On the structure of finite memory and separable 2D systems

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Abstract

Several characterizations of finite memory and separability properties for a 2D system are presented, in terms of both the characteristic polynomial and the matrix pair that describes the state evolution. Necessary and sufficient conditions for a finite memory or separable 2D system to have an inverse with the same properties are given; these involve only the structure of the transfer matrix.
Keywords two dimensional systems, inverse system, finite memory property, separable systems.

1 Introduction

As well known, a quarter plane causal 2D system is represented by the following first order equations [2][1]

$$\begin{aligned} x(h+1,k+1) &= A_1x(h,k+1)+A_2x(h+1,k)+B_1u(h,k+1)+B_2u(h+1,k) \\ y(h,k) &= Cx(h,k)+Du(h,k) \end{aligned} \tag{1.1}$$

where the local state x is a n-dimensional vector over the real field $\mathbf{R}$, input and output functions take values in $\mathbf{R}^m$ and $\mathbf{R}^p$ respectively, and A_1, A_2, B_1, B_2, C and D are matrices of suitable dimensions, with entries in $\mathbf{R}$. Initial conditions are usually given by the so called "initial global state"

$$\mathcal{X}_0 = \{x(i,-i); i \in \mathbf{Z}\} \tag{1.2}$$

In this paper we analyze some properties of two classes of 2D state models which are of great interest from an applicative point of view: finite memory and separable systems.

The main feature of finite memory systems [1,4] is that of reaching the zero-state in a finite number of steps after the zeroing of the input signal. Therefore they constitute a state model suitable for implementing dead-beat controllers and estimators, and for realizing two-dimensional convolutional encoders and decoders.

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Separable systems [3] are usually thought of as the simplest example of I.I.R. 2D systems. Actually many properties that one may hope to extrapolate from an understanding of 2D finite memory systems do, in fact, carry over to separable 2D systems. Indeed, just the knowledge that the system is separable allows one to make some fairly strong statements about its behaviour. In particular, internal stability can be quickly deduced from the general theory of discrete time 1D systems, since the eigenvalues of largest absolute value in A_1 and A_2 determine the long term performance of separable systems.

Our investigation will focus essentially on the analysis of the algebraic structure of the pairs (A_1, A_2) which correspond to either finite memory or separable systems, and on the conditions which guarantee that these properties are inherited by the correspondig inverse systems. The possibility of synthesizing finite memory and/or separable 2D inverse systems is of great importance in 2D precompensation and decoding.

2 Finite memory systems

A 2D system (1.1) is finite memory if there exists an integer $N > 0$ such that the unforced state evolution satisfies

$$x(h,k) = 0 \tag{2.1}$$

for all $(h,k) \in \mathbf{Z} \times \mathbf{Z}$ with $h + k \geq N$ and for all initial global states $\mathcal{X}_0$. As in the 1D case, the finite memory property can be expressed as a condition on the characteristic polynomial of the pair (A_1, A_2), as stated in the following proposition.

Proposition 1 [1] A 2D system (1.1) is finite memory if and only if its characteristic polynomial is unitary, i.e.

$$\Delta(z_1, z_2) = \det(I - A_1 z_1 - A_2 z_2) = 1. \tag{2.2}$$

Proof If (1.1) is finite memory, for all initial global states $\mathcal{X}_0$ with $x(i, -i) = 0$, $\forall i \neq 0$, the power series $(I - A_1 z_1 - A_2 z_2)^{-1} x(0,0)$ reduces to a polynomial, and therefore both $(I - A_1 z_1 - A_2 z_2)$ and $(I - A_1 z_1 - A_2 z_2)^{-1}$ are polynomial matrices. Thus $(I - A_1 z_1 - A_2 z_2)$ is unimodular and (2.2) holds.

