2D SYSTEMS: CHARACTERISTIC POLYNOMIAL STRUCTURE AND STATE SPACE GEOMETRY

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1. INTRODUCTION

During the last few years, a considerable research effort has been devoted to dynamical patterns that evolve in the discrete plane $\mathbf{Z}^2$, partially ordered by the product of the orderings, i.e. $(r,s) \leq (h,k)$ iff $r \leq h$ and $s \leq k$. The easiest nontrivial instance of these dynamical behaviours is provided by autonomous 2D systems, where the local configuration at $(h+1,k+1)$ linearly depends only on the nearest past configurations at $(h,k+1)$ and $(h+1,k)$. This is formalized by the following first order recursive equation [2]

$$x(h+1,k+1) = A_1 x(h,k+1) + A_2 x(h+1,k), \tag{1.1}$$

where the “local state” x is an n-dimensional vector over the real field $\mathbf{R}$, and A_1, A_2 are $n \times n$ real matrices. Initial conditions are usually given by the so called “initial global state”, namely the set $\{x(i,-i), i \in \mathbf{Z}\}$.
The nature of the pair (A_1, A_2) strongly influences the dynamics of the system. In particular, the characteristic polynomial

$$\Delta_{A_1,A_2}(z_1,z_2) := \det(I - A_1 z_1 - A_2 z_2) \tag{1.2}$$

plays a fundamental role in the analysis of some properties, such as finite memory, separability, asymptotic stability etc.[3 ÷ 5], which give valuable insights into the qualitative pattern of the trajectories.

On the other hand, $\Delta_{A_1,A_2}(z_1,z_2)$ does not summarize all relevant information on the system dinamics. Actually two pairs of matrices with the same characteristic polynomial may have different sizes and, even when their dimensions coincide, they are not necessarily related by a similarity transformation.

The scope of this paper is twofold. In the next two sections we feature some classes of 2D systems which admit a nice description in terms of their characteristic polynomials. This requires a detailed investigation of the equivalence relation existing between pairs of matrices (A_1, A_2) with the same characteristic polynomial, and involves the construction of suitable families of complete invariants.

In the last section we investigate how some spectral properties of the pair (A_1, A_2) reflect into the dynamical behaviour of the corresponding 2D system. As we shall see, these properties need not be captured by the characteristic polynomial, and require a finer analysis of the structure of the pair. Special attention will be deserved to the possibility of characterizing properties P and L, by resorting to suitable power series in commuting and noncommuting variables.

2. CHARACTERISTIC POLYNOMIAL AND TRACES OF A MATRIX PAIR

For any matrix $A \in \mathbf{C}^{n\times n}$, assigning the traces of A, A^2, ..., A^n is equivalent to give the polynomial $\det(I - Az)$ and, consequently, the spectrum of A. In this section we aim to extend this result to a pair of $n \times n$ matrices (A_1, A_2) with elements in $\mathbf{C}$. More precisely, we will investigate the correspondence between the characteristic polynomial of the pair and the traces of some elements of the matrix algebra generated by A_1 and A_2.
To that purpose, we need a standard property of matrices, we shall often refer to in the sequel.

Lemma 2.1 [6] *Let A be in $\mathbf{C}^{n\times n}$ and assume $\det(I - Az) = 1 - d_1 z - d_2 z^2 - ... - d_n z^n$. Then we have*

$$\mathrm{tr}A - d_1 = 0 \mathrm{tr}A^2 - d_1 \mathrm{tr}A - 2d_2 = 0 \cdots \mathrm{tr}A^n - d_1 \mathrm{tr}A^{n-1} - \ldots - n d_n = 0 \tag{2.1}$$

and, for $k > 0$,

$$\mathrm{tr}A^{n+k} - d_1 \mathrm{tr}A^{n+k-1} - \ldots - d_n \mathrm{tr}A^k = 0 \;\blacksquare \tag{2.2}$$

The following proposition provides a set of equivalent conditions guaranteeing that two pairs of matrices $(A_1, A_2) \in \mathbf{C}^{n\times n} \times \mathbf{C}^{n\times n}$ and $(\hat{A}_1, \hat{A}_2) \in \mathbf{C}^{\hat{n}\times\hat{n}} \times \mathbf{C}^{\hat{n}\times\hat{n}}$ have the same characteristic polynomial. These involve

a) the linear combinations $\alpha A_1 + \beta A_2$, $\alpha, \beta \in \mathbf{C}$ and the set $\Lambda_0(\alpha A_1 + \beta A_2)$ of the nonzero eigenvalues;

b) the matrix coefficients of the power series expansion of $(I - A_1 z_1 - A_2 z_2)^{-1}$, i.e. the matrices $A_1{}^{i}\sqcup\!\sqcup{}^{j} A_2$, $(i, j) \in \mathbf{N}^2$, inductively defined as

$$A_1{}^{i}\sqcup\!\sqcup{}^{0} A_2 = A_1^i, \qquad A_1{}^{0}\sqcup\!\sqcup{}^{j} A_2 = A_2^j \tag{2.3}$$

and, when i and j are both greater than zero,

$$A_1{}^{i}\sqcup\!\sqcup{}^{j} A_2 = A_1(A_1{}^{i-1}\sqcup\!\sqcup{}^{j} A_2) + A_2(A_1{}^{i}\sqcup\!\sqcup{}^{j-1} A_2), \tag{2.4}$$

and the corresponding matrices for the pair $(\hat{A}_1, \hat{A}_2)$.

