ASYMPTOTIC BEHAVIOUR OF 2D MARKOV CHAINS

E.Fornasini

ABSTRACT The paper analyzes the matrix representation structure of the probability transition map in a 2D Markov chain and some properties of the associated characteristic polynomial in two variables. These allow to show how the long term behaviour depends on the intersections between the variety of the characteristic polynomial and the distinguished boundary of the unit closed bidisk.

Key Words: Markov chains, 2D systems, positive matrices, stochastic matrices

1. INTRODUCTION

During the last few years a considerable research effort has been devoted to dynamical patterns that evolve in the discrete plane $\mathbf{Z} \times \mathbf{Z}$, partially ordered by the product of the orderings

$$(r,s) \leq (h,k) \quad \text{iff} \quad r \leq h \quad \text{and} \quad s \leq k \tag{1}$$

The causality constraints that (1) naturally induces on the dynamical patterns imply that the configuration attained at (h,k) only depends on configurations and input values at $(r,s) \leq (h,k)$. Autonomous 2D systems [1-3] constitute the easiest nontrivial instance of these dynamical behaviours. Here the local configuration $\mathbf{x}(h+1,k+1)$ is linearly determined by the nearest past configurations $\mathbf{x}(h,k+1)$ and $\mathbf{x}(h+1,k)$. We therefore have the following first order updating equation

$$\mathbf{x}(h+1,k+1) = \mathbf{x}(h,k+1)\mathbf{A}^{(1)} + \mathbf{x}(h+1,k)\mathbf{A}^{(2)} \tag{2}$$

where $\mathbf{x}$ is an n-dimensional real valued row vector and $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}$ are $n \times n$ real matrices. The separation property of classical (one-dimensional) Markov chains is inherited by system (2) in a two-dimensional environment. Actually, the computation of the local configuration at $(h+1,k+1)$ does'nt require information about system history in the "past cone" $\{(r,s) < (h+1,k+1)\}$, with the exception of the nearest points $(h,k+1)$ and $(h+1,k)$. So, although no particular probability meaning is associated with the local vector $\mathbf{x}$ in the general theory of 2D systems, it seems rather natural to obtain a 2D theory of Markov chains by introducing suitable constraints in equation (2). These must guarantee that any pair of probability vectors $\mathbf{x}(h,k+1)$ and $\mathbf{x}(h+1,k)$ leads in turn to a new probability vector at $(h+1,k+1)$, so that the components of $\mathbf{x}(h+1,k+1)$ can be viewed as probabilities of the various states at point $(h+1,k+1)$.

Multidimensional Markov models (hidden Markov mesh random fields) have been recently considered in the image processing literature, with the purpose of developing coherent approaches to both problems of image segmentation and model acquisition [4]. In this paper we emphasize the algebraic structure of 2D Markov chains and some properties of their characteristic polynomials, our final goal being a general result on their asymptotic behaviour.

2. THE STRUCTURE OF A 2D MARKOV CHAIN

By a 2D Markov chain $\mathcal{M}$ with n states $S_1, S_2, \ldots, S_n$ we will mean:

1. an *autonomous 2D system*

$$\mathbf{x}(h+1,k+1) = \mathbf{x}(h,k+1)\mathbf{A}^{(1)} + \mathbf{x}(h+1,k)\mathbf{A}^{(2)} \tag{3}$$

of dimension n, with the property that $\mathbf{x}(h+1,k+1)$ is a probability row vector for every pair of probability row vectors $\mathbf{x}(h,h+1)$ and $\mathbf{x}(h+1,k)$.

2. a sequence of *initial probability vectors* $\mathcal{X}_0 = \Big\{\mathbf{x}(h,k) \mid (h,k) \in \mathcal{C}_0, \mathbf{x}(h,k) \in X\Big\}$, where $\mathcal{C}_0 = \Big\{(h,k) \in \mathbf{Z}\times\mathbf{Z} \mid h+k=0\Big\}$ is a separation set in $\mathbf{Z}\times\mathbf{Z}$ and $x_i(h,k), i = 1,2\ldots,n$, denotes the probability that S_i is the state of the system at the initial point (h,k).

