On the asymptotic dynamics of 2D positive systems

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1 Introduction

A 2D positive system is a 2D system [1] in which the value of the state variables is always nonnegative. The positivity constraint arises quite naturally when modelling real systems, whose state variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc..
In this contribution we consider *homogeneous* positive 2D systems, described by the equation:

$$x(h+1,k+1) = A\ x(h,k+1) + B\ x(h+1,k), \tag{1.1}$$

where the doubly indexed local state sequence $x(\cdot,\cdot)$ takes values in the positive cone $\mathbf{R}^n_+ := \{x \in \mathbf{R}^n : x_i \geq 0,\ i = 1,2,...,n\}$, A and B are nonnegative $n \times n$ matrices, and the initial conditions are assigned by specifying the nonnegative values of the local states on the separation set $\mathcal{C}_0 := \{(i,-i) : i \in \mathbf{Z}\}$. Our aim is to explore how some properties of nonnegative matrix pairs (A,B) influence the asymptotic behaviour of the associated homogeneous 2D systems. More precisely, in section 2 we analyse under which conditions on the initial local states and on the pair (A,B) the states $x(h,k)$ eventually become strictly positive, while in section 3 a couple of results on the convergence towards a constant asymptotic distribution is presented. Some proofs are rather long and, for sake of brevity, will be omitted. The interested reader is referred to [2] and [3].

Before proceeding, it is convenient to introduce some notation. If $M = [m_{ij}]$ is a matrix (in particular, a vector), we write $M \gg 0$ (M *strictly positive*), if $m_{ij} > 0$ for all i,j; $M > 0$ (M *positive*), if $m_{ij} \geq 0$ for all i,j, and $m_{hk} > 0$ for at least one pair (h,k); $M \geq 0$ (M *nonnegative*), if $m_{ij} \geq 0$ for all i,j.
The Hurwitz products of two square matrices A and B are inductively defined as $A^i {\sqcup\!\sqcup}^0 B = A^i$, $A^0 {\sqcup\!\sqcup}^j B = B^j$ and, when i and j are both greater than zero, $A^i {\sqcup\!\sqcup}^j B = A(A^{i-1} {\sqcup\!\sqcup}^j B) + B(A^i {\sqcup\!\sqcup}^{j-1} B)$.
One easily sees that $A^i {\sqcup\!\sqcup}^j B = \sum w(A,B)$, where the summation is extended to all matrix products that include the factors A and B, i and j times respectively. Assuming zero initial conditions on $\mathcal{C}_0$, except at $(0,0)$, it is clear that $x(h,k)$, $h,k \geq 0$, is given by $x(h,k) = A^h {\sqcup\!\sqcup}^k B\ x(0,0)$.

2 Strictly positive asymptotic dynamics

An issue that arises quite naturally when considering the asymptotic behaviour of positive systems is that of guaranteeing that the states eventually become strictly positive vectors.

For 1D positive systems

$$x(h+1) = A\,x(h), \qquad x(0) > 0,$$

the primitivity [4] of the system matrix A is necessary and sufficient for $x(h)$ beeing strictly positive when h is large enough.
For 2D systems described as in (1.1), we say that the state evolution eventually becomes strictly positive if there exists a positive integer T such that $x(h,k) \gg 0$ for all (h,k), $h+k \geq T$. Clearly it's impossible that every nonzero initial global state $\mathcal{X}_0 = \{x(i,-i) : i \in \mathbf{Z}\}$ produces a strictly positive asymptotic dynamics. Actually, when $\mathcal{X}_0$ includes only a finite number of nonzero states, the support of the free evolution is included in a quarter plane causal cone of $\mathbf{Z} \times \mathbf{Z}$. As a consequence, we have to take into account not only the properties of the matrix pair (A,B), but also the zero-pattern of $\mathcal{X}_0$, and we will confine our attention to global states which satisfy the following condition: there exists an integer M such that

$$\sum_{h=1}^{M} x(i+h, -i-h) > 0, \qquad \forall\, i \in \mathbf{Z}. \tag{2.1}$$

In other words, the maximal distance between two consecutive positive states on the separation set $\mathcal{C}_0$ is upper bounded by M. In the sequel we will provide a set of sufficient conditions on the pair (A,B) guaranteeing a strictly positive asymptotic dynamics for all initial global states satisfying (2.1).

Proposition 2.1 Suppose that $A > 0$ and $B > 0$ are $n \times n$ positive matrices and there exists (i,j) such that $A^i \sqcup^j B$ is primitive. Then, for each initial global state satisfying (2.1), there exists an integer $T \geq 0$ such that $x(h,k) \gg 0$ whenever $h+k \geq T$.