Proposition 2.2 *The following statements are equivalent:*
i) $\Delta_{A_1,A_2}(z_1, z_2) = \Delta_{\hat{A}_1,\hat{A}_2}(z_1, z_2)$*;*
ii) $\Lambda_0(\alpha A_1 + \beta A_2) = \Lambda_0(\alpha \hat{A}_1 + \beta \hat{A}_2), \quad \forall \alpha, \beta \in \mathbf{C}$*;*
iii) $\mathrm{tr}(\alpha A_1 + \beta A_2)^k = \mathrm{tr}(\alpha \hat{A}_1 + \beta \hat{A}_2)^k, \forall \alpha, \beta \in \mathbf{C}, k > 0$*;*
iv) $\mathrm{tr}(A_1{}^{i}\sqcup\!\sqcup{}^{j} A_2) = \mathrm{tr}(\hat{A}_1{}^{i}\sqcup\!\sqcup{}^{j} \hat{A}_2), \quad \forall\ (i, j) \neq (0, 0)$.

PROOF *i)* $\Leftrightarrow$ *ii)* As both i) and ii) are equivalent to
$\det[I - (\alpha A_1 + \beta A_2)z] = \det[I - (\alpha \hat{A}_1 + \beta \hat{A}_2)z]$, $\forall \alpha, \beta \in \mathbf{C}$, they are equivalent each other, too.
i) $\Leftrightarrow$ *iii)* It's an immediate consequence of Lemma 2.1.
iii) $\Leftrightarrow$ *iv)* Because of the linearity of the trace operator, we have $\mathrm{tr}(\alpha A_1 + \beta A_2)^k = \sum_{i=0}^k \alpha^i \beta^{k-i} \mathrm{tr}(A_1{}^{i}\sqcup\!\sqcup{}^{k-i} A_2)$. Therefore condition iii) is equivalent to

$$\sum_{i=0}^{k} \alpha^i \beta^{k-i} [\mathrm{tr}(A_1{}^{i}\sqcup\!\sqcup{}^{k-i} A_2) - \mathrm{tr}(\hat{A}_1{}^{i}\sqcup\!\sqcup{}^{k-i} \hat{A}_2)] = 0,$$

for all $\alpha, \beta \in \mathbf{C}$, which, in turn, is equivalent to iv) $\blacksquare$

The equivalences stated in Prop.2.2 are better understood by investigating the recursive equations which connect the set $\{\mathrm{tr}(A_1{}^{i}\sqcup\!\sqcup{}^{j} A_2)\}$ with the coefficients of the characteristic polynomial $\Delta_{A_1,A_2}(z_1, z_2)$. These relations can be viewed as an extension to the 2D case of Lemma 2.1, which relates the traces of the powers of a matrix A with the coefficients of $\det(I - Az)$.
Consider the polynomial

$$\Delta(z_1, z_2) = 1 - \sum_{h=1}^{n} \Big(\sum_{i+j=h} d_{ij} z_1^i z_2^j \Big) = 1 - \sum_{h=1}^{n} \delta_h(z_1, z_2) \tag{2.5}$$

where $\delta_h(z_1, z_2)$ denotes the homogeneus component of degree h in Δ. Assume that (A_1, A_2) is a matrix pair whose characteristic polynomial coincides with $\Delta(z_1, z_2)$, and let $z_1 = \alpha z$ and $z_2 = \beta z$. Then we have

$$\det[I - (\alpha A_1 + \beta A_2)z] = 1 - \sum_{h=1}^{n} \delta_h(\alpha, \beta) z^h.$$

Denoting by $\tau_h(\alpha, \beta)$ the trace of $(\alpha A_1 + \beta A_2)^h$

$$\tau_h(\alpha, \beta) = \sum_{i+j=h} \mathrm{tr}(A_1{}^{i}\sqcup\!\sqcup{}^{j} A_2) \alpha^i \beta^j, \tag{2.6}$$

equations (2.1) and (2.2) become

$\tau_1(\alpha, \beta) - \delta_1(\alpha, \beta) = 0 \tau_2(\alpha, \beta) - \delta_1(\alpha, \beta)\tau_1(\alpha, \beta) - 2\delta_2(\alpha, \beta) = 0 \qquad (2.7) \cdots \tau_n(\alpha, \beta) - \delta_1(\alpha, \beta)\tau_{n-1}(\alpha, \beta) - \ldots - n\delta_n(\alpha, \beta$

and, for all $k > 0$,

$$\tau_{n+k}(\alpha, \beta) - \sum_{i=1}^{n} \tau_{n+k-i}(\alpha, \beta)\delta_i(\alpha, \beta) = 0. \tag{2.8}$$

As (2.7) and (2.8) hold for every $\alpha, \beta \in \mathbf{C}$, by zeroing the coefficients of every monomial in α and β, and using (2.5) and (2.6), one gets

$$\operatorname{tr}(A_1{}^i \sqcup\!\sqcup^j A_2) = \sum_{0<r+s<i+j} d_{rs} \operatorname{tr}(A_1{}^{i-r} \sqcup\!\sqcup^{j-s} A_2) + (i+j) d_{ij} \tag{2.9}$$

where $d_{rs} = 0$ for $r + s > n$ and $A_1{}^r \sqcup\!\sqcup^s A_2$ is the zero matrix whenever r or s is negative.

Equation (2.9) has some simple but useful consequences. First of all, it provides an algorithm for recursively computing the traces of $A_1{}^i \sqcup\!\sqcup^j A_2$ from the coefficients of the characteristic polynomial. On the other hand, once these traces are given, also the converse, i.e. the computation of the coefficients of Δ, is possible. Actually, if an upper bound $\bar{n}$ on the degree of Δ is known, assigning $\operatorname{tr}(A_1{}^i \sqcup\!\sqcup^j A_2)$ for $i + j \le \bar{n}$ allows to obtain both Δ_{A_1,A_2} and the traces of $A_1{}^i \sqcup\!\sqcup^j A_2$ for $i + j > \bar{n}$.

