $S_i = \{(h,k) : h+k = i\}, i = 0,1,...$ and to associate with $z$ a formal power series

$$X = \sum_{i=0}^{+\infty} \sum_{j=-\infty}^{+\infty} x(i+j,i-j)\xi^i\lambda^j$$

So, the "bilateral" formal power series $X' = \sum_{i=0}^{+\infty} \sum_{j=-\infty}^{+\infty} x(i+j,i-j)\xi^i\lambda^j$, $i = 0,1,...$ are associated with the restrictions of the signal $x$ to the separation sets $S_i$, $i = 0,1,...$

Let denote by $F^n$ and $G^n$ respectively the spaces of polynomials in $\xi, \xi^{-1}, \lambda$ and of formal power series in $\xi, \lambda^{-1}$, with coefficients in $\mathbb{R}$. Introduce in $F^n \times G^n$ a nondegenerate bilinear function $\langle \cdot, \cdot \rangle_n$ that associates with a polynomial $p = \sum_{i=0}^{+\infty} \sum_{j=-\infty}^{+\infty} p_{ij}\xi^i\lambda^j$ in $F^n$ and a series $X = \sum_{i=0}^{+\infty} \sum_{j=-\infty}^{+\infty} x(i+j,i-j)\xi^i\lambda^j$ in $G^n$ the coefficient of the constant term in the Cauchy product $pX$

$$\langle p, X \rangle_n = \sum_{i=0}^{+\infty} \sum_{j=-\infty}^{+\infty} p_{ij}x(i,j).$$

Every series $X \in G^n$ induces a linear function $\varphi_X$ on $F^n$, defined by $\varphi_X : p \mapsto \langle p, X \rangle_n$. Moreover, the linear mapping that associates $X$ with the linear function $\varphi_X$ is an isomorphism of $G^n$ onto the space $L(F^n)$ of linear functions on $F^n$ and, consequently, each series in $G^n$ (or, equivalently, each signal $z : \mathbb{H} \rightarrow \mathbb{R}^n$) can be identified with an element of the algebraic dual space $L(F^n)$. This accounts for the possibility of expressing many features of signal spaces with support in $\mathbb{H}$ in terms of properties of suitable subspaces of $F^n$.

Let $M(\xi, \xi^{-1}, \lambda)$ be a q x n matrix with entries in $R(\xi, \xi^{-1}, \lambda)$ and consider the linear mappings

$$\mu : F^n \rightarrow F^n : p \mapsto M(\xi, \xi^{-1}, \lambda)p$$

$$\mu' : G^n \rightarrow G^n : X \mapsto M(\xi, \xi^{-1}, \lambda)X$$

Here $\sigma$ is the shift operator in $G^1$

$$\sum_{i=0}^{+\infty} \sum_{j=-\infty}^{+\infty} w(i+j,i-j)\xi^{-i}\lambda^{-j} = \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} w(i+1,j+1)\xi^{-i-1}\lambda^{-j-1}$$

and $\mu$ and $\mu'$ are dual mappings [6], as $(\mu p, X)_n = \langle p, \mu' X \rangle_n$ holds for all $X \in G^n$ and $p$ in $F^n$. We therefore have

$$\ker \mu' = (\im \mu)^{-1},$$

where $\im \mu$ denotes the $R(\xi, \xi^{-1}, \lambda)$-module generated by the columns of the matrix $M(\xi, \xi^{-1}, \lambda)$.