Proof If $A^i \sqcup^j B$ is primitive, then $(A^i \sqcup^j B)^p \gg 0$ for some $p > 0$, and therefore $A^{pi} \sqcup^{pj} B \geq (A^i \sqcup^j B)^p \gg 0$. So in the sequel we will assume that there exists (h,k) such that $A^h \sqcup^k B$ is strictly positive. If we suppose $x(0,0) > 0$, we have $x(h,k) \geq (A^h \sqcup^k B)\; x(0,0) \gg 0$, and, as A and B are nonzero matrices, both $x(h+1,k)$ and $x(h,k+1)$ are nonzero vectors. Consequently, if M denotes the maximal distance between two consecutive nonzero local states on $\mathcal{C}_0$, the maximal distance on the separation set $\mathcal{C}_{h+k+1} = \{(i,j) : i+j = h+k+1\}$ is not greater than $M-1$. An inductive argument shows that all local states on $\mathcal{C}_{(M-1)(h+k+1)}$ are nonzero, and, recalling that $A^h \sqcup^k B \gg 0$, we see that all local states on $\mathcal{C}_{M(h+k+1)-1}$ are strictly positive. Consequently, we can choose $T = M(h+k+1) - 1$ ■

Remark The existence of a primitive Hurwitz product $A^i \sqcup^j B$ implies that, for a suitable $p > 0$, $(A^i \sqcup^j B)^p$, and hence $(A+B)^{(i+j)p}$, are strictly positive. Therefore $A+B$ is primitive. The converse in general is not true, as shown by the following example. The pair of irreducible matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.2}$$

has a primitive sum. However, as $B = A^2$, the Hurwitz products are expressed as $A^h \sqcup^k B = \binom{h+k}{h} A^{h+2k}$, and hence are not primitive.

Corollary 2.2 If $A > 0$ and $B > 0$ are $n \times n$ matrices and there exists a primitive matrix product $w(A,B) = A^{h_1} B^{k_1} \cdots A^{h_r} B^{k_r}$, then for all initial global states satisfying (2.1) the asymptotic behaviour of system (1.1) is strictly positive.

Proof If $h_1+h_2+...+h_r = i$ and $k_1+k_2+...+k_r = j$, then $A^i \sqcup^j B \geq w(A,B)$ is primitive too, and we can resort to Proposition 2.1 ■

In order to apply Corollary 2.2 above, in some cases it is enough to have at disposal a quite poor information on the zero pattern of A and B. For instance, it's sufficient to know that A is primitive and $B > 0$, or, when dealing with a pair of irreducible matrices, it is enough to check whether their imprimitivity indexes are coprime, as shown by the following proposition.

Proposition 2.3 [2] Assume that A and B are irreducible matrices with imprimitivity indexes h_A and h_B. If g.c.d.$(h_A, h_B) = 1$, then there exists a primitive matrix product $w(A,B)$ ■

3 Further aspects of the asymptotic dynamics

The problems that will be addressed in this section concern some aspects of the two-dimensional dynamics which entail a finer analysis of its asymptotic behaviour. Indeed our interest here does not merely concentrate on nonzero patterns; it involves also the values of the local state and a qualitative description of the vectors distribution along the separation sets $\mathcal{C}_t = \{(i,j) : i+j = t\}$ as t goes to infinity.
The first problem is that concerning the zeroing of state oscillations on the separation sets $\mathcal{C}_t$, $t \rightarrow +\infty$, when scalar positive systems are considered.

Definition 1 : A scalar (nonnecessarily nonnegative) global state $\mathcal{X}_0 = \{x(i,-i) : \ i \in \mathbf{Z}\}$ has (finite) mean value μ if, given any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that, for all $\nu \geq N(\varepsilon)$ and $h \in \mathbf{Z}$

$$\left| \frac{1}{\nu} \sum_{i=h}^{h+\nu-1} x(i,-i) - \mu \right| < \varepsilon. \tag{3.1}$$

It is not difficult to prove that if $\mathcal{X}_0$ has a (finite) mean value μ, then $\mathcal{X}_0$ is bounded, i.e. there exists a positive integer M such that $|x(i,-i)| < M, \ i \in \mathbf{Z}$, and $\mathcal{X}_t = \{x(i,j), i+j = t\}$ has mean value $(A+B)^t \mu$.
Given a scalar global state $\mathcal{X}_0$ with mean value $\mu \neq 0$, the *oscillation rate* of $\mathcal{X}_0$ is defined as

$$\text{osc}(\mathcal{X}_0) := \frac{\sup_{i,j \in \mathbf{Z}} |x(i,-i) - x(j,-j)|}{|\mu|} \tag{3.2}$$