Remark Consider the set $\mathcal{M}$ of all pairs of square matrices (A_1, A_2) with elements in $\mathbf{C}$, where A_1 and A_2 have the same dimension, and introduce in $\mathcal{M}$ the equivalence $(A_1, A_2) \sim (\bar{A}_1, \bar{A}_2) \Leftrightarrow \Delta_{A_1,A_2}(z_1, z_2) = \Delta_{\bar{A}_1,\bar{A}_2}(z_1, z_2)$. The above proposition shows that both $\{\operatorname{tr}(A_1{}^i \sqcup\!\sqcup^j A_2), i + j > 0\}$ and $\{\operatorname{tr}(\alpha A_1 + \beta A_2)^k, k > 0, \alpha, \beta \in \mathbf{C}\}$ constitute complete families of invariants for $\sim$.

Given a matrix pair (A_1, A_2), we associate the doubly indexed sequence $\{t_{ij}\}_{i+j>0} := \{\operatorname{tr}(A_1{}^i \sqcup\!\sqcup^j A_2)\}_{i+j>0}$ with the following "trace series" in the commutative va-riables z_1 and z_2

$$T := \sum_{(i,j)\neq(0,0)}^{\infty} t_{ij} z_1^i z_2^j. \tag{2.10}$$

Every characteristic polynomial, and, consequently, every equivalence class in $\mathcal{M}$, is biuniquely associated with a trace series. As the coefficients of T satisfy the recursive equations (2.9), it's easy to realize that T is a rational power series. In the remaining part of this section we aim to make explicit the rational structure of T and its connections with the characteristic polynomial.

Proposition 2.3 *Let $\Delta(z_1, z_2) = 1 - \sum_{h=1}^{r} \delta_h(z_1, z_2)$ be the characteristic polynomial of the matrix pair (A_1, A_2). The corresponding trace series T can be expressed as*

$$T = \frac{\delta_1(z_1, z_2) + 2\delta_2(z_1, z_2) + \ldots + n\delta_n(z_1, z_2)}{\Delta(z_1, z_2)}. \tag{2.11}$$

PROOF Consider the linear system defined on the ring of polynomials in the indeterminates α, β, $\mathbf{C}[\alpha, \beta]$,

$$x_{i+1} = F x_i + g u_i \qquad y_i = H x_i,$$

with

$$F = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & 1 \\ \delta_n(\alpha,\beta) & \delta_{n-1}(\alpha,\beta) & \cdots & \delta_1(\alpha,\beta) \end{bmatrix}, \; g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, H = [0 \quad 0 \quad 0 \quad \cdots \quad 1].$$

Assuming $x_0 = 0$ and

$$u_i = \begin{cases} (i+1)\delta_{i+1}(\alpha,\beta) & \text{i = 0, 1, ... n-1} \\ 0 & \text{otherwise,} \end{cases}$$

it's a matter of direct computation to check that the corresponding output is given by $y_i = \tau_i(\alpha, \beta)$, $i = 1, 2, ...$ As the system transfer function is

$$H(I - zF)^{-1} g z = \frac{z}{1 - \sum_{i=1}^{n} \delta_i(\alpha,\beta) z^i},$$

the input $U(z) = u_0 + u_1 z + \ldots + u_{n-1} z^{n-1}$ produces the output

$$Y(z) = \sum_{i=0}^{\infty} \tau_i(\alpha,\beta) z^i = \frac{\sum_{i=1}^{n} i\delta_i(\alpha,\beta) z^i}{1 - \sum_{i=1}^{n} \delta_i(\alpha,\beta) z^i} \tag{2.12}$$

and, letting $z_1 = \alpha z$ and $z_2 = \beta z$, one gets

$$T = \frac{\sum_{i=1}^{n} i\delta_i(z_1, z_2)}{1 - \sum_{i=1}^{n} \delta_i(z_1, z_2)} \; \blacksquare$$

The representation (2.11) of T is not necessarily irreducible; its special structure, however, makes it quite easy to obtain an irreducible representation. To this purpose, consider the injective homomorphism $\phi : \mathbf{C}[\alpha, \beta] \rightarrow \mathbf{C}[\alpha, \beta, z] : \sum_{i=0}^{n} \delta_i(\alpha, \beta) \mapsto \sum_{i=0}^{n} \delta_i(\alpha, \beta) z^i$ where, as usual, $\delta_i(\alpha, \beta)$ denotes an homogeneus polynomial of degree i, and introduce the derivation map

$$D_z : \mathbf{C}[\alpha, \beta, z] \rightarrow \mathbf{C}[\alpha, \beta, z] : \sum_{i=0}^{m} p_i(\alpha, \beta) z^i \mapsto \sum_{i=0}^{m} i p_i(\alpha, \beta) z^i. \tag{2.13}$$

Clearly (2.12) can be rewritten as

$$Y(z) = \frac{-D_z(\phi(\Delta(\alpha, \beta)))}{\phi(\Delta(\alpha, \beta))}. \tag{2.14}$$

Assuming that Δ factorizes as $\Delta = \prod_{i=1}^{t} \Delta_i(z_1, z_2)^{\nu_i}$, where Δ_i are irreducible distinct factors and $\Delta_i(0, 0) = 1$, $i = 1, 2, ..., t$, one easily gets

$$Y(z) = \sum_{i=1}^{t} \nu_i \frac{-D_z(\phi(\Delta_i(\alpha, \beta)))}{\phi(\Delta_i(\alpha, \beta))}. \tag{2.15}$$