Conversely (2.2) implies that $(I - A_1 z_1 - A_2 z_2)^{-1}$ is polynomial and (2.1) is satisfied for all global initial states $\mathcal{X}_0$ ■

Next proposition provides a set of equivalent conditions on the structure of the pairs (A_1, A_2), that characterize the finite memory property. These involve

i) the property L [5] of the pair (A_1, A_2), i.e. the possibility of ordering the eigenvalues of A_1 and A_2 in two complex n-tuples

$$\Lambda(A_1) = (\lambda_1, ..., \lambda_n) \text{ and } \Lambda(A_2) = (\mu_1, ..., \mu_n)$$

such that, for all α and β in $\mathbf{C}$, the eigenvalues of $\alpha A_1 + \beta A_2$ are given by

$$\Lambda(\alpha A_1 + \beta A_2) = (\alpha\lambda_1 + \beta\mu_1, ..., \alpha\lambda_n + \beta\mu_n) = \alpha\Lambda(A_1) + \beta\Lambda(A_2); \tag{2.3}$$

ii) the matrix coefficients of the power series expansion of $(I - A_1z_1 - A_2z_2)^{-1}$, i.e. the matrices ${A_1}^i\sqcup\!\sqcup^j A_2$, $(i, j) \in \mathbf{N} \times \mathbf{N}$, inductively defined as follows:

$$ {A_1}^i\sqcup\!\sqcup^0 A_2 = A_1^i, \quad {A_1}^0\sqcup\!\sqcup^j A_2 = A_2^j \tag{2.4.1} $$

and, if both i and j are greater than 0,

$$ {A_1}^i\sqcup\!\sqcup^j A_2 = A_1({A_1}^{i-1}\sqcup\!\sqcup^j A_2) + A_2({A_1}^i\sqcup\!\sqcup^{j-1} A_2). \tag{2.4.2} $$

Proposition 2 The following statements are equivalent:
i) system (1.1) is finite memory;
ii) the linear subspace generated by A_1 and A_2 consists of nilpotens;
iii) A_1 and A_2 are nilpotent matrices and satisfy property L;
iv) the additive semigroup generated by A_1 and A_2 consists of nilpotens;
v) for all $(h, k) \in \mathbf{N} \times \mathbf{N} \setminus (0, 0)$, $\mathrm{tr}({A_1}^h\sqcup\!\sqcup^k A_2) = 0$. (2.5)

Proof *i) ⇒ ii)* Assume α, β in $\mathbf{C}$ and let in (2.2) $z_1 = \alpha z$, $z_2 = \beta z$. Then for all α, β in $\mathbf{C}$, $\det[I - (\alpha A_1 + \beta A_2)z] = 1$, which corresponds to the nilpotency of $\alpha A_1 + \beta A_2$.

ii) ⇒ iii) Since $\alpha A_1 + \beta A_2$ is nilpotent for all α and β in $\mathbf{C}$, we have $\Lambda(A_1) = (0, ..., 0)$, $\Lambda(A_2) = (0, ..., 0)$ and $\Lambda(\alpha A_1 + \beta A_2) = (0, ..., 0) = \alpha\Lambda(A_1) + \beta\Lambda(A_2)$, which corresponds to property L.

iii) ⇒ i) Because of property L and the nilpotency of A_1 and A_2, we get

$$ \Lambda(\alpha A_1 + \beta A_2) = \alpha\Lambda(A_1) + \beta\Lambda(A_2) = (0, ..., 0). $$

This gives $\det[I - (\alpha A_1 + \beta A_2)z] = 1$ for all α, $\beta \in \mathbf{C}$, which, in turn, implies $\det(I - A_1z_1 - A_2z_2) = 1$ for all z_1, z_2 in $\mathbf{C}$.

ii) ⇒ iv) Obvious.

iv) ⇒ ii) As $m_1A_1 + m_2A_2$ is nilpotent for all (m_1, m_2) in $\mathbf{N} \times \mathbf{N} \setminus (0, 0)$, the homogeneous polynomial of degree $m > 0$ in the complex variables z_1 and z_2, $\mathrm{tr}(z_1A_1 + z_2A_2)^m$, annihilates for all non negative integer values of z_1, z_2 and, consequently, for all complex values of z_1 and z_2.
Fix α and β in $\mathbf{C}$. Then $\mathrm{tr}(\alpha A_1 + \beta A_2)^m = 0, \quad \forall m \in \mathbf{N}_+$ implies that $\alpha A_1 + \beta A_2$ is nilpotent.