A basic question concerning equation (3) is the following: if $\mathbf{x}(h,k+1)$ and $\mathbf{x}(h+1,k)$ are probability vectors, but otherwise arbitrary, under what circumstances can one be certain that the new vector $\mathbf{x}(h+1,k+1)$ will also be of the same type? It is not difficult to show that, given any pair $\mathbf{P}$ and $\mathbf{Q}$ of $n\times n$ stochastic matrices and any real number a in the interval $[0,1]$, then $\mathbf{A}^{(1)} = a\mathbf{P}$, $\mathbf{A}^{(2)} = (1-a)\mathbf{Q}$ are matrices of a 2D Markov chain. The converse of this result, however, is not true, since 2D Markov chains need not be represented by the convex combination of a pair of stochastic matrices. In fact, suppose that an n-states 2D Markov chain $\mathcal{M}$ has been given via the assignment of a one step transition probability map $\pi : X\times X \to X$, i.e. via the restriction to $X\times X$ of a suitable linear map from $\mathbf{R}^n\times\mathbf{R}^n$ into $\mathbf{R}^n$ represented by a pair of $n\times n$ matrices $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})$. Then, for any $n\times n$ matrix $\mathbf{M}$ with all rows the same vector, the pair $(\mathbf{A}^{(1)}+\mathbf{M}, \mathbf{A}^{(2)}-\mathbf{M})$ realizes the same transition map. Viceversa, if $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})$ and $(\bar{\mathbf{A}}^{(1)}, \bar{\mathbf{A}}^{(2)})$ realize π, then there exists a matrix $\mathbf{M}$ with all rows the same vector such that

$$\bar{\mathbf{A}}^{(1)} = \mathbf{A}^{(1)} + \mathbf{M}, \qquad \bar{\mathbf{A}}^{(2)} = \mathbf{A}^{(2)} - \mathbf{M} \tag{4}$$

So, given a 2D Markov chain with n states, there are infinitely many chains equivalent to it (i.e. realizing the same probability map) and the natural question arises as to what extent convex combinations are "canonical", in the sense that each equivalence class includes a 2D Markov chain represented by a convex combination of two stochastic matrices. This is answered by the following theorem.

Theorem 1 [5] *A 2D Markov chain with n states can be represented as*

$$\mathbf{x}(h+1,k+1) = \mathbf{x}(h,k+1)a\mathbf{P} + \mathbf{x}(h+1,k)(1-a)\mathbf{Q} \tag{5}$$

where $\mathbf{P}$ *and* $\mathbf{Q}$ *are* $n\times n$ *stochastic matrices and* $0 \le a \le 1$.

In the sequel a chain in form (5) will be called a *canonical 2D Markov chain* and will be denoted as $\mathcal{M} = (a, \mathbf{P}, \mathbf{Q})$ (note that each equivalence class needs not include just one canonical chain).
Theorem 1 completely clarifies the class of dynamical models described by equation (3). Actually we may visualize the process which moves from states S_f at $(h, k+1)$ and S_g at $(h+1, k)$ to some state at $(h+1, k+1)$, according to the following rules:
1. The probability vectors $\mathbf{x}(h, k+1)$ and $\mathbf{x}(h+1, k)$ are thought of as giving the probabilities for the various possible starting states. Then an experiment in two stages takes place at $(h+1, k+1)$:
2. The first stage of the experiment exhibits two possible outcomes, e.g. $\theta(h+1, k+1) = 0$ and $\theta(h+1, k+1) = 1$, with probability a and $1-a$ respectively. The random variable $\theta(h+1, k+1)$ is independent of $\theta(l, m)$, for all $(l, m) \neq (h+1, k+1)$
3. At the second stage a state transition occurs that uniquely depends on the state at $(h, k+1)$ if $\theta(h+1, k+1) = 0$, and on the state at $(h+1, k)$ if $\theta(h+1, k+1) = 1$. The process moves from S_f at $(h, k+1)$ into S_m with probability P_{fm} and from S_g at $(h+1, k)$ into S_m with probability Q_{gm}.

3. ASYMPTOTIC BEHAVIOUR

Since the matrices of a 2D Markov chain are given by the convex combination of a pair of stochastic matrices, it is expected that the strong spectral properties of stochastic matrices play a central role in the theory of 2D Markov chains. To make our intuition precise, it is convenient to introduce the 2D characteristic polynomial and the algebraic variety of its zero set, that constitute the basic tools for much of the internal stability analysis of general 2D systems [2,6].

