State models of 2D positive systems

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ABSTRACT

Homogeneous 2D state space models whose variables are always nonnegative are described by a pair of nonnegative square matrices (A, B). In the paper, we discuss some spectral and combinatorial properties under particular assumptions on the structure of the matrix pair, like finite memory, separability and property L.

1. INTRODUCTION

Positive state models are widely applied in representing physical, biological and economical dynamical evolutions in which the variables are always nonnegative in value. In this contribution we consider *2D positive systems*, i.e. discrete positive state models whose variables depend on two integer indices, according to a quarter plane causality law. The investigation of this class of systems is quite recent [1,2] and several challenging problems remain still open. As a point has not reached yet where a general survey can be attempted, we have preferred to concentrate on some basic topics which underlie the analysis ot the unforced 2D state equation

$$\mathbf{x}(h+1,k+1) = A\,\mathbf{x}(h,k+1) + B\,\mathbf{x}(h+1,k), \quad (1.1)$$

where the doubly indexed local state sequence $\mathbf{x}(\cdot,\cdot)$ takes values in the positive cone $\mathbf{R}_+^n := \{\mathbf{x} \in \mathbf{R}^n : x_i \geq 0,\ i = 1,2,...,n\}$, A and B are nonnegative $n \times n$ matrices, and the initial conditions are assigned on $\mathcal{C}_0 := \{(i,-i) : i \in \mathbf{Z}\}$.

The results we are going to present fall in two classes, which correspond to the sections of the paper. In section 2, we investigate various connections among the spectral properies (nilpotency, dominant eigenvalues, common dominant eigenvectors) a positive matrix pair (A, B) may exhibit. Different hypotheses on its structure are introduced, such as finite memory, separability, commutativity etc., which are frequently used for characterizing the behavior of a 2D system. An usual method for studying invariants defined on matrices is to simplify the structure of the matrices by linear transformations that preserve the invariants. Following this philosophy, in section 3 we use permutation matrices for obtaining canonical matrix pairs cogredient to positive pairs endowed with special properties.

Before proceeding, we introduce some notation. If $M = [m_{ij}]$ is a matrix , we write $M \gg 0$ (M *strictly positive*), if $m_{ij} > 0$ for all i,j; $M > 0$ (M *positive*), if $m_{ij} \geq 0$ for all i,j, and $m_{hk} > 0$ for at least one pair (h,k); $M \geq 0$ (M *nonnegative*), if $m_{ij} \geq 0$ for all i,j. In some cases, it will be useful to denote the (i,j)-th entry of a matrix M as $[M]_{ij}$. The Hurwitz products of two square matrices A and B are inductively defined as $A^i \sqcup\!\sqcup^0 B = A^i$, $A^0 \sqcup\!\sqcup^j B = B^j$ and, when i and j are both greater than zero, $A^i \sqcup\!\sqcup^j B = A(A^{i-1} \sqcup\!\sqcup^j B) + B(A^i \sqcup\!\sqcup^{j-1} B)$.

The characteristic polynomial of a pair of square matrices (A, B) is $\Delta_{A,B}(z_1,z_2) = \det(I - Az_1 - Bz_2)$

Given an alphabet $\Xi = \{\xi_1, \xi_2\}$, the free monoid Ξ^* is the set of all words $w = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}$, $m \in \mathbf{N}$, $\xi_{i_h} \in \Xi$. The integer m is called the length of the word w and denoted by $|w|$, while $|w|_i$ represents the number of occurencies of ξ_i in w, $i = 1,2$. For each pair of matrices $A, B \in \mathbf{C}^{n\times n}$, the map ψ defined on $\{1, \xi_1, \xi_2\}$ by the assignements $\psi(1) = I_n$, $\psi(\xi_1) = A$ and $\psi(\xi_2) = B$, uniquely extends to a monoid morphism of Ξ^* into $\mathbf{C}^{n\times n}$. The ψ-image of $w \in \Xi^*$ is denoted by $w(A, B)$.