Proposition 3.1 [2] Consider an homogeneous 2D system (1.1) with $n = 1$ (scalar local states) and $A, B > 0$. Assume moreover that the initial global state $\mathcal{X}_0$ has mean value $\mu > 0$. Then the oscillation rate, $\text{osc}(\mathcal{X}_t)$, goes to zero as t goes to infinity ■

If we drop the hypothesis that (1.1) is a scalar system, the qualitative description of the asymptotic dynamics is by far more interesting, and more difficult. The results so far available deal with two rather restrictive classes of positive 2D systems, that is 2D Markov chains and 2D systems with commutative A and B [3]. Further research will lead, it is hoped, to more comprehensive theorems. For sake of brevity, we discuss only some aspects of commutative 2D systems.

Lemma 3.2 [2] Let $A > 0$ and $B > 0$ be $n \times n$ commutative matrices, whose sum $A+B$ is irreducible. Then A and B have a strictly positive common eigenvector v

$$Av = r_A v, \qquad Bv = r_B v \tag{3.3}$$

and r_A, r_B are the spectral radii of A and B, respectively ■

Lemma 3.3 [3] Suppose that in system (1.1) A, B and the initial global state $\mathcal{X}_0$ satisfy the following assumptions:
(i) A and B are positive commuting matrices
(ii) A and B have a strictly positive common dominant eigenvector v
(iii) There exists ℓ and L, both positive, such that

$$0 < \ell[1 \quad 1 \quad \ldots \quad 1]^T \leq x(i,-i) \leq L[1 \quad 1 \quad \ldots \quad 1]^T, \quad \forall i \in \mathbf{Z}. \tag{3.4}$$

Then

$$\lim_{h+k\to+\infty} \frac{x(h,k)}{||x(h,k)||} = \frac{v}{||v||} \ ■$$

Proposition 3.4 Suppose that in system (1.1)
a) A and B are primitive commuting matrices
b) there exist an integer $M > 0$ and two positive real numbers r and R such that

$$r \leq [1 \quad 1 \quad \ldots \quad 1] \sum_{h=1}^{M} x(i+h, -i-h) \leq R, \quad \forall i \in \mathbf{Z}$$

Then $\lim_{h+k\to+\infty} x(h,k)/||x(h,k)|| = v/||v||$ where $v \gg 0$ is a common eigenvector of A and B.

Proof As $A+B$ is irreducible, by Lemma 3.2 there exists $v \gg 0$ that satisfies equations (3.3). The primitivity assumption guarantees that v is a dominant eigenvector of both A and B. Thus conditions (i) and (ii) of Lemma 3.3 are fulfilled. On the other hand, when N is large enough, all matrices $A^{\nu} \sqcup\!\sqcup^{N-\nu} B,\ 0 \leq \nu \leq N$, are strictly positive. So, denoting by s_N and S_N their minimum and maximum entries

$$s_N := \min_{\nu} \min_{h,k} [A^{\nu} \sqcup\!\sqcup^{N-\nu} B]_{hk} > 0, \qquad S_N := \max_{\nu} \max_{h,k} [A^{\nu} \sqcup\!\sqcup^{N-\nu} B]_{hk} > 0$$

respectively, and assuming $N \geq M$, we have

$$x_j(i+N,-i) \geq s_N \sum_{\nu=0}^{N} [1 \quad 1 \quad \ldots \quad 1] x(i+N-\nu, -i-N+\nu) \geq s_N r \qquad j = 1, 2, \ldots, n$$

and

$$x_j(i+N,-i) \leq S_N \sum_{\nu=0}^{N} [1 \quad 1 \quad \ldots \quad 1] x(i+N-\nu, -i-N+\nu) \leq S_N N R \qquad j = 1, 2, \ldots, n.$$

Therefore, for large values of N, $\mathcal{X}_N$ fulfills condition (iii) of Lemma 3.3 , with $\ell = s_N r$ and $L = S_N N R$, and the proof is complete ■

4 References

1. E.Fornasini, G.Marchesini, (1978), *Doubly indexed dynamical systems: state space models and structural properties*, Math. Sys. Theory, vol.12, pp. 59-72

2. M.E.Valcher, E.Fornasini, (1993) *State models and asymptotic behaviour of 2D positive systems*, to appear in IMA Journal
3. E.Fornasini, G.Marchesini, (1993), *Properties of pairs of matrices and state models for 2D systems. Pt.1: state dynamics and geometry of the pairs*, in "Multivariate Analysis: Future Directions", C.R.Rao ed., Elsevier Sci.Publ.
4. H.Minc, (1988), *Nonnegative matrices*, J.Wiley Publ.