Consider a 2D Markov chain with n states $\mathcal{M}$ given by equation (3). The following polynomial in two indeterminates

$$\Delta(z_1, z_2) = \det(\mathbf{I} - z_1\mathbf{A}^{(1)} - z_2\mathbf{A}^{(2)}) \tag{6}$$

is called the *characteristic polynomial* of $\mathcal{M}$ and the solutions of the corresponding equation $\Delta(z_1, z_2) = 0$ constitute the variety $\mathcal{V}(\Delta)$ of the chain. The peculiar structure of $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ induces some constraints on the polynomial variety $\mathcal{V}(\Delta)$, that are summarized in the following theorem.

Theorem 2 [5] 1) *The characteristic polynomial of a 2D Markov chain with n states factorizes into the product of a first order polynomial $h_1(z_1, z_2) = 1 - bz_1 - (1-b)z_2$ and a polynomial $h_2(z_1, z_2)$ of degree not greater than $n-1$*

$$\Delta(z_1, z_2) = h_1(z_1, z_2)h_2(z_1, z_2) \tag{7}$$

While h_2 is invariant under 2D chains equivalence (4), h_1 is not, and its orbit is obtained by varying arbitrarily the parameter b over the real numbers (over the interval $[0, 1]$ in case of canonical Markov chains).
In a canonical 2D Markov chain $\mathcal{M} = (a, \mathbf{P}, \mathbf{Q})$, with $a(1-a) \neq 0$,

2) *$\mathcal{V}(\Delta)$ does not intersect the unit closed polydisk $\mathcal{P}_1 = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}$, except at $(1, 1)$ and, possibly, at some other points of its distinguished boundary $\mathcal{T}_1 = \{(z_1, z_2); |z_1| = |z_2| = 1\}$.*

3) *the following facts are equivalent:* i) $\lambda_0 = 1$ *is a multiple eigenvalue of* $\mathbf{A} = a\mathbf{P} + (1-a)\mathbf{Q}$ ii) *when evaluated at* $(1, 1)$, $\partial\Delta/\partial z_1$ *is zero* iii) *when evaluated at* $(1, 1)$, $\partial\Delta/\partial z_2$ *is zero*

The third theorem establishes a remarkable connection between the intersection $\mathcal{V}(\Delta) \cap \mathcal{T}_1$ and the long term behaviour of the probability vectors $\mathbf{x}(h,k)$. An interesting question we shall answer in this context is the following: does there exist a probability vector $\mathbf{w}$ such that $\mathbf{x}(h,k)$ approaches $\mathbf{w}$ as $h+k$ tends to infinity? As we shall see, for certain types of 2D Markov chains there exists a unique limiting probability vector, independently of the distribution of the probability vectors $\mathbf{x}(h,-h)$ on the separation set $\mathcal{C}_0$. These chains, which can be regarded as the 2D analogue of 1D Markov chains with a single aperiodic class, have a deep but intuitive body of theory.

In the foregoing developments we shall consider only nontrivial canonical 2D Markov chains $\mathcal{M} = (a, \mathbf{P}, \mathbf{Q})$, i.e. with $0 < a < 1$. Without loss of generality, we assume also that the states of the 1D chain associated with the stochastic matrix $\mathbf{A} = a\mathbf{P} + (1-a)\mathbf{Q}$ have been permuted so that all the ergodic states are listed before the transient states. This amounts to say that $\mathbf{A}$, $\mathbf{P}$ and $\mathbf{Q}$ are block triangular

$$\mathbf{A} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{R} & \mathbf{T} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{0} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{0} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \tag{8}$$

$\mathbf{E}$, $\mathbf{T}$ are a stochastic and a substochastic matrix respectively, representing the transition probabilities within the ergodic classes and the transition probabilities among the transient states of a 1D Markov chain, and the polynomial $\det\big(\mathbf{I} - az_1\mathbf{P}_{22} - (1-a)z_2\mathbf{Q}_{22}\big)$ is devoid of zeros in $\mathcal{P}_1$ [5].

Definition Let $\mathcal{M} = (a, \mathbf{P}, \mathbf{Q})$ be a 2D Markov chain and $\mathcal{X}_0$ a sequence of initial probability vectors. A probability vector $\mathbf{w} \in X$ is a *limiting probability vector* (LPV) of $\mathcal{X}_0$ if $\lim_{h+k\to+\infty} \mathbf{x}(h,k) = \mathbf{w}$. If this property holds for all sequences $\mathcal{X}_0$ of initial probability vectors, $\mathbf{w}$ is termed *global limiting probability vector* (GLPV).