2. SPECTRAL PROPERTIES

The dynamics of a 2D system (1.1) is essentially determined by the matrix pair (A, B). Unfortunately, the algebraic tools for studying a pair of linear transformations are not as simple and effective as those available for the investigation of a single linear transformation. In particular, the modal analysis approach to the unforced dynamics does not extend to 2D systems. Interestingly enough, however, some natural assumptions on the structure of the pair (A, B) allow to single out important classes of positive systems, whose spectral properties are easily investigated.

As a first instance, we consider *finite memory* systems, i.e. systems whose unforced state evolution goes to zero in a finite number of steps. As proved in [3] a generic (i.e. nonnecessarily positive) 2D system (1.1) is finite memory if and only if $\Delta_{A,B}(z_1,z_2) = 1$. The nonnegativity assumption leads to some penetrating characterizations, as shown in the following Proposition.

Proposition 2.1 For a pair of $n \times n$ nonnegative matrices (A, B), the following statements are equivalent:
i) $\Delta_{A,B}(z_1,z_2) = 1$; ii) $A + B$ is a nilpotent (and, a fortiori, an irreducible) matrix; iii) $A^i \sqcup\!\sqcup^j B$ is nilpotent,

for all $(i,j) \neq (0,0)$; iv) $w(A,B)$ is nilpotent, for all $w \in \Xi^* \setminus \{1\}$.

PROOF $i) \Rightarrow ii)$ Letting $z_1 = z_2 = z$ in $\Delta_{A,B}(z_1,z_2) = 1$, we get $\det(I - (A+B)z) = 1$, which implies the nilpotency of $A+B$.

$ii) \Rightarrow iii)$ For all $\nu \geq n$ we have $0 = (A+B)^\nu = \sum_{h+k=\nu} A^h \sqcup\!\sqcup^k B$. The nonnegativity assumption further implies that $A^h \sqcup\!\sqcup^k B$ is zero whenever $h+k \geq n$. Consequently, when $(i,j) \neq (0,0)$, one gets $0 \leq (A^i \sqcup\!\sqcup^j B)^n \leq A^{in} \sqcup\!\sqcup^{jn} B = 0$, which proves the nilpotency of $A^i \sqcup\!\sqcup^j B$

$iii) \Rightarrow iv)$ Let $|w|_1 = i, |w|_2 = j$. As $[w(A,B)]^n \leq (A^i \sqcup\!\sqcup^j B)^n = 0$, we see that $w(A,B)$ is nilpotent.

$iv) \Rightarrow iv)$ By a theorem of Levitzki [4], assumption (iv) corresponds to the existence of a similarity trsnsformation that reduces both A and B to upper triangular form. Clearly, the characteristic polynomial of a pair of upper triangular nilpotent matrices is 1. ■

Remark In the general case of matrices whose entries assume both positive and negative values, condition (ii) is necessary, but not sufficient, for guaranteeing the finite memory property, which depends on the nilpotency of all linear combinations $\alpha A + \beta B$, $\alpha, \beta \in \mathbf{C}$ [5]. By contrast, anyone of conditions (iii) and (iv) is sufficient, but not necessary, for the finite memory property.

A fairly complete description is also available for 2D positive systems whose characteristic polynomial factorizes into the product of a polynomial in z_1 and a polynomial in z_2 (*separable* positive systems).

Proposition 2.2 For a pair of $n \times n$ matrices $A > 0$ and $B > 0$, the following statements are equivalent:
i) $\Delta_{A,B}(z_1,z_2) = r(z_1)s(z_2)$; ii) $A^i \sqcup\!\sqcup^j B$ is nilpotent for all (i,j) with $i,j \neq 0$; iii) $w(A,B)$ is nilpotent, for all $w \in \Xi^*$ such that $|w|_i > 0$, $i = 1,2$; iv) there exists a complex valued nonsingular matrix T such that $\hat{A} = T^{-1}AT$ and $\hat{B} = T^{-1}BT$ are upper triangular matrices and $\hat{a}_{hh} \neq 0$ implies $\hat{b}_{hh} = 0$.