ii) ⇒ v) For $m \in N_+$ and for all α, β in $\mathbf{C}$, the nilpotency of $\alpha A_1 + \beta A_2$ implies

$$ 0 = \mathrm{tr}(\alpha A_1 + \beta A_2)^m = \sum_{i=0}^{m} \alpha^{m-i}\beta^i \mathrm{tr}({A_1}^{m-i}\sqcup\!\sqcup^i A_2), \tag{2.6} $$

so that $\mathrm{tr}({A_1}^{m-i}\sqcup\!\sqcup^i A_2) = 0, \quad i = 0, 1, ..., m$. As in (2.6) m is an arbitrary positive integer, $\mathrm{tr}({A_1}^h\sqcup\!\sqcup^k A_2)$ is zero for all $(h, k) \neq (0, 0)$ in $\mathbf{N} \times \mathbf{N}$.

v) ⇒ ii) Because of (2.5),

$$ \mathrm{tr}(\alpha A_1 + \beta A_2)^m = \sum_{i=0}^{m} \alpha^{m-i}\beta^i \mathrm{tr}({A_1}^{m-i}\sqcup\!\sqcup^i A_2) = 0 $$

holds for all $m > 0$, which proves the nilpotency of $\alpha A_1 + \beta A_2$ ■

3 Separable systems

A 2D system is separable if the characteristic polynomial of the $n \times n$ matrix pair (A_1, A_2) factorizes into the product of two polynomials in one variable, i.e.

$$\det(I - A_1 z_1 - A_2 z_2) = r(z_1)s(z_2). \tag{3.1}$$

Letting $\rho = \deg r$ and $\sigma = \deg s$, clearly $\rho + \sigma \leq n$. Moreover, there's no loss of generality in assuming $r(0) = s(0) = 1$. In Proposition 3 below, the separability property will be related to equivalent properties expressed in terms of the spectrum of $\alpha A_1 + \beta A_2$ and the traces of the matrices $A_1{}^h{\sqcup\!\sqcup}^k A_2$. In the proof we find it convenient to exploit the following:

Lemma [3] Let a_1, a_2,...,a_n be complex numbers. Then the system

$$\xi_1^i + \xi_2^i + ... + \xi_n^i = a_i \quad i = 1, 2, ..., n$$

in the indeterminates ξ_1, ξ_2,...,ξ_n admits one solution, which is uniquely determined up to a permutation.

Proposition 3 The following statements are equivalent:
i) $\det(I - A_1 z_1 - A_2 z_2) = r(z_1)s(z_2)$;
ii) A_1 and A_2 satisfy property L w.r.t. the orderings

$$\begin{array}{ll} \Lambda(A_1) & = (\;\lambda_1, \;\lambda_2, \;..., \;\lambda_\rho, \;0, \;0, \;..., \;0, \;0, \;..., \;0) \\ \Lambda(A_2) & = (\;0, \;0, \;..., \;0, \;\mu_1, \;\mu_2, \;..., \;\mu_\sigma, \;0, \;..., \;0) \end{array} \tag{3.2}$$

of the spectra, so that, for all α, β in $\mathbf{C}$, one gets

$$\Lambda(\alpha A_1 + \beta A_2) = (\alpha\lambda_1, \alpha\lambda_2, ..., \alpha\lambda_\rho, \beta\mu_1, \beta\mu_2, ..., \beta\mu_\sigma, 0, ..., 0); \tag{3.3}$$

iii) for all α, β in $\mathbf{C}$ and ν in $\mathbf{N}_+$

$$\mathrm{tr}(\alpha A_1 + \beta A_2)^\nu = \mathrm{tr}(\alpha A_1)^\nu + \mathrm{tr}(\beta A_2)^\nu; \tag{3.4}$$

iv)

$$\mathrm{tr} A_1{}^h{\sqcup\!\sqcup}^k A_2 = 0, \forall h, k > 0. \tag{3.5}$$

Proof *i)* $\Rightarrow$ *ii)* Let $1/\lambda_1$, $1/\lambda_2$,..., $1/\lambda_\rho$ be the zeros of $r(z_1)$ and $1/\mu_1$, $1/\mu_2$,..., $1/\mu_\sigma$ the zeros of $s(z_2)$. Then for all α and β in $\mathbf{C}$, we have