Thus, letting $z_1 = \alpha z$ and $z_2 = \beta z$, we have proved the following

Proposition 2.4 *Let $\Delta(z_1, z_2) = \prod_{i=1}^{t} \delta_i(z_1, z_2)^{\nu_i}$, with $\Delta_i(z_1, z_2) = 1 - \sum_{j=1}^{r_i} \delta_j^{(i)}(z_1, z_2)$, $i = 1, 2, ..., t$ irreducible distincts polynomials. For every pair of matrices (A_1, A_2) such that $\Delta_{A_1, A_2}(z_1, z_2) = \Delta(z_1, z_2)$, the corresponding trace series is given by*

$$T = \sum_{i=1}^{t} \nu_i \frac{\sum_{j=1}^{r_i} j \delta_j^{(i)}(z_1, z_2)}{1 - \sum_{j=1}^{r_i} \delta_j^{(i)}(z_1, z_2)} \blacksquare \tag{2.16}$$

Equation (2.16) represents the trace series T as a partial fraction expansion, whose i-th term is the trace series of the irreducible factor $\Delta_i(z_1, z_2)$, weighted by the corresponding multiplicity ν_i. So the characteristic polynomial Δ can be uniquely recovered from the partial fraction expansion. Moreover, in any irreducible representation of T the denominator factorizes into the product of all Δ_i's, $i = 1, 2, ..., t$, each of them with multiplicity one.

3. INFLUENCE OF THE CHARACTERISTIC POLYNOMIAL ON 2D STATE DYNAMICS

In this section we aim to analyse more closely how the local states update in a 2D system and which is the role played by the characteristic polynomial in influencing the structure of the state evolution. To that purpose, we associate to the doubly indexed sequence of local states $\{x(h, k)\}$ induced by a global initial state $\mathcal{X}_0 = \sum_i x(i, -i) z_1^i z_2^{-i}$, the formal power series

$X(z_1, z_2) = \sum_{h,k} x(h, k) z_1^h z_2^k = (I - A_1 z_1 - A_2 z_2)^{-1} \mathcal{X}_0 = \sum_{i,j=0}^{\infty} (A_1{}^i \sqcup\!\sqcup^j A_2 \; z_1^i z_2^j) \mathcal{X}_0.$

The linear subspace of $\mathbf{R}^{n \times n}$ generated by the matrices $A_1{}^i \sqcup\!\sqcup^j A_2$ satisfies some interesting properties, which can be viewed as an extension of Cayley-Hamilton theorem.

Proposition 3.1 [4] *Let $\Delta_{A_1, A_2}(z_1, z_2) = 1 - \sum_{i+j \leq n} d_{ij} z_1^i z_2^j$ be the characteristic polynomial of the $n \times n$ matrix pair (A_1, A_2). Then for all pairs (h, k) with $h + k \geq n$*

i) $A_1{}^h \sqcup\!\sqcup^k A_2 = \sum_{i+j \leq n} d_{ij} \; A_1{}^{h-i} \sqcup\!\sqcup^{k-j} A_2$
(where $A_1{}^r \sqcup\!\sqcup^s A_2$ is assumed to be nonzero whenever r or s is negative);
ii) $A_1{}^h \sqcup\!\sqcup^k A_2 \in \mathrm{span}\{A_1{}^i \sqcup\!\sqcup^j A_2; i \leq h, j \leq k, i + j < n\}$.
Moreover
iii) $\mathrm{span}\{A_1{}^h \sqcup\!\sqcup^k A_2; h, k \geq 0\} = \mathrm{span}\{A_1{}^h \sqcup\!\sqcup^k A_2, h, k < n\}$ $\blacksquare$

The recursive properties of the pair (A_1, A_2), as expressed by Proposition 3.1, hold independently of the structure of the matrices A_1 and A_2 and, in particular, of their characteristic polynomial. We restrict now our attention to polynomials $\Delta(z_1, z_2)$ whose supports are subsets of straight lines, i.e. there exists $(l, m) \neq (0, 0)$ in $\mathbf{N}^2$ such that $\mathrm{supp}(\Delta) \subset \{(kl, km), k \in \mathbf{N}\}$. A 2D system having Δ as characteristic polynomial exhibits several features which strictly resemble those of a 1D system. Indeed, the dynamics induced by a local state $x(0, 0)$ is zero except on a "strip" that includes the straight line $\{(kl, km), k \in \mathbf{Z}\}$. So each local state $x(h, k)$ is always determined by a finite subset of the initial global state, whose cardinality does not exceed a fixed integer N, no matter how far is (h, k) and how we choose the initial conditions.

Proposition 3.2 *Let A_1, A_2 be a pair of $n \times n$ matrices with elements in $\mathbf{C}$ and (l, m) a pair of nonnegative integers with g.c.d.$(l, m) = 1$. The followings are equivalent*

i) $\Delta_{A_1,A_2}(z_1,z_2)=1-\sum_{h=1}^{r} d_h z_1^{lh} z_2^{mh}$;
ii) there exist $c_1,c_2,...,c_n$ *in* $\mathbf{C}$ *such that, for every* α,β *in* $\mathbf{C}$ *and every* $(l+m)$*-th root of* $\alpha^l\beta^m$,

$$\Lambda(\alpha A_1+\beta A_2)=\Big(c_1(\alpha^l\beta^m)^{\frac{1}{l+m}},...,c_n(\alpha^l\beta^m)^{\frac{1}{l+m}}\Big);$$

iii) there exist $c_1,c_2,...,c_n\in\mathbf{C}$ *such that*

$$\Lambda(\nu^m A_1+\nu^{-l}A_2)=(c_1,c_2,...,c_n)$$

for every integer $\nu\in[1,(l+m)n+1]$;
iv) $\mathrm{tr}(A_1{}^i\sqcup{}^jA_2)=0$, *for all* $(i,j)\neq(kl,km)$, $k>0$;
v) for all α,β *and for suitable* b_k *in* $\mathbf{C}$
$\mathrm{tr}(\alpha A_1+\beta A_2)^k=\begin{cases} b_k(\alpha^l\beta^m)^\nu & \text{if } k=(l+m)\nu\\ 0 & \text{otherwise;}\end{cases}$
vi) $A_1{}^i\sqcup{}^jA_2=0$ *for all* (i,j) *out of the strip*