PROOF $i) \Rightarrow ii)$ We refer to a characterization of separability [5], which states that a pair (A,B) is separable if and only if $\mathrm{tr}(A^i \sqcup\!\sqcup^j B) = 0$, for all (i,j) with $i,j > 0$. When (A,B) is a positive pair, one gets $\mathrm{tr}[(A^i \sqcup\!\sqcup^j B)^\nu] \leq \mathrm{tr}(A^{i\nu} \sqcup\!\sqcup^{j\nu} B) = 0$, $\nu = 1,2,\ldots$ which implies (ii).

$ii) \Rightarrow iii)$ Let $|w|_1 = i \geq 1$, $|w|_2 = i \geq 1$. As $w(A,B) \leq A^i \sqcup\!\sqcup^j B$, we have $[w(A,B)]^n \leq (A^i \sqcup\!\sqcup^j B)^n = 0$.

$iii) \Rightarrow iv)$ For all pairs $w, \bar{w} \in \Xi^*$, we have that $|w|_i = |\bar{w}|_i$, $i = 1,2$, implies $\mathrm{tr}[w(A,B)] = \mathrm{tr}[\bar{w}(A,B)]$. This guarantees [6] that A and B are simultaneously triangularizable. As the trace is invariant under similarity, we obtain $\sum_{h=1}^n \hat{a}^i_{hh} \hat{b}^j_{hh} = 0$ for all $i,j > 0$. This equation is satisfied if and only if $\hat{a}_{hh} \neq 0 \Rightarrow \hat{b}_{hh} = 0$.

$iv) \Rightarrow i)$ Obvious. ■

A pair of $n \times n$ matrices (A,B) has *property L* if the eigenvalues of A and B can be ordered into two n-tuples

$$\Lambda(A) = (\lambda_1, \lambda_2, ..., \lambda_n), \ \Lambda(B) = (\mu_1, \mu_2, ..., \mu_n) \quad (2.1)$$

such that, for all α, β in $\mathbf{C}$, the spectrum of $\Lambda(\alpha A + \beta B)$ is given by $\Lambda(\alpha A + \beta B) = (\alpha\lambda_1 + \beta\mu_1, ..., \alpha\lambda_n + \beta\mu_n)$. It is not difficult to show that property L corresponds to the possibility of factorizing the characteristic polynomial into linear terms [7]. Under appropriate irreducibility assumptions, the nonnegativity of A and B allows for some precise statements concerning the coupling of their maximal eigenvalues.

Proposition 2.3 Let (A,B) be a nonnegative $n \times n$ matrix pair, endowed with property L w.r.t. the orderings (2.1), and assume $A+B$ irreducible.
Then there exists a unique index i such that $\lambda_i, \mu_i \in \mathbf{R}_+$, $\lambda_i \geq |\lambda_j|$, $\mu_i \geq |\mu_j|$, $j = 1,2,...,n$, and, for each $\alpha, \beta > 0$, $\alpha\lambda_i + \beta\mu_i$ is the maximal positive eigenvalue of the irreducible matrix $\alpha A + \beta B$.

PROOF Denoting by $\nu_1(\alpha), \nu_2(\alpha), ..., \nu_n(\alpha)$ the eigenvalues of $\alpha A + (1-\alpha)B$, property L implies that

$$\nu_j(\alpha) = \alpha\lambda_j + (1-\alpha)\mu_j, \qquad j = 1,2,...,n. \quad (2.2)$$