$$\det[I - (\alpha A_1 + \beta A_2)z] = c \prod_{i=1}^{\rho} (\alpha z - 1/\lambda_i) \prod_{j=1}^{\sigma} (\beta z - 1/\mu_j)$$

with $c = (-1)^{\rho+\sigma} \prod_i \lambda_i \prod_j \mu_j$. Hence $\alpha\lambda_1, \alpha\lambda_2, ..., \alpha\lambda_\rho$, $\beta\mu_1, \beta\mu_2, ..., \beta\mu_\sigma$ are the non zero eigenvalues of $\alpha A_1 + \beta A_2$ and property L holds.

$ii) \Rightarrow i)$ Since $\Lambda(\alpha A_1 + \beta A_2) = (\alpha\lambda_1, ..., \alpha\lambda_\rho, \beta\mu_1, ..., \beta\mu_\sigma, 0, ..., 0)$, we get

$$\det[I - (\alpha A_1 - \beta A_2)z] = \prod_{i=1}^{\rho}(1 - \alpha\lambda_i z)\prod_{j=1}^{\sigma}(1 - \beta\mu_j z),$$

and, letting $z_1 = \alpha z$ and $z_2 = \beta$, we obtain

$$\det(I - A_1 z_1 - A_2 z_2) = \prod_{i=1}^{\rho}(1 - \lambda_i z_1)\prod_{j=1}^{\sigma}(1 - \mu_j z_2).$$

$ii) \Rightarrow iii)$ From (3.2) and (3.3) one gets

$$\begin{array}{rcl} \mathrm{tr}(\alpha A_1)^\nu & = & \alpha^\nu(\lambda_1^\nu + \lambda_2^\nu + ... + \lambda_\rho) \\ \mathrm{tr}(\beta A_2)^\nu & = & \beta^\nu(\mu_1^\nu + \mu_2^\nu + ... + \mu_\sigma^\nu) \\ \mathrm{tr}(\alpha A_1 + \beta A_2)^\nu & = & \alpha^\nu \sum_{i=1}^{\rho}\lambda_i^\nu + \beta^\nu \sum_{j=1}^{\sigma}\mu_j^\sigma \end{array}$$

which implies (3.4).

$iii) \Rightarrow ii)$ Assume that $\xi_1(\alpha,\beta), \xi_2(\alpha,\beta), ..., \xi_n(\alpha,\beta)$ are the eigenvalues of $\alpha A_1 + \beta A_2$, and $\lambda_1, \lambda_2, ..., \lambda_\tau$ and $\mu_1, \mu_2, ..., \mu_k$ are the nonzero eigenvalues of A_1 and A_2, respectively. Because of iii), for all $\nu \in \mathbf{N}_+$ we have

$$\xi_1^\nu + \xi_2^\nu + ... + \xi_n^\nu = \mathrm{tr}(\alpha A_1 + \beta A_2)^\nu = \alpha^\nu \mathrm{tr} A_1^\nu + \beta^N \mathrm{tr} A_2^\nu. \tag{3.6}$$

This implies $\tau + k \leq n$, otherwise the system

$$\xi_1^h + \xi_2^h + ... + \xi_n^h + ... + \xi_{\tau+k}^h = \sum_{i=1}^{h}\alpha^\tau\lambda_i^h + \sum_{j=1}^{k}\beta^h\mu_j^h, \quad h = 1, 2, ...\tau + k \tag{3.7}$$

would have as solution both

$$\xi_1(\alpha,\beta), \xi_2(\alpha,\beta), ..., \xi_n(\alpha,\beta), 0, ..., 0$$

and

$$\alpha\lambda_1, \alpha\lambda_2, ..., \alpha\lambda_\tau, \beta\mu_1, \beta\mu_2, ..., \beta\mu_k$$

for all α, $\beta \in \mathbf{C}$. This would contradict the uniqueness of the solution of (3.6), up to a permutation, as stated in the Lemma.
As $\tau + k \leq n$, the system

$$\xi_1^h + \xi_2^h + ... + \xi_n^h = \sum_{i=1}^{h}\alpha^\tau\lambda_i^h + \sum_{j=1}^{k}\beta^h\mu_j^h, \quad h = 1, 2, ..., n$$

admits the solution

$$\alpha\lambda_1, \alpha\lambda_2, ..., \alpha\lambda_\tau, \beta\mu_1, ..., \beta\mu_k, 0, ..., 0 \tag{3.8}$$

and also

$$\xi_1(\alpha,\beta), \xi_2(\alpha,\beta), ..., \xi_n(\alpha,\beta). \tag{3.9}$$