In order to analyze the trajectories of system (5), we introduce the following series

$$X_\ell = \sum_{i=0}^{+\infty} \sum_{j=-\infty}^{+\infty} x_\ell(i+j,i-j)\xi^{-i}\lambda^{-j}, \quad \ell = 1,2$$

and the matrix

$$M(\xi, \sigma) = \begin{bmatrix} \sigma i & -A_{11} - B_{11} \xi & -A_{12} - B_{12} \xi \\ -A_{21} - B_{21} \xi & -A_{22} - B_{22} \xi \\ -A_{21} - B_{21} \xi & -A_{22} - B_{22} \xi \end{bmatrix}$$

The constraints induced on $x$ by equation (5) are expressed as

$$M(\xi, \sigma) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

Therefore the behaviour of (5) can be viewed as the kernel of the linear operator $\mu'$, or, alternatively, as the orthogonal subspace to the $R(\xi, \xi^{-1}, \lambda)$-module $M$ generated by the columns of the matrix $M(\xi, \sigma)$:

$$B = \left\{ \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in G^n : M(\xi, \sigma) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \right\} = \ker \mu' = M$$

In our context an important consequence stems directly from the fact that $G^n$ is the algebraic dual $L(F^n)$, namely

$$B = (M^1)^{-1} = M$$
Actually (14) shows that the module \( M \) is uniquely determined by \( B \), so that \( B \) can be described as the kernel of some matrix \( \overline{M}(\xi, \sigma) \) if and only if the columns of both \( \overline{M}(\xi, \lambda) \) and \( \overline{M}(\xi, \lambda) \) generate the same \( \mathbb{R}[\xi, \xi^{-1}, \lambda] \)-module.

The duality theory provides also an useful tool for analyzing the restrictions \( B^{(a)} \) of the behaviour \( B \) to the sets \( S_0 \cup S_1 \cup \cdots \cup S_k \). This is easily seen by considering the linear mappings

\[
\begin{align*}
F^a &\xrightarrow{i} F^a \xrightarrow{\pi} F^a/\text{im} \mu \\
G^n/\sigma^a G^n &\xrightarrow{\phi} G^n \xrightarrow{\text{ker} \mu} 
\end{align*}
\]

where \( F^a \) is the \( \mathbb{R}[\xi, \xi^{-1}] \)-submodule of the polynomial columns in \( F^n \) having degree less than or equal to \( k \) in the indeterminate \( \lambda \). \( G^n/\sigma^a G^n \) is (isomorphic to) the \( \mathbb{R}[\xi, \xi^{-1}] \)-submodule obtained by truncating each series of \( G^n \) terms with degree greater than \( k \) w.r.t. \( \lambda^{-1} \), the maps \( \phi \) and \( \pi \) are canonical injections, \( \sigma \) and \( \pi \) are canonical projections.

Obviously \( G^n/\sigma^a G^n \) is isomorphic with the space \( L[F^a] \) of linear functions on \( F^a \). Moreover \( F^n/\text{im} \mu \) is isomorphic with a direct complement of \( \text{im} \mu \) in \( F^n \), and using the duality theory on direct decompositions \([6]\) gives \( \text{ker} \mu^* = (\text{im} \mu)^\perp \cong L[F^a/\text{im} \mu] \). The first and the last space on the second row of (15) can be viewed as the algebraic duals of the corresponding spaces on the first row and the maps \( \sigma \circ i \), \( \pi \circ i \) in (15) are dual linear maps w.r.t. to the bilinear function induced on the pairs \((F^a, G^n/\sigma^a G^n)\) and \((F^n/\text{im} \mu, \text{ker} \mu^*)\). Consequently the restriction \( B^{(a)} \) is given by

\[
B^{(a)} = \ker (\pi \circ i)^\perp
\]

The above relation characterizes \( B^{(a)} \) as the subspace of all signals with support in \( S_0 \cup S_1 \cup \cdots \cup S_k \) and values in \( \mathbb{R}^n \) that correspond to formal power series \( \sum_{i=0}^n x_i \lambda^{-i} \) satisfying the orthogonality condition

\[
\left( \sum_{i=0}^n c_i(x)_i, \sum_{i=0}^n x_i \lambda^{-i} \right)_n = \left[ \begin{array}{c} c_0(x) \ c_1(x) \cdots c_k(x) \end{array} \right] \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} = 0
\]

for all polynomial vectors \( \sum_{i=0}^n c_i(x)_i \lambda^i \) in \( \mathbb{R}[\xi, \xi^{-1}, \lambda] \)-module \( \text{im} \mu \).