Moreover, for all $\alpha \in (0,1)$, the matrix $\alpha A + (1-\alpha)B$, having the same zero-pattern as $A+B$, is irreducible and hence has a simple maximal eigenvalue $\nu_{\max}(\alpha)$. We aim to prove that there exists an integer i such that for all α, $\nu_{\max}(\alpha) = \alpha\lambda_i + (1-\alpha)\mu_i$, where λ_i and μ_i are real positive eigenvalues of A and B, respectively.
Note first that the characteristic polynomial $\Delta_{A,B}(z_1,z_2) = \prod_{i=1}^n (1 - \lambda_i z_1 - \mu_i z_2)$ belongs to $\mathbf{R}[z_1,z_2]$. So, if one factor $1 - \lambda_i z_1 - \mu_i z_2$ has not real coefficients, also $1 - \bar{\lambda}_i z_1 - \bar{\mu}_i z_2$ appears in $\Delta_{A,B}$. That amounts to say that, when a nonreal pair (λ_j, μ_j) appears in (2.2), also the conjugate pair $(\bar{\lambda}_j, \bar{\mu}_j)$ does, and hence both $\nu_j(\alpha) = \alpha\lambda_j + (1-\alpha)\mu_j$ and $\nu_k(\alpha) = \alpha\bar{\lambda}_j + (1-\alpha)\bar{\mu}_j$ belong to $\Lambda(\alpha A + (1-\alpha)B)$. Moreover, $\nu_j(\alpha)$ is real if and only if $\nu_k(\alpha)$ is, and they take the same value. As $\nu_{\max}(\alpha)$, $0 < \alpha < 1$, has to be simple, it cannot coincide with any eigenvalue $\nu_j(\alpha)$ associated with a nonreal pair (λ_j, μ_j).
Therefore, an integer $j(\alpha)$ exists, possibly depending on α, such that (λ_j, μ_j) is a real pair and $\nu_{\max}(\alpha) = \nu_{j(\alpha)}(\alpha)$. Because of the linear structure of (2.2), we can determine finitely many points, $\alpha_1, \alpha_2, ..., \alpha_r$, $0 < \alpha_1 < \alpha_2 < ... < \alpha_r < 1$, with the property that the index $j(\alpha)$ remains constant on each interval $(\alpha_\mu, \alpha_{\mu+1})$, $\mu = 1,2,...,r-1$, and takes different values on different intervals. If r were greater than zero, $\nu_{\max}(\alpha_\mu)$, $\mu = 1,2,...,r$, would be a multiple eigenvalue of the irreducible matrix $\alpha A + (1-\alpha)B$, a contradiction. So r has to be zero and $j(\alpha)$ takes in $(0,1)$ a unique value i.
Next, we show that λ_i and μ_i are maximal eigenvalues of A and B. Suppose, for instance, that A possesses

a positive eigenvalue $\lambda_h > \lambda_i$. As the eigenvalues of $\alpha A + (1-\alpha)B$ are continuous functions of α, $|\nu_h(\alpha)|$ would be greater than $|\nu_i(\alpha)|$ for all values of α in a suitable neighbourhood of 1, a contradiction.

Finally, letting $\bar{\alpha} = \alpha/(\alpha+\beta)$ and $1-\bar{\alpha} = \beta/(\alpha+\beta)$, we have that $\bar{\alpha}\lambda_i + (1-\bar{\alpha})\mu_i$ is the maximal positive eigenvalue of $\bar{\alpha}A + (1-\bar{\alpha})B = \frac{1}{\alpha+\beta}(\alpha A + \beta B)$ and, consequently, $\alpha\lambda_i + (1-\alpha)\mu_i$ is the maximal positive eigenvalue of $\alpha A + \beta B$ ■

We conclude this section with some results on the existence of a common positive eigenvector $\mathbf{v}$ of A and B, which corresponds to their maximal eigenvalues.

Proposition 2.4 Let $A > 0$ and $B > 0$ be $n \times n$ commutative matrices, whose sum $A+B$ is irreducible. Then A and B have a strictly positive common eigenvector $\mathbf{v}$, which corresponds to the maximal eigenvalues r_A and r_B of A and B, respectively.

Proof Assume first that A is irreducible, and let $\mathbf{v} \gg 0$ be the eigenvector of A corresponding to the eigenvalue r_A, that is $A\mathbf{v} = r_A\mathbf{v}$. The commutativity of A and B and the assumption $B > 0$ imply $A(B\mathbf{v}) = r_A(B\mathbf{v})$ and $B\mathbf{v} > 0$ respectively. Since an irreducible matrix has exactly one eigenvector [8] in $E^n := \{\mathbf{x} \in \mathbf{R}^n_+ : \sum_{i=1}^n x_i = 1\}$, and both $\mathbf{v}$ and $B\mathbf{v}$ are positive eigenvectors of A, we have $B\mathbf{v} = \lambda\mathbf{v},\ \lambda > 0$ Consequently, $\mathbf{v}$ is a strictly positive eigenvector of B, corresponding to its maximal eigenvalue r_B, and $\lambda = r_B$.