$$\mathcal{S}_n:=\{(i,j)\in\mathbf{N}^2:\mid mi-lj\mid<n\}.$$

PROOF *i)* $\Rightarrow$ *ii)* As $\Delta_{A_1,A_2}(z_1,z_2)\in\mathbf{C}[z_1^l z_2^m]$, there exist $\lambda_1,...,\lambda_r\in\mathbf{C}$ such that $\Delta_{A_1,A_2}(z_1,z_2)=\prod_{h=1}^r(1-\lambda_h z_1^l z_2^m)$ and, consequently,

$$\det(zI-\alpha A_1-\beta A_2)=z^{n-r(l+m)}\prod_{h=1}^{r}(z^{l+m}-\lambda_h\alpha^l\beta^m).$$

Let $(\lambda_h)^{\frac{1}{l+m}}$ and $(\alpha^l\beta^m)^{\frac{1}{l+m}}$ be arbitrary $(l+m)$-th roots of λ_h and $\alpha^l\beta^m$ respectively, and ε a primitive $(l+m)$-th root of 1. The spectrum of $(\alpha A_1+\beta A_2)$ is given by $\Lambda(\alpha A_1+\beta A_2)=\Big(c_1(\alpha^l\beta^m)^{\frac{1}{l+m}},...,c_n(\alpha^l\beta^m)^{\frac{1}{l+m}}\Big)$, where $c_{r\nu+h}=(\lambda_h)^{\frac{1}{l+m}}\varepsilon^\nu$, for $h=1,...,r$ and $\nu=1,...,l+m$, while $c_\mu=0$ if $\mu>(l+m)r$.

ii) $\Rightarrow$ *iii)* Obvious.

iii) $\Rightarrow$ *iv)* For all integers $\nu\in[1,(l+m)n+1]$
$\mathrm{tr}(\nu^mA_1+\nu^{-l}A_2)^h=\sum_{i=0}^h\nu^{(l+m)i-hl}\mathrm{tr}(A_1{}^i\sqcup{}^{h-i}A_2)=\sum_{i=1}^n c_i^h=:f_h,\quad h=1,2,...,$
which implies $\sum_{i=0}^h\nu^{(l+m)i}\mathrm{tr}(A_1{}^i\sqcup{}^{h-i}A_2)-f_h\nu^{hl}=0$. As in the polynomials $p_h(x):=\sum_{i=0}^h x^{(l+m)i}\,\mathrm{tr}(A_1{}^i\sqcup{}^{h-i}A_2)$ $-f_hx^{hl}$ the number of the zeros exceeds the degree, for $h=1,2,...,n$, all their coefficients have to be zero. We distinguish two cases.

• Case 1: $k(l+m)=hl$, for some $k\in\mathbf{N}$. As g.c.d.$(l,m)=1$, there exists $t>0$ such that $k=lt$ and $h-k=mt$. Therefore $\mathrm{tr}(A_1{}^i\sqcup{}^{h-i}A_2)$ coincides with f_h if $(i,h-i)=(tl,tm)$, and is zero otherwise.

• Case 2: $k(l+m)\neq hl$ for all $h\in\mathbf{N}$. Then, for $0\le i\le h$, tr $(A_1{}^i\sqcup{}^{h-i}A_2)=0$.

iv) $\Rightarrow$ *v)* Obvious.

v) $\Rightarrow$ *i)* Equations (2.7) and (2.8) show that the homogeneus components δ_k of the characteristic polynomial satisfy $\delta_k(\alpha,\beta)=d_\nu\alpha^{l\nu}\beta^{m\nu}$, if $k=(l+m)\nu$, and are zero otherwise.

i) $\Rightarrow$ *vi)* Note that $\sum_{i,j=0}^\infty A_1{}^i\sqcup{}^jA_2z_1^iz_2^j=(I-A_1z_1-A_2z_2)^{-1}=\mathrm{adj}(I-A_1z_1-A_2z_2)/(1-\sum_{h=1}^r d_hz_1^{lh}z_2^{mh}$.
As
$\mathrm{supp}(\mathrm{adj}(I-A_1z_1-A_2z_2))\subseteq\{(i,j)\in\mathbf{N}^2;i+j\le n-l\}\mathrm{supp}((1-\sum_{h=1}^r d_hz_1^{lh}z_2^{mh})^{-1})\subseteq\{(i,j)\in\mathbf{N}^2;mi=lj\}$,
it's clear that $\mathrm{supp}((I-A_1z_1-A_2z_2)^{-1})\subseteq\mathcal{S}_n$.

vi) $\Rightarrow$ *i)* Consider the injective ring homomorphism
$\phi:\mathbf{C}[[z_1,z_2]]\to\mathbf{C}[[\eta,\xi,\xi^{-1}]]$ obtained by linearly extending the map that associates $z_1^iz_2^j$ with $\eta^k\xi^h$, where

$$\begin{bmatrix}h\\k\end{bmatrix}=\begin{bmatrix}-m & l\\ 0 & 1\end{bmatrix}\begin{bmatrix}i\\j\end{bmatrix}.$$

ϕ maps any series in $\mathbf{C}[[z_1,z_2]]$ with support in $\mathcal{S}_r$ into an element of the ring $\mathbf{C}[[\eta]][\xi,\xi^{-1}]$ of Laurent polynomials [10] in the indeterminate ξ, with coefficients in $\mathbf{C}[[\eta]]$. By assumption, $\mathrm{supp}\Big((I-A_1z_1-A_2z_2)^{-1}\Big)\subseteq\mathcal{S}_n$ and therefore $\mathrm{supp}\Big(\det(I-A_1z_1-A_2z_2)^{-1}\Big)\subseteq\mathcal{S}_{n^2}$.