The \( \mathbb{R}[\xi, \xi^{-1}] \)-submodule of \( \mathbb{R}^{\lambda n(k+1)}[\xi, \xi^{-1}] \) whose elements are the rows \( (c_0(x) \ c_1(x) \cdots c_k(x)) \) that satisfy the condition \( \sum_{i=0}^n c_i(x)_i \lambda^i \in \text{im} \mu \) is finitely generated. Therefore there exists a polynomial matrix \( C^{0,1}(\xi) \) with \( n(k+1) \) columns such that \( B^{(a)} = \ker C^{0,1}(\xi) \).

In the next section we shall take advantage of the particular structure of \( M(\xi, \sigma) \) given by (12), when determining the \( \mathbb{R}[\xi, \xi^{-1}] \)-submodule \( B^{(a)} \).

3 Computation of trajectories

The following lemma directly provides a matrix \( C^{0,1}(\xi) \) whose rows are given in terms of submatrices \( A_{ij} \) and \( B_{ij} \) that appear in the partition (12). The proof is based on Cayley-Hamilton theorem and can be found in [7].

Lemma Let \( A_{ij} \) and \( B_{ij} \) be as in (12) and define the polynomial matrices:

\[
A_{ij} := A_{ij} + B_{ij} \xi, \quad i, j = 1, 2
\]

Then \( C^{0,1}(\xi) = [C_0(\xi), C_1(\xi)] \) and \( B^{(a)} = \ker C^{0,1}(\xi) \) or, equivalently,

\[
\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} \in B^{(a)} \iff C_0 x^0 = -C_1 x^1
\]

Premultiplying both \( C_0 \) and \( C_1 \) by a suitable unimodular matrix \( U \), one gets

\[
UC_0 = \begin{bmatrix} D_0 \\ D_0 \end{bmatrix}, \quad -UC_1 = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}
\]

where both \( D_0 \) and \( D_1 \) have full row rank. Just rewriting (18) as

\[
D_0 x^0 = 0, \quad D_1 x^1 = D_0 x^0
\]

we now see that all solutions of equation (20.1) can be viewed as restrictions of admissible trajectories to the separation set \( S^0 \). In fact \( D_1 \) has full row rank and, therefore, given any \( x^0 \), eq. (20.2) can be fulfilled by suitably chosen values of \( x^1 \).

We are now in a position for establishing the following theorem.

**Theorem** A signal \( X = \sum_{i=0}^\infty \overline{X}^{i+1} \lambda^{-i} \) belongs to \( B \) if and only if \( X^i \) satisfies the following equations

\[
D_0 x^0 = 0, \quad D_1 x^{i+1} = [D_0] x^i, \quad i = 0, 1, \ldots
\]

**Proof** Suppose that \( X \) satisfies (21). Then we have

\[
C^{0,1}(\xi) \begin{bmatrix} x^0 \\ x^{i+1} \end{bmatrix} = 0, \quad i = 0, 1, \ldots
\]

which implies

\[
x^0 + x^1 \lambda^{i+1} X^{i+1} = 0, \quad i = 0, 1, \ldots
\]

The degree of all columns in \( M^2(\xi, \lambda) \) w.r.t. \( \lambda \) is less than or equal to one. So, any such column can be written as \( c_0(\xi) + \lambda c_1(\xi) \) and we have

\[
\begin{align*}
(c_0(\xi) + \lambda c_1(\xi))X^0 &= (c_0(\xi) + \lambda c_1(\xi))x^0, \quad X^0 + x^1 \lambda^{i+1} X^{i+1} = 0, & \quad \text{(23)}
\end{align*}

as a consequence of (16) and (23). This shows that \( X \) is orthogonal to \( \text{im} \mu \) and therefore \( X \in B \). The converse is obvious.

Equations (21) provide a recursive procedure for generating the system trajectories. Moreover, the difference \( n - \text{rank} D_0 \) gives the number of free variables that appear in system (6), i.e. the variables that can be arbitrarily chosen on all separation sets \( S^i \).

4 References