Assume next that $A+B$ is irreducible, and let $A_\varepsilon := A + \varepsilon B, B_\varepsilon := B + \varepsilon A$, where ε is an arbitrary positive real number. As A_ε and B_ε commute and are both irreducible, the first part of the proof gives, for all $\varepsilon > 0$ $A_\varepsilon\mathbf{v}^{(\varepsilon)} = r_{A_\varepsilon}\mathbf{v}^{(\varepsilon)}, \quad B_\varepsilon\mathbf{v}^{(\varepsilon)} = r_{B_\varepsilon}\mathbf{v}^{(\varepsilon)}$ where $\mathbf{v}^{(\varepsilon)} \gg 0$ is a common eigenvector of A_ε and B_ε, uniquely determined by the condition $\mathbf{v}^{(\varepsilon)} \in E^n$, and $r_{A_\varepsilon}, r_{B_\varepsilon}$ are the spectral radii of A_ε and B_ε respectively.

Now the eigenvalues are continuous functions of the entries of the matrices. Hence $r_{A_\varepsilon} \to r_A$ and $r_{B_\varepsilon} \to r_B$ as $\varepsilon \to 0^+$. Moreover, a compactness argument shows that there exists $\mathbf{v} \in E^n$ such that $\mathbf{v}^{(\varepsilon)} \to v$, and $\mathbf{v}$ is a common eigenvector of A and B relative to r_A and r_B. To conclude the proof, it remains to show that the limiting vector $\mathbf{v}$ is strictly positive. Indeed, $(A+B)\mathbf{v} = (r_A + r_B)\mathbf{v}$ shows that $\mathbf{v}$ is a positive eigenvector of the irreducible matrix $A+B$, which implies $\mathbf{v} \gg 0$ ■

The analysis of nonnegative matrix pairs that admit a strictly positive common eigenvector essentially restricts to stochastic matrix pairs.

Proposition 2.5 Let $A > 0, B > 0, A+B$ irreducible. A and B have a common positive eigenvector if and only if their maximal eigenvalues r_A and r_B are positive and there exists a diagonal positive full rank matrix D such that $r_A^{-1}D^{-1}AD$ and $r_B^{-1}D^{-1}BD$ are row stochastic.

Proof Assume that r_A, r_B are positive, $r_A^{-1}D^{-1}AD, r_B^{-1}D^{-1}BD$ are row stochastic, and let $D = \operatorname{diag}\{d_1, d_2, \cdots, d_n\},\ d_i > 0$. Clearly $[1\ \ 1\ \ \ldots\ \ 1]^T \gg 0$ is a common eigenvector of $D^{-1}AD$ and $D^{-1}BD$, relative to r_A and r_B. Thus $[d_1\ \ d_2\ \ \cdots\ \ d_n]^T \gg 0$ is a common eigenvector of A and B, associated with their maximal eigenvalues. Conversely, suppose that A and B have a common eigenvector $\mathbf{v} = [v_1\ \ v_2\ \ \ldots\ \ v_n] > 0$. As $A+B$ is irreducible, $(A+B)\mathbf{v} = r_{A+B}\mathbf{v}$ and $\mathbf{v} > 0$ imply $\mathbf{v} \gg 0$. Moreover $A, B \neq 0$ together with $r_A\mathbf{v} = A\mathbf{v} \neq 0$ and $r_B\mathbf{v} = B\mathbf{v} \neq 0$ imply $r_A, r_B > 0$. Then [11] $r_A^{-1}A$ and $r_B^{-1}B$ simultaneously reduce to row stochastic matrices via the similarity induced by $D = \operatorname{diag}\{d_1, d_2, \cdots, d_n\}$. ■

3. CANONICAL FORMS OF MATRIX PAIRS

A pair of $n \times n$ matrices (A, B) is said to be cogredient to a pair $(\bar{A}, \bar{B})$ if there exists a permutation matrix P such that $\bar{A} = P^TAP$ and $\bar{B} = P^TBP$.