Applying the map ϕ on both sides of the identity $1 = \det(I - A_1 z_1 - A_2 z_2)^{-1} \Delta_{A_1,A_1}(z_1, z_2)$ one gets

$$1 = \phi(\det(I - A_1 z_1 - A_2 z_2)^{-1} \phi(\Delta_{A_1,A_2}(z_1, z_2)). \tag{3.1}$$

As both factors on the right hand side of (3.1) can be viewed as elements of $\mathbf{C}[[\eta]][\xi, \xi^{-1}]$, we have that $\phi(\Delta_{A_1,A_2})$ is a unit of the ring, i.e. $\phi(\Delta_{A_1,A_2}) = \xi^h s(\eta)$ for some $h \in \mathbf{Z}$ and $s(\eta) \in \mathbf{C}[[\eta]]$.
Condition $\Delta_{A_1,A_2}(0,0) = 1$ implies $h = 0$ and, therefore, Δ_{A_1,A_2} is a polynomial in $z_1^l z_2^m$ ■

An immediate consequence of property ii) in Prop. 3.2 is the following

Corollary 3.3 *Consider A_1, A_2 in $\mathbf{C}^{n\times n}$. If* $\mathrm{supp}(\Delta_{A_1,A_2})$ *is a subset of a straight line, different from the coordinate axes, then both A_1 and A_2 are nilpotent* ■

A 2D system is called "finite memory" [8] if there exists an integer $N > 0$ such that, for any initial global state $\mathcal{X}_0$, $x(h,k) = 0$ when $h + k \geq N$. Obviously the finite me-mory property corresponds to assume $(I - A_1 z_1 - A_2 z_2)^{-1}$ polynomial or, equivalently, $\Delta_{A_1,A_2}(z_1, z_2) = 1$.
As $\mathrm{supp}(\Delta_{A_1,A_2})$ is a subset of both $\{(i,0); i \in \mathbf{N}\}$ and $\{(0,j); j \in \mathbf{N}\}$, finite memory systems are those which satisfy properties $i) \div vi)$ of Prop. 3.2 both for $(l,m) = (1,0)$ and $(l,m) = (0,1)$. This immediately implies

Corollary 3.4 [Finite memory systems] *Let A_1, A_2 be in $\mathbf{C}^{n\times n}$. The following statements are equivalent*
$FM_1)$ $\Delta_{A_1,A_2}(z_1, z_2) = 1$;
$FM_2)$ $\Lambda(\alpha A_1 + \beta A_2) = (0, 0, ..., 0)$, $\forall \alpha, \beta \in \mathbf{C}$;
$FM_3)$ $\Lambda(\nu A_1 + A_2) = \Lambda(A_1 + \nu A_2) = (0, 0, ..., 0)$, *for* $\nu = 1, ..., n+1$;
$FM_4)$ $\mathrm{tr} A_1{}^{i}\sqcup\!\sqcup^{j} A_2 = 0$, $\forall\ (i,j) \neq (0,0)$;
$FM_5)$ $A_1{}^{i}\sqcup\!\sqcup^{j} A_2 = 0$, *for* $i + j \geq n$ ■

The results of Prop. 3.2 partially extend to the case of a characteristic polynomial which factorizes into irreducible factors, each of them having support included in a straight line. For sake of simplicity, we confine ourselves to the case when $\Delta(z_1, z_2)$ factorizes as $\Delta = \Delta_1 \Delta_2$ where

$$\Delta_i(z_1, z_2) = 1 - \sum_{j=1}^{r_i} d_j^{(i)} (z_1^{l_i} z_2^{m_i})^j, \tag{3.2}$$

and g.c.d.$(l_i, m_i) = 1$, for $i = 1, 2$. If (A_1, A_2) is an $n \times n$ matrix pair with characteristic polynomial Δ,
i) there exist two positive integers ρ and σ, $\rho + \sigma \leq n$, and $\rho + \sigma$ complex numbers $c_1, ..., c_\rho, d_1, ..., d_\sigma$, such that, for all $\alpha, \beta \in \mathbf{C}$

$$\Lambda(\alpha A_1 + \beta A_2) = \Big(c_1 (\alpha^{l_1} \beta^{m_1})^{\frac{1}{l_1+m_1}}, ..., c_\rho (\alpha^{l_1} \beta^{m_1})^{\frac{1}{l_1+m_1}},$$

$$d_1 (\alpha^{l_2} \beta^{m_2})^{\frac{1}{l_2+m_2}}, ..., d_\sigma (\alpha^{l_2} \beta^{m_2})^{\frac{1}{l_2+m_2}}, 0, ..., 0 \Big);$$

ii) $\mathrm{tr}(A_1{}^{i}\sqcup\!\sqcup^{j} A_2) \neq 0$ implies either $(i,j) = (kl_1, km_1)$ or $(i,j) = (hl_2, hm_2)$, for some $h, k \in \mathbf{N}_+$.
Viceversa, each of the above properties guarantees that Δ factorizes as $\Delta = \Delta_1 \Delta_2$ with Δ_i, $i = 1, 2$, as in (3.2).

The previous extension of Prop.3.2 provides a fairly complete description of 2D systems whose characteristic polynomials factorize into the product of a polynomial in z_1 and a polynomial in z_2. Such systems are called "separable" and are usually thought of as the simplest example of I.I.R. 2D systems. Actually, many properties one may hope to extrapolate from an understanding of 1D systems carry over to separable systems [4,7].