The strategy we follow in studying the existence of a GLPV is to derive first some constraints on the values of its entries and on the structure of $\mathbf{E}$ (lemma 1). Then we show that the variety $\mathcal{V}(\Delta)$ must be regular at $(1,1)$ (lemma 2) and, by a perturbation argument, cannot intersect the distinguished boundary $\mathcal{T}_1$ except at $(1,1)$ (lemma 4 and part of thm. 3). The above constraints on $\mathcal{V}(\Delta)$ finally provide a necessary and sufficient condition for $\mathcal{M}$ having a GLPV (thm. 3).

Lemma 1 [5] *Let* $\mathbf{w}$ *be a GLPV of* $\mathcal{M} = (a, \mathbf{P}, \mathbf{Q})$ *and assume that in* (8) *the matrix* $\mathbf{T}$ *has dimension* $r \times r$. *Then* i) *the first* $n-r$ *entries of* $\mathbf{w}$ *are stricly positive and the last* r *are zero;* ii) *the matrix* $\mathbf{E}$ *is fully regular*

As far as the structure of $\mathcal{V}(\Delta)$ is concerned, we show first that it is impossible to find a GLPV for a chain $\mathcal{M} = (a, \mathbf{P}, \mathbf{Q})$ when its characteristic polynomial exhibits repeated roots at $(1,1)$.

Lemma 2 *Let* $\mathcal{M} = (a, \mathbf{P}, \mathbf{Q})$ *have a GLPV. Then the variety* $\mathcal{V}(\Delta)$ *of its characteristic polynomial is regular at* $(1,1)$.

PROOF Suppose $(1,1)$ be a singular point of $\mathcal{V}(\Delta)$. By thm.2, $\lambda_0 = 1$ is a multiple eigenvalue od $\mathbf{A}$ and consequently $\mathbf{E}$ cannot be fully regular. By lemma 1, this would contradict the existence of a GLPV.

Next we consider the possibility that the variety $\mathcal{V}(\Delta)$ and the distinguished boundary $\mathcal{T}_1$ have intersections other than $(1,1)$ or, equivalently, the matrix $\mathbf{I} - az_1\mathbf{P} - (1-a)\mathbf{Q}$ may be not full rank at $(e^{i\omega_1}, e^{i\omega_2}) \neq (1,1)$. To discuss this property, we need the following technical lemma

Lemma 3 [5] *Suppose that* $\mathbf{I} - az_1\mathbf{P} - (1-a)z_2\mathbf{Q}$ *is not full rank at* $(e^{i\omega_1}, e^{i\omega_2}) \neq (1,1)$. *If* $\mathbf{v} = [v_1\ v_2 \dots v_n] \in \mathbf{C}^n$ *satisfies* $\mathbf{v}[\mathbf{I} - ae^{i\omega_1}\mathbf{P} - (1-a)e^{i\omega_2}\mathbf{Q}] = \mathbf{0}$ *then its entries sum up to zero, i.e.* $\sum_{k=1}^{n} v_k = 0$

In view of the above lemma, our original assumption on the existence of an intersection between $\mathcal{V}(\Delta)$ and $\mathcal{T}_1 \setminus \{(1,1)\}$ can be restated as follows: there exists a complex valued nonzero vector $\mathbf{v} = [\, v_1 \quad v_2 \quad \cdots \quad v_{n-r}\,]$, with $\sum_{h=1}^{n-r} v_h = 0$ that satisfies $\mathbf{v}\Big(\mathbf{I} - ae^{i\omega_1}\mathbf{P}_{11} - (1-a)e^{i\omega_2}\mathbf{Q}_{11}\Big) = \mathbf{0}$. If we partition the probability vectors conformably with the block structure of (8), i.e. $\mathbf{x}(h,k) = [\mathbf{x}_1(h,k) \ \ \mathbf{x}_2(h,k)]$ and assume that the initial probability vectors satisfy $\mathbf{x}_2(h,-h) = \mathbf{0},\ h \in \mathbf{Z}$, then the first $n-r$ entries of $\mathbf{x}(\cdot,\cdot)$ evolve according to the equation of a 2D Markov chain with $n-r$ states

$$\mathbf{x}_1(h+1,k+1) = \mathbf{x}_1(h,k+1)a\mathbf{P}_{11} + \mathbf{x}(h+1,k)(1-a)\mathbf{Q}_{11}. \tag{9}$$