Again, by the Lemma, (3.8) and (3.9) coincide, so that the elements listed in (3.8) coincide with the eigenvalues of $\alpha A_1 + \beta A_2$ for all α, β in $\mathbf{C}$.

iii) $\Rightarrow$ *iv)* Condition (3.4) implies

$$\begin{aligned} \mathrm{tr}(\alpha A_1)^\nu \quad + \quad & \mathrm{tr}(\beta A_2)^\nu = \mathrm{tr}(\alpha A_1 + \beta A_2)^\nu \\ = \quad & \mathrm{tr}(\alpha A_1)^\nu + \alpha^{\nu-1}\beta \mathrm{tr}(A_1{}^{\nu-1} \sqcup\!\sqcup {}^1 A_2) + ... + \alpha\beta^{\nu-1}\mathrm{tr}(A_1{}^1 \sqcup\!\sqcup {}^{\nu-1} A_2) + \mathrm{tr}(\beta A_2)^\nu \end{aligned}$$

for every $\alpha, \beta \in \mathbf{C}$. Hence $\mathrm{tr}(A_1{}^h \sqcup\!\sqcup {}^k A_2) = 0$ for all $h, k > 0$ with $h + k = \nu$. Because of the arbitrariety of ν, (3.5) follows.

iv) $\Rightarrow$ *iii)* Obvious ■

Corollary If the $n \times n$ matrix pair (A_1, A_2) is separable, then

$$\mathrm{tr}(A_1^k A_2) = \mathrm{tr}(A_1 A_2^k) = 0, \ \forall k > 0. \tag{3.10}$$

Proof Recalling that the trace of a matrix product is invariant w.r.t. a cyclic permutation of its factors, from (3.5) we have

$$\begin{aligned} 0 \quad &= \quad \mathrm{tr}(A_1{}^k \sqcup\!\sqcup {}^1 A_2) \\ &= \quad \mathrm{tr}(A_2 A_1 ... A_1) + \mathrm{tr}(A_1 A_2 A_1 ... A_1) + ... + \mathrm{tr}(A_1 ... A_1 A_2) \\ &= \quad (k+1)\mathrm{tr}(A_1^k A_2) \ \blacksquare \end{aligned}$$

As well known, property L corresponds to assume that the characteristic polynomial of the matrix pair (A_1, A_2) factorizes into the product of first order terms as

$$\det(I - A_1 z_1 - A_2 z_2) = \prod_{i=1}^{n} (1 - \lambda_i z_1 - \mu_i z_2). \tag{3.11}$$

It follows that property L, separability and finite memory can be seen as the result of a progressive reinforcement of the constraints on the structure of $\Delta(z_1, z_2)$. On the other side, the set of commutative matrix pairs is properly included in the set of all pairs with property P, which, in turn, is properly included in the set of all pairs with property L [5].
So we may ask to what extent the properties we have expressed in terms of the characteristic polynomial can be related to property P and to the commutativity of (A_1, A_2). We first observe that commutativity and property P do not impose further constraints on the structure of (3.11), so that there exist commutative pairs and pairs with property P which are not finite memory, and not even separable. Moreover, examples can be found of finite memory pairs which are not simultaneously triangularizable [4] and hence do not satisfy property P. See, for instance, the pair

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This shows that there is no implication between commutativity and property P on one hand, and finite memory and separability conditions on the other.

4 Inverse systems

In this section we shall explore to what extent the finite memory and separability properties of a 2D system, $\Sigma = (A_1, A_2, B_1, B_2, C, D)$, with m inputs and p outputs, are inherited by the inverse system $\Sigma_{D^{-1}} = (A_1 - B_1 D^{-1} C, A_2 - B_2 D^{-1} C, B_1 D^{-1}, B_2 D^{-1}, -D^{-1} C, D^{-1})$.