Corollary 3.5 [Separable systems] *Let A_1, A_2 be in $\mathbf{C}^{n\times n}$. The following statements are equivalent*
i) $\Delta_{A_1,A_2}(z_1, z_2) = r(z_1) s(z_2)$;
ii) there exist two appropriate orderings of the spectra of A_1 and A_2, $\Lambda(A_1) = (\lambda_1, ..., \lambda_\rho, 0, ..., 0, 0, ..., 0)$ *and* $\Lambda(A_2) = (0, ..., 0, \mu_1, ..., \mu_\sigma, 0, ..., 0)$, *such that, for every* $\alpha, \beta \in \mathbf{C}$

$$\Lambda(\alpha A_1 + \beta A_2) = (\alpha\lambda_1, ..., \alpha\lambda_\rho, \beta\mu_1, ..., \beta\mu_\sigma, 0, ..., 0);$$

iii) $\mathrm{tr}(A_1{}^{i}\sqcup\!\sqcup^{j} A_2) = 0$ *if both i and j are nonzero;*
iv) $\mathrm{tr}(\alpha A_1 + \beta A_2)^k = \mathrm{tr}(\alpha A_1)^k + \mathrm{tr}(\beta A_2)^k$, $\forall \alpha, \beta \in \mathbf{C}, k > 0$ ■

4. PROPERTIES P AND L

In this section we assume a point of view somehow complementary to that of the previous sections, where the behaviour of a 2D system has been related to the properties of its characteristic polynomial, and consider particular classes of matrix pairs, which can be described in terms of their spectral properties.

The matrix pairs (A_1, A_2) we analyse are those endowed with property L or property P [11], which correspond to the possibility of ordering the eigenvalues of A_1 and A_2 into two n-tuples

$$\Lambda(A_1) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \quad \Lambda(A_2) = (\mu_1, \mu_2, \ldots, \mu_n) \tag{4.1}$$

such that

i) *[PROPERTY L]* for every $\alpha, \beta \in \mathbf{C}$

$$\Lambda(\alpha A_1 + \beta A_2) = (\alpha\lambda_1 + \beta\mu_1, \ldots, \alpha\lambda_n + \beta\mu_n) \tag{4.2}$$

ii) *[PROPERTY P]* for every polynomial $\mathcal{P}(\xi_1, \xi_2)$ in the noncommuting indeterminates [1] ξ_1 and ξ_2,

$$\Lambda(\mathcal{P}(A_1, A_2)) = (\mathcal{P}(\lambda_1, \mu_1), \ldots, \mathcal{P}(\lambda_n, \mu_n)). \tag{4.3}$$

It's clear from the definition itself that property P implies property L. However examples can be given showing that the converse is not true [11]. Property L is shared by several important classes of 2D systems, in particular finite memory systems, 2D systems with characteristic polynomials in $\mathbf{R}[z_1]$ or in $\mathbf{R}[z_2]$, separable systems and, more generally, systems whose characteristic polynomials split into a product of linear factors. The following proposition provides a set of conditions equivalent to property L.

Proposition 4.1 [6] *Let A_1, A_2 be in $\mathbf{C}^{n\times n}$, and consider the orderings of their spectra given in* (4.1)*. The followings are equivalent:*
L) (A_1, A_2) has property L (w.r.t. the orderings (4.1)*);*
L_1) $\Delta_{A_1,A_2}(z_1, z_2) = \prod_{i=1}^n (1 - \lambda_i z_1 - \mu_i z_2)$;
L_2) $\mathrm{tr}(\alpha A_1 + \beta A_2)^k = \sum_{i=1}^n (\alpha\lambda_i + \beta\mu_i)^k$, $\forall\ \alpha, \beta \in \mathbf{C}$ and $k \in \mathbf{N}$;
L_3) $\mathrm{tr}(A_1{}^h \sqcup\!\sqcup{}^k A_2) = \binom{h+k}{h} \sum_{i=1}^n \lambda_i^h \mu_i^k$, $\forall\ (h, k) \in \mathbf{N}^2$;
L_4) $T = \sum_{h+k>0} t_{hk} z_1^h z_2^k = \sum_{i=1}^n \dfrac{\lambda_i z_1 + \mu_i z_2}{1 - \lambda_i z_1 - \mu_i z_2}$.

Proof The matrix pair $\bar{A}_1 = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, and $\bar{A}_2 = \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_n\}$ fulfills all conditions L) $\div$ L_4) of the proposition. Any other pair (A_1, A_2), of the same dimension with property L w.r.t. the orderings (4.1) satisfies $\Lambda(\alpha A_1 + \beta A_2) = \Lambda(\alpha\bar{A}_1 + \beta\bar{A}_2)$, which corresponds to ii) of Prop. 2.2. Therefore all equivalent statements in Prop. 2.2 hold true ■

Unlike property L, property P cannot be completely described neither in terms of characteristic polynomial nor in terms of the trace series (2.10). Non commutative polynomials and power series [1] turn out to be the appropriate tools for analysing the structure of matrix pairs with property P.

Proposition 4.2 *Let A_1, A_2 be in $\mathbf{C}^{n\times n}$, and consider the orderings of the spectra given in* (4.1)*. The followings statements are equivalent:*
P) (A_1, A_2) has property P (w.r.t. the orderings (4.1)*);*
P_1) $\mathrm{tr}(w(A_1, A_2)) = \sum_{i=1}^n \lambda_i^{|w|_1} \mu_i^{|w|_2}$, for any $w \in \Xi^*$;
P_2) the noncommutative power series whose coefficients are the traces of the matrices $w(A_1, A_2)$,

$$\mathcal{T} := \sum_{w \in \Xi^* \setminus \emptyset} \mathrm{tr}(w(A_1, A_2))w,$$

is recognizable [1] *and represented as*

$$\mathcal{T} = \sum_{i=1}^n (\lambda_i\xi_1 + \mu_i\xi_2)(1 - \lambda_i\xi_1 - \mu_i\xi_2)^{-1}; \tag{4.4}$$