Suppose, for the moment, that in (10) all $\mathbf{x}_1$s are allowed to be complex valued vectors and consider the sequence $\tilde{\mathcal{X}}_0 = \{\mathbf{x}_1(h,-h) = \mathbf{v}e^{i\omega h},\ h \in \mathbf{Z}\}$, with $\omega = \omega_2 - \omega_1$. It is clear that the updating equation (9) produces at (h,k), with $h+k \geq 0$, a vector $\mathbf{x}_1(h,k) = \mathbf{v}e^{-i\omega_1 h - i\omega_2 k}$ and, consequently, the vector sequence $\tilde{\mathcal{X}}_m$ on $\mathcal{C}_m = \{(h,k) \mid h+k = m\}$ is given by $\tilde{\mathcal{X}}_m = \{\mathbf{x}_1(h,-h+m) = \mathbf{v}e^{i\omega h - i\omega_2 m},\ h \in \mathbf{Z}\}$.

When $\mathbf{v}$ is expressed in polar form $\mathbf{v} = [\, p_1 e^{i\beta_1} \quad p_2 e^{i\beta_2} \quad \cdots \quad p_{n-r}e^{i\beta_{n-r}}\,]$, the sequence $\tilde{\mathcal{X}}_0$ breaks apart into a real and an imaginary sequence $\tilde{\mathcal{X}}_0 = \tilde{\mathcal{X}}_0^R + i\tilde{\mathcal{X}}_0^I$

$$\begin{aligned} \tilde{\mathcal{X}}_0^R &= \{[p_1\cos(\beta_1 + h\omega) \ \ p_2\cos(\beta_2 + h\omega) \ \cdots \ p_{n-r}\cos(\beta_{n-r} + h\omega)], h \in \mathbf{Z}\} \\ \tilde{\mathcal{X}}_0^I &= \{[p_1\sin(\beta_1 + h\omega) \ \ p_2\sin(\beta_2 + h\omega) \ \cdots \ p_{n-r}\sin(\beta_{n-r} + h\omega)], h \in \mathbf{Z}\} \end{aligned} \tag{10}$$

Since the transition matrices $a\mathbf{P}_{11}$ and $(1-a)\mathbf{Q}_{11}$ are real valued, assuming $\tilde{\mathcal{X}}_0^R$ or $\tilde{\mathcal{X}}_0^I$ as initial conditions will produce separately

$$\begin{aligned} \tilde{\mathcal{X}}_m^R &= \{[p_1\cos(\beta_1 + h\omega - m\omega_2) \cdots\cdots p_{n-r}\cos(\beta_{n-r} + h\omega - m\omega_2)], h \in \mathbf{Z}\} \\ \tilde{\mathcal{X}}_m^I &= \{[p_1\sin(\beta_1 + h\omega - m\omega_2) \cdots\cdots p_{n-r}\sin(\beta_{n-r} + h\omega - m\omega_2)], h \in \mathbf{Z}\} \end{aligned} \tag{11}$$

Owing to assumption $\mathbf{v} \neq \mathbf{0}$, $\tilde{\mathcal{X}}_0^R$ and $\tilde{\mathcal{X}}_0^I$ cannot be simultaneously zero. Furthermore, the property $\sum_h v_h = 0$ implies that the entries of every real vector of the sequences $\tilde{\mathcal{X}}_m^R$ and $\tilde{\mathcal{X}}_m^R$ sum up to zero.

Suppose now to start the chain from $\tilde{\mathcal{X}}_0^R \neq 0$. Then the sequences $\tilde{\mathcal{X}}_m^R$ cannot converge to zero as m goes to infinity. Actually, if $\omega_2/2\pi$ is rational the sequences $\tilde{\mathcal{X}}_m^R$ vary periodically with m; if not, there are sequences $\tilde{\mathcal{X}}_m^R$ arbitrarily close to $\tilde{\mathcal{X}}_0^R$ for arbitrarily large values of m. The above discussion is summarized in the following lemma.