Although the existence of a polynomial inverse of a transfer matrix $W(z_1, z_2)$ guarantees that both W and W^{-1} admit state space realizations with the finite memory property, in general it is not true that there exists a finite memory realization $\bar{\Sigma}$ of $W(z_1, z_2)^{-1}$ which can be viewed as the inverse system of some finite memory realization Σ of $W(z_1, z_2)$. The same applies when considering bicausal recognizable transfer matrices and separable systems, reapectively. So the natural problem arises to determine under what circumstances a 2D system and its inverse share the finite memory or the separability properties.

The following proposition shows that the solution only depends on the transfer matrix $W(z_1, z_2)$ and, possibly, on the constant matrix D^{-1}, whereas the particular structure of the state space realization does not play any role.

Proposition 4 Let $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be a finite memory 2D system which realizes a $p \times m$ polynomial transfer matrix $W(z_1, z_2)$, with $p > m$ and $W(0,0) = D$ full column rank. For every left inverse D^{-1} of D, the following statements are equivalent
i) the inverse system $\Sigma_{D^{-1}} = (A_1 - B_1 D^{-1} C, A_2 - B_2 D^{-1} C, B_1 D^{-1}, B_2 D^{-1}, -D^{-1} C, D^{-1})$ is finite memory;

ii) $D^{-1} W(z_1, z_2)$ is a unimodular matrix;

iii) $W(z_1, z_2)$ can be represented in the following form

$$W(z_1, z_2) = D U(z_1, z_2) + B(z_1, z_2) \tag{4.1}$$

with $U(z_1, z_2)$ unimodular, $U(0,0) = I_m$, $B(z_1, z_2)$ polynomial, $D^{-1} B(z_1, z_2) = 0$ and $B(0,0) = 0$;

iv) $W(z_1, z_2)$ can be column bordered into a unimodular $p \times p$ matrix $V(z_1, z_2)$

$$V(z_1, z_2) = [\, W(z_1, z_2) \quad K \,], \tag{4.2}$$

by any constant full column rank $p \times (p - m)$ matrix K such that $D^{-1} K = 0$;

PROOF *i)* $\Rightarrow$ *ii)* Assume that there exists a left inverse D^{-1} with the property that the corresponding inverse system $\Sigma_{D^{-1}}$ is finite memory, i.e.

$$\det[I - (A_1 - B_1 D^{-1} C) z_1 - (A_2 - B D^{-1} C) z_2] = 1.$$

This implies

$$\begin{aligned} 1 &= \det[I - A_1 z_1 - A_2 z_2] \det[I + (B_1 z_1 + B_2 z_2)\; D^{-1} C (I - A_1 z_1 - A_2 z_2)^{-1}] \\ &= \det[I + D^{-1} C (I - A_1 z_1 - A_2 z_2)^{-1} (B_1 z_1 + B_2 z_2) D^{-1}] = \det[D^{-1} W(z_1, z_2)] \end{aligned} \tag{4.3}$$

and therefore $D^{-1}W(z_1, z_2)$ is a unimodular matrix.

ii) $\Rightarrow$ *iii)* Let's assume that D^{-1} is a unimodular matrix and let A be a $(p-m) \times p$ matrix such that $\begin{bmatrix} D^{-1} \\ A \end{bmatrix}$ is nonsingular and $AD = 0$. Then

$$\begin{bmatrix} D^{-1} \\ A \end{bmatrix} W(z_1, z_2) = \begin{bmatrix} U(z_1, z_2) \\ T(z_1, z_2) \end{bmatrix},$$

with $U(z_1, z_2)$ unimodular and $U(0,0) = I_m$, $T(z_1, z_2)$ polynomial and $T(0,0) = 0$. If we denote by $[\, D \quad K \,]$ the inverse of $\begin{bmatrix} D^{-1} \\ A \end{bmatrix}$, we have

$$W(z_1, z_2) = [\, D \quad K \,] \begin{bmatrix} U(z_1, z_2) \\ T(z_1, z_2) \end{bmatrix} = DU(z_1, z_2) + KT(z_1, z_2). \tag{4.4}$$