[1] We shall denote by Ξ^* the free monoid generated by the alphabet $\Xi = \{\xi_1, \xi_2\}$ and, for every word $w \in \Xi^*$, by $|w|_i$ the number of occurrencies of ξ_i in w, $i = 1, 2$.
$\mathbf{C}\langle\xi_1, \xi_2\rangle$ and $\mathbf{C}\langle\langle\xi_1, \xi_2\rangle\rangle$ are the algebras of polynomials and formal power series in the noncommuting indeterminates ξ_1 and ξ_2, respectively. For each pair of matrices A_1, A_2 in $\mathbf{C}^{n\times n}$, the map $\psi : \Xi \to \mathbf{C}^{n\times n} : \xi_i \mapsto A_i,\ i = 1, 2$, uniquely extends to an algebra morphism of $\mathbf{C}\langle\xi_1, \xi_2\rangle$ into $\mathbf{C}^{n\times n}$. The ψ-image of a polynomial $\mathcal{P}(\xi_1, \xi_2) \in \mathbf{C}\langle\xi_1, \xi_2\rangle$ is denoted by $\mathcal{P}(A_1, A_2)$.

$P_3)$ $\det(zI - w(A_1, A_2)) = \prod_{i=1}^n (z - \lambda_i^{|w|_1} \mu_i^{|w|_2})$, *for any* $w \in \Xi^*$.

PROOF P) $\Rightarrow P_1$) If $|w|_1 = h$ and $|w|_2 = k$, definition of property P implies $\Lambda(w(A_1, A_2)) = (\lambda_1^h \mu_1^k, \ldots, \lambda_n^h \mu_n^k)$. Therefore $\mathrm{tr}(w(A_1, A_2)) = \sum_{i=1}^n \lambda_i^h \mu_i^k$.

$P_1) \Rightarrow P)$ Consider any noncommutative polynomial $\mathcal{P} = \sum_w p_w w \in \mathbf{C}\langle \xi_1, \xi_2 \rangle$. For all $h > 0$ we have $\mathrm{tr}(\mathcal{P}(A_1, A_2))^h = \mathrm{tr}(\sum_w p_w(A_1, A_2))^h$. So, letting $\mathcal{Q} = \mathcal{P}^h = \sum_w q_w w$, one gets

$$\mathrm{tr}(\mathcal{P}^h(A_1, A_2)) = \sum_w q_w \mathrm{tr}(w(A_1, A_2)) = \sum_w q_w \sum_{i=1}^n \lambda_i^{|w|_1} \mu_i^{|w|_2} = \sum_{i=1}^n \mathcal{Q}(\lambda_i, \mu_i) = \sum_{i=1}^n \mathcal{P}^h(\lambda_i, \mu_i).$$

Thus $\Lambda(\mathcal{P}(A_1, A_2)) = (\mathcal{P}(\lambda_1, \mu_1), \ldots, \mathcal{P}(\lambda_n, \mu_n))$.

$P_1) \Leftrightarrow P_2)$ Assuming P_1, we get $\mathcal{T} = \sum_{w \in \Xi^* \setminus \{\emptyset\}} \mathrm{tr}(w(A_1, A_2)w = \sum_{i=1}^n \sum_{w \in \Xi^* \setminus \{\emptyset\}} \lambda_i^{|w|_1} \mu_i^{|w|_2} w$. On the other hand,

$$\sum_{i=1}^n (\lambda_i \xi_1 + \mu_i \xi_2)(1 - \lambda_i \xi_1 - \mu_i \xi_2)^{-1} = \sum_{i=1}^n \sum_{j=1}^{+\infty} (\lambda_i \xi_1 + \mu_i \xi_2)^j = \sum_{i=1}^n \sum_{j=1}^{+\infty} \sum_{\substack{w \in \Xi^* \\ |w|_1 + |w|_2 = j}} \lambda_i^{|w|_1} \mu_i^{|w|_2} w = \sum_{i=1}^n \sum_{w \in \Xi^* \setminus \{\emptyset\}} \lambda_i^{|w|_1} \mu_i^{|w|_2} w$$

which proves (4.4). The converse can be shown in the same way.

$P_1) \Rightarrow P_3)$ Given $w \in \Xi^*$, we have $\mathrm{tr}(w(A_1, A_2))^h = \sum_{i=1}^n \lambda_i^{h|w|_1} \mu_i^{h|w|_2} = \sum_{i=1}^n (\lambda_i^{|w|_1} \mu_i^{|w|_2})^h$ for all $h \in \mathbf{N}$. As h is arbitrary, $(\lambda_1^{|w|_1} \mu_1^{|w|_2}, \ldots, \lambda_n^{|w|_1} \mu_n^{|w|_2})$ is the spectrum of $w(A_1, A_2)$, which proves P_2).

$P_3) \Rightarrow P_1)$ Obvious ■

Remark 1 As a consequence of P_1), property P can be stated by referring only to the words of the free monoid Ξ^*, instead of $\mathbf{C}\langle \xi_1, \xi_2 \rangle$. Actually, (A_1, A_2) has property P if and only if, for all $w \in \Xi^*$, we have

$$\Lambda\Big(w(A_1, A_2)\Big) = (\lambda_1^{|w|_1} \mu_1^{|w|_2}, \ldots, \lambda_n^{|w|_1} \mu_n^{|w|_2}),$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $(\mu_1, \mu_2, \ldots, \mu_n)$ are the spectra of A_1 and A_2, suitably ordered.

Remark 2 A celebrated result of McCoy [9] is the equi-valence of property P with simultaneous triangularizability. So, implications P) $\Rightarrow P_i)$, $i = 1, 2, 3$, can be alternatively derived after reducing A_1 and A_2 to triangular form.

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