Lemma 4 *Let* $\det\Big(\mathbf{I} - az_1\mathbf{P} - (1-a)z_2\mathbf{Q}\Big) = 0$ *at* $(z_1, z_2) = (e^{i\omega_1}, e^{i\omega_2}) \neq (1,1)$. *Then there exists a nonzero sequence of real vectors*

$$\mathcal{X}_0 = \{\mathbf{x}(h,-h) = [\underbrace{\mathbf{x}_1}_{n-r} \mid \mathbf{0}],\ h \in \mathbf{Z}\} \tag{12}$$

and two positive real mumbers $l \leq L$ with the following properties: the vectors we obtain from $\mathcal{X}_0$ according to (5) *satisfy* i) $x_j(h,k) = 0, \quad n-r < j \leq n$ ii) $\sum_{j=1}^{n} x_j(h,k) = 0$ iii) $\|\mathcal{X}_m\| = \sup_{h \in \mathbf{Z}} \|x(h,-h+m)\|_\infty \in [l, L]$

We are now in a position for giving the main result of this section.

Theorem 3 *Let $\mathcal{M} = (a, \mathbf{P}, \mathbf{Q})$ be a 2D Markov chain. Then $\mathcal{M}$ admits a GLPV if and only if $(1,1)$ is a regular point of $\mathcal{V}(\Delta)$ and is the unique intersection of $\mathcal{V}(\Delta)$ with the distinguished boundary $\mathcal{T}_1$.*

PROOF For the sufficiency part the interested reader is referred to [5]. To prove the necessity part, we only need to show that $\mathcal{V}(\Delta) \cap \mathcal{T}_1 = \{(1,1)\}$. So, assume that $\mathbf{w} = [\underbrace{\mathbf{w}_1}_{n-r} \mid \mathbf{0}]$ is the GLPV of $\mathcal{M}$ and suppose that $\mathcal{V}(\Delta)$ intersects $\mathcal{T}_1$ at $(e^{i\omega_1}, e^{i\omega_2}) \neq (1,1)$. Then $n-r > 1$ and, by lemma 1, $m_1 := \min_{1 \leq h \leq n-r} w_h$ as well as $m_2 := \min_{1 \leq h \leq n-r}(1 - w_h)$ are strictly positive quantities. If we assume $\mathcal{X}'_0 = \{\mathbf{x}'(h,-h) = \mathbf{w}, \ h \in \mathbf{Z}\}$ as a sequence of initial probability vectors of $\mathcal{M}$, we obtain $\mathbf{x}'(h,k) = \mathbf{w}$, for any (h,k) with $h+k \geq 0$.

Consider now the sequence of initial vectors $\mathcal{X}''_0 = \mathcal{X}'_0 + \frac{\mu}{2L}\mathcal{X}_0$, with $\mathcal{X}_0$ and L defined in lemma 4 and $\mu := \min(m_1, m_2)$. The perturbation term $\frac{\mu}{2L}\mathcal{X}_0$ is small enough to guarantee that all vectors of the sequence $\mathcal{X}''_0$ are nonnegative. Moreover property ii) of lemma 4 implies that the entries of each vector in $\mathcal{X}''_0$ sum up to 1, so that $\mathcal{X}''_0$ may be considered as a sequence of probability vectors. The corresponding dynamical evolution of $\mathcal{M}$ is obtained as the superposition of $\mathbf{x}'(h,k)$, that provides a constant pattern in the half plane $\{(h,k) : h+k \geq 0\}$, and the evolution induced by (12), scaled down by $\frac{\mu}{2L}$, that does not converge to zero as $k+h \to 0$. This shows that $\mathbf{w}$ is not a LPV of $\mathcal{X}''_0$

REFERENCES

[1] Fornasini E. and Marchesini G. (1976). State space realization theory of two dimensional filters. *IEEE Trans.Aut.Contr.*, **AC-21**, 4, 284-92.

[2] Fornasini E. and Marchesini G. (1978). Doubly indexed dynamical systems: state space models and structural properties. *Math.Sys.Th.*,**12,** 59-72.

[3] Kung S., Lévy B., Morf M., and Kailath T. (1977). New results in 2D systems theory, part I and part II *Proc. IEEE,* **4l,** 861-72 and 945-61

[4] Devijver P.A. (1988). Modelling of digital images.*Signal Processing: Theories and Applications,* J.L.Lacoune et al. eds., Elsevier Sci.Publ., 23-30

[5] Fornasini E. (1990). 2D Markov chains. *Lin. Alg. Appl.***24,** 101-27

[6] Fornasini E. and Marchesini G. (1980). Stability analysis of 2D systems *IEEE Trans. Circ. Sys.,* **CAS-27,** 1210-17