Then $B(z_1, z_2) := KT(z_1, z_2)$ satisfies $D^{-1}B(z_1, z_2) = D^{-1}KT(z_1, z_2) = 0$ and $B(0,0) = 0$.

iii) $\Rightarrow$ *iv)* Let K be a real $p \times (p-m)$ matrix such that $[\, D \quad K \,]$ is nonsingular and $D^{-1}K = 0$. If we call $\begin{bmatrix} D \; -1 \\ A \end{bmatrix}$ the inverse of $[\, D \quad K \,]$, we get

$$\begin{bmatrix} D \; -1 \\ A \end{bmatrix} [\, W(z_1, z_2) \quad K \,] = \begin{bmatrix} U(z_1, z_2) & 0 \\ AW(z_1, z_2) & I \end{bmatrix}$$

and consequently

$$[\, W(z_1, z_2) \quad K \,] = [\, D \quad K \,] \begin{bmatrix} U(z_1, z_2) & 0 \\ AW(z_1, z_2) & I \end{bmatrix}.$$

So $[\, W(z_1, z_2) \quad K \,]$, which is the product of two unimodular matrices, is unimodular too.

iv) $\Rightarrow$ *i)* As $V(z_1, z_2)$ is unimodular, $V(0,0) = [\, D \quad K \,]$ is nonsingular and, because of $D^{-1}K = 0$, we have

$$\begin{bmatrix} D^{-1} \\ A \end{bmatrix} = [\, D \quad K \,]^{-1}$$

Thus

$$\begin{bmatrix} D^{-1} \\ A \end{bmatrix} [\, W(z_1, z_2) \quad K \,] = \begin{bmatrix} D^{-1}W(z_1, z_2) & 0 \\ AW(z_1, z_2) & I \end{bmatrix},$$

and consequently $D^{-1}W(z_1, z_2)$, are unimodular matrices. It follows that

$$\det[I - (A_1 - B_1 D^{-1} C) z_1 - (A_2 - B_2 D^{-1} C) z_2] = \det[I - A_1 z_1 - A_2 z_2] \det[D^{-1} W(z_1, z_2)] = 1$$

and therefore $\Sigma_{D^{-1}}$ is a finite memory system ■

Remark Even if it has not been explicitly stated, the zero primeness of $W(z_1, z_2)$ is essential in order to fulfill conditions $(i) \div (iv)$ of Proposition 4. Actually, the possibility of column bordering W into a 2D unimodular matrix implies that $W(z_1, z_2)$ is full rank for every (z_1, z_2) in $\mathbf{C} \times \mathbf{C}$. It is worthwhile to remark, however, that zero primeness of W is by no means a sufficient condition for guaranteeing that the inverse of a finite memory system is finite memory. Actually, the zero primeness of W is equivalent, by the Quillen-Suslin theorem [6], to the possibility of completing W into a unimodular matrix by resorting to a suitable set of polynomial columns, whereas Proposition 4 requires that the column bordering has to be performed using $\mathbf{R}$-valued columns only. From a different point of view, a matrix is zero prime if its maximum order minors have no common zeros. Thus a polynomial combination of the minors can be found which satisfies the Bézout identity. If we suppose that condition (ii) holds, and apply the Binet-Cauchy theorem, we can express $\det[D^{-1}W(z_1,z_2)]$ as

$$\sum_{1}^{\binom{p}{m}} \det[D^{-1}S_i]\det[S_i^T W(z_1,z_2)] \equiv 1,$$

where S_i is a $p \times m$ selection matrix, all zero except for an $m \times m$ permutation submatrix. So the Bézout identity reduces to a linear combination over $\mathbf{R}$ of the maximal order minors.