Proposition 3.1 A pair of $n \times n$ nonnegative matrices (A, B) is finite memory if and only if it is cogredient to a pair of upper triangular nonnegative nilpotent matrices.

Proof Assume first that (A, B) is finite memory. Thus $A+B$ is a nilpotent and, a fortiori, a reducible matrix. Consequently, there exists a permutation matrix P_1 such that

$$P_1^T(A+B)P_1 = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}.$$

As C_{11} and C_{22} are nilpotent, we can apply the above procedure to both diagonal blocks. By iterating this reasoning, we end up with one dimensional nilpotent diagonal blocks and, therefore, with an upper triangular matrix $P^T(A+B)P = P^TAP + P^TBP$. As P^TAP and P^TBP are nonnegative, both of them are upper triangular with zero diagonal. The converse is obvious. ■

The combinatorial structure of separable matrix pairs is extremely simple, and easily determined as a consequence of the following lemma.

Lemma 3.2 [1] Let $A > 0$ and $B > 0$ constitute a separable pair; then $A+B$ is a reducible matrix.

Proposition 3.3 A pair of $n \times n$ nonnegative matrices (A, B) is separable if and only if there exists a permutation matrix P such that P^TAP and P^TBP are conformably partitioned into block triangular matrices:

$$\begin{bmatrix} A_{11} & * & * & * \\ & A_{22} & * & * \\ & & \ddots & * \\ & & & A_{tt} \end{bmatrix}, \quad \begin{bmatrix} B_{11} & * & * & * \\ & B_{22} & * & * \\ & & \ddots & * \\ & & & B_{tt} \end{bmatrix}, \tag{3.1}$$

where $A_{ii} \neq 0$ implies $B_{ii} = 0$ and the nonzero diagonal blocks are irreducible.

PROOF Assume that A and B is a separable pair. If one of the matrices is zero, there is nothing to prove. In case A and B are both nonzero, by the previous Lemma there exists a permutation matrix P_1 s.t.

$$P_1^T A P_1 + P_1^T B P_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

where A_{ii} and B_{ii}, $i = 1, 2$, are square submatrices. As the nonnegative matrix pairs (A_{ii}, B_{ii}) are separable, we apply the same procedure to both of them. By iteration, we end up with a pair of matrices with the structure (3.1). The converse is obvious. ■

We conclude this section by investigating the combinatorial structure of a nonnegative matrix pair with property L, when one of the matrices is diagonal.

Lemma 3.4 Let M be an $n \times n$ nonnegative matrix such that $[M^r]_{ii} = ([M]_{ii})^r$, $i = 1, 2, \ldots, n$, $r = 0, 1, 2, \ldots$. Then M is cogredient to a triangular matrix

PROOF We prove first that M is reducible. If not, for any pair (i, j) with $i \neq j$ there exist integers h and k such that $[M^h]_{ij} > 0$, $[M^k]_{ji} > 0$. Consequently we have $[M^{h+k}]_{ii} \geq [M^h]_{ij}[M^k]_{ji} + ([M]_{ii})^{h+k} > ([M]_{ii})^{h+k}$, which contradicts the assumption of the lemma. Next we remark that $(P^T M P)^r = P^T M^r P$, for any permutation matrix P and for any positive integer r. This implies that the diagonal elements in $(P^T M P)^r$ and in M^r are connected by the same index permutation which connects the diagonal elements in $P^T M P$ and in M. So, we get $[(P^T M P)^r]_{ii} = [P^T M^r P]_{ii} = ([P^T M P]_{ii})^r$ for all nongative integers r and for $i = 1, 2, \ldots n$. Now we apply a cogredience transformation which reduces M to a block triangular matrix

$$P^T M P = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}$$

and notice that, by the previous remark, both M_1 and M_2 fulfill the hypothesis of the lemma. We iterate the procedure until a triangular matrix is obtained. ■