Before considering the inversion of separable systems, it seems appropriate to discuss some properties of the ring $\mathbf{R}^{\mathrm{rec}}(z_1,z_2)$ of recognizable functions in two variables, i.e. rational functions with separable denominators, which play in the separable case the same role as that of $\mathbf{R}[z_1,z_2]$ in the finite memory case.
An $m \times m$ rational matrix in two variables $U(z_1,z_2)$ is unimodular w.r.t. $\mathbf{R}^{\mathrm{rec}}(z_1,z_2)$ if $U(z_1,z_2)$ is full rank and the entries of both $U(z_1,z_2)$ and $U(z_1,z_2)^{-1}$ are recognizable functions. In the sequel, "unimodularity" will always be meant w.r.t. the ring $\mathbf{R}^{\mathrm{rec}}(z_1,z_2)$.

Proposition 5 The following statements are equivalent:
i) $U(z_1,z_2) \in \mathbf{R}^{\mathrm{rec}}(z_1,z_2)^{m\times m}$ is unimodular;
ii) $\det U(z_1,z_2)$ is the product of a rational function in $\mathbf{R}(z_1)$ and a rational function in $\mathbf{R}(z_2)$, i.e.

$$\det U(z_1,z_2) = \frac{n_1(z_1)}{d_1(z_1)}\frac{n_2(z_2)}{d_2(z_2)}; \tag{4.5}$$

iii) in every left or right coprime MFD, $U(z_1,z_2) = N_R(z_1,z_2)D_R^{-1}(z_1,z_2) = D_L^{-1}(z_1,z_2)$ $N_L(z_1,z_2)$, $\det D_R$, $\det N_R$, $\det D_L$ and $\det N_L$ are separable polynomials.

Proof *i) $\Rightarrow$ ii)* Since $U(z_1,z_2)$ and $U(z_1,z_2)^{-1}$ have recognizable entries, the same holds true for $\det U$ and $\det U^{-1}$:

$$\det U(z_1,z_2) = \frac{h(z_1,z_2)}{d_1(z_1)d_2(z_2)} \qquad \det U(z_1,z_2)^{-1} = \frac{k(z_1,z_2)}{f_1(z_1)f_2(z_2)}. \tag{4.6}$$

The identity

$$1 = \det U \det U^{-1} = \frac{h(z_1,z_2)k(z_1,z_2)}{d_1(z_1)d_2(z_2)f_1(z_1)f_2(z_2)}$$

shows that both $h(z_1,z_2)$ and $k(z_1,z_2)$ are separable polynomials.

$ii) \Rightarrow iii)$ The irreducibility of the MFD implies that $\det N_R$ and $\det D_R$ are coprime polynomials. Therefore the identity

$$\det U = \frac{n_1(z_1)}{d_1(z_1)} \frac{n_2(z_2)}{d_2(z_2)} = \frac{\det N_R(z_1, z_2)}{\det D_R(z_1, z_2)} \tag{4.7}$$

shows that both N_R and D_R are separable.

$iii) \Rightarrow i)$ Obvious ■

Unlike unimodular polynomial matrices, unimodular recognizable matrices need not be (quarter plane) causal and causally invertible. However it is easy to check that a unimodular recognizable matrix $U(z_1, z_2)$ is realizable with its inverse if and only if $U(0,0)$ exists and is full rank.

We are now in a position for providing a complete extension of Proposition 4 to the case of separable systems. The proof is essentially the same and, for sake of brevity, will be omitted.

Proposition 6 Let $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be a separable 2D system which realizes a $p \times m$ separable transfer matrix $W(z_1, z_2)$, with $p > m$ and $W(0,0) = D$ full column rank. For every left inverse D^{-1} of D, the following statements are equivalent

i) the inverse system $\Sigma_{D^{-1}}$ is separable;

ii) $D^{-1}W(z_1, z_2)$ is a unimodular matrix w.r.t. $\mathbf{R}^{\text{rec}}(z_1, z_2)$;

iii) $W(z_1, z_2)$ can be represented as in (4.2) with $U(z_1, z_2)$ unimodular, $U(0,0) = I_m$, $B(z_1, z_2)$ recognizable, $D^{-1}B(z_1, z_2) = 0$ and $B(0,0) = 0$;

iv) $W(z_1, z_2)$ can be column bordered into a bicausal unimodular $p \times p$ matrix $V(z_1, z_2)$ by any constant full column rank $p \times (p - m)$ matrix K such that $D^{-1}K = 0$ ■

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