Proposition 3.5 Let $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ be a non negative matrix with $a_i \neq a_j$ if $i \neq j$, and let $B \geq 0$. The following statements are equivalent:
i) (A, B) has property L; ii) B is cogredient to a triangular matrix; iii) $\Lambda(B) = \{b_{11}, b_{22}, \ldots, b_{nn}\}$

PROOF $(i) \Rightarrow (ii)$ Property L implies [5] that there exists a suitable ordering $(\mu_1, \mu_2, \ldots, \mu_n)$ of $\Lambda(B)$ such that, for all h, $\mathrm{tr}(A^h \sqcup\!\sqcup^1 B) = \binom{h+1}{h} \sum_{i=1}^n a_i^h \mu_i$ On the other hand we have $\mathrm{tr}(A^h \sqcup\!\sqcup^1 B) = (h+1)\mathrm{tr}(A^h B) = (h+1) \sum_{i=1}^n a_i^h b_{ii}$, As a consequence, we obtain $\sum_i a_i^h (\mu_i - b_{ii}) = 0$, $h = 0, 1, \ldots, n-1$, and, taking into account that $[a_{ii}^h]$ is a nonsingular Vandermonde matrix, $\mu_i = b_{ii}$, $i = 1, 2, \ldots n$.

$(ii) \Rightarrow (iii)$ The assumption on $\Lambda(B)$ implies $\sum_{i=1}^n b_{ii}^r = \mathrm{tr}(B^r) = \sum_{i=1}^n [B^r]_{ii}$, $r = 0, 1, \ldots$ On the other hand, since B is nonnegative, we have also $b_{ii}^r \leq [B^r]_{ii}$. Hence $b_{ii}^r = [B^r]_{ii}$, $r = 0, 1, \ldots$ $1 = 1, 2, \ldots, n$ and B is cogredient to a triangular matrix by Lemma 3.4.

$(iii) \Rightarrow (i)$ Obvious. ■

Lemma 3.4 and Proposition 3.5 above generalize [9] to the case when the diagonal elements of A are nonnecessarily distinct. The proofs will be omitted for sake of brevity.

Lemma 3.4 and Prop. 3.5 generalize [9] to the case when the diagonal elements of A need not be distinct.

Lemma 3.6 Let $n = \nu_1 + \ldots + \nu_k$, and suppose that the $n \times n$ nonnegative matrix M is partitioned into blocks M_{ij} of dimension $\nu_i \times \nu_j$. If the blocks M_{ii} are irreducible and $\mathrm{tr}([M^r]_{ii}) = \mathrm{tr}((M_{ii})^r)$ $i = 1, 2, ..., k, r = 0, 1, ...,$ M is cogredient to a block-triangular matrix whose diagonal blocks coincide (except for the order) with the M_{ii}'s.

Proposition 3.7 Let $A = \mathrm{diag}\{a_1 I_{\nu_1}, \ldots, a_k I_{\nu_k}\} \geq 0$ be a block diagonal matrix, with $a_i \neq a_j$ if $i \neq j$, and let $B \geq 0$ be partitioned, conformably with the partition of A, into blocks B_{ij}. The following are equivalent:
i) (A, B) has property L; ii) $\det(zI_n - B) = \prod_{i=1}^k \det(zI_{\nu_i} - B_{ii})$; iii) there exists a permutation matrix P such that $P^T A P = \mathrm{diag}\{\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{pp}\}$, and

$$P^T B P = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \ldots & \hat{B}_{1p} \\ & \hat{B}_{22} & & \hat{B}_{2p} \\ & & \ddots & \vdots \\ & & & \hat{B}_{pp} \end{bmatrix},$$

where the $\hat{A}_{ii}$'s are scalar matrices and each $\hat{B}_{ii}$'s is a diagonal block of the Frobenius normal form of B_{jj}, for some j.

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