Multidimensional Signals in Communication and Control

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1 Introduction

Multidimensional (nD) signal processing and transmission constitute important topics in today's communication and control systems. This kind of signals is encountered, for instance, in the discretization of partial differential equations which represent industrial or environmental processes, simultaneously depending on space and time, or in image coding and processing. Many different techniques are available for the representation and the analysis of multidimensional signals, each of them suitable within a particular range of applications and for achieving certain goals.

A very powerful tool is constituted by "behavior theory", which provides a general framework for the study of the trajectories a dynamical system produces according to its evolution laws. It originated in the analysis of 1D systems, and was developed in a complete and useful form by J.C.Willems in the last two decades. In a series of landmark papers [12, 13, 14], Willems provided a thorough description of the ways a system interacts with its environment, as well as a clear conceptual apparatus for analysing and identifying the attributes a family of trajectories possibly exhibits.

A second stage in the development of behavior theory initiated by P.Rocha and J.C.Willems at the end of the eighties [8, 9], resulted in the absorption of two-dimensional (2D) signals into the theory. The analysis of 2D behaviors has led to new insights in the classical theory of 2D systems and to new investigations of Laurent polynomial operators, centering around the algebra of matrix pairs and various primeness notions for polynomial matrices.
Another major development in behavior theory is G.D.Forney's work on the behavioral approach to group systems [4]. Like the original work on minimal bases of rational spaces [3], Forney's papers find significant applications in the theory of convolutional codes. At the same time, however, they draw on duality theory, and suggest new problems on observability and memory span.

In the last few years, there has been an increasing interest in convolutional coding of multidimensional (nD) data [1, 11], motivated to large extent by the possibility of investigating code performances and properties in a behavior context. Also, multidimensional convolutional codes have been a fruitful source of problems and conjectures, both in polynomial modules algebra and in signal processing of discrete data arrays [10].

Basing on the theory of multidimensional behaviors, this paper aims to present a unifying approach to the description of nD signals which allows for concise statements and analytic solutions of many problems arising in different areas of communications and control.
Particular attention has been devoted to the support structure of multidimensional signals, and to certain elementary operations (restriction, extension and concatenation) which have a concrete meaning from the signal processing standpoint. These provide a strong link between the parity checks description of nD behaviors and the concepts of observability and extendability; indeed, the support of the parity check matrix "measures" the range of action of the system laws and provides useful bounds on the region where parity checks apply when detecting if some sequence is a legal behavior trajectory.

A point of view somewhat complementary to detection calls for an input/output analysis of the trajectories generation, and of the way the supports of the trajectories are related to the corresponding inputs. This problem appears particularly relevant when the behavior sequences have to be injectively generated, as it always happens in coding theory. Although no general statement can be made, specific assumptions on the structure of the generating matrices allow to uniformly confine the support of each input signal into a suitable extension of the support of the associated output trajectory.

The paper is organized as follows. The first part introduces the basic definitions and properties of nD finite behaviors; in particular, operations which involve only the supports allow to define the notions of (external) controllability and observability. While controllability is well-estabilished [4, 9], and in the context of finite support signals it follows from linearity and shift invariance, the observability definition we will adopt comes from duality issues, and is fully justified when a parity checks description of the behavior trajectories is adopted. Actually, as shown in Section 3, an observable behavior is characterized by a finite set of parity checks one has to apply in order to recognize its trajectories, and hence it is the kernel of a polynomial matrix operator.

Extendability is introduced in the same section as a significant strengthening of observability; the binding requirement that the sequences involved in the definition have to be behavior elements is replaced - when defining extendability - by weaker conditions on the parity checks applied to the signals. As a consequence, extendable behaviors prove to be kernels of very particular polynomial operators.

In the last part of the paper we develop the theory of input-output generation of nD behaviors, and present some relevant connections between support conditions on the input-output pairs and primeness requirements on the generator matrices.

2 Controllability and observability of finite support behaviors

Let $\mathbb{F}$ be an arbitrary field and denote by $\mathbf{z}$ the n-tuple $(z_1, z_2, ..., z_n)$, so that $\mathbb{F}[\mathbf{z}]$ and $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ are shorthand notations for the polynomial and the Laurent polynomial (*L-polynomial*) rings in the indeterminates $z_1, ..., z_n$, respectively.
For any sequence $\mathbf{w} = \{\mathbf{w}(\mathbf{h})\}_{\mathbf{h}\in\mathbb{Z}^n}$, taking values in $\mathbb{F}^p$, the *support* of $\mathbf{w}$ is the set of points where $\mathbf{w}$ is nonzero, i.e. $\mathrm{supp}(\mathbf{w}) := \{\mathbf{h} = (h_1, h_2, ..., h_n) \in \mathbb{Z}^n : \mathbf{w}(\mathbf{h}) \neq 0\}$. Also, $\mathbf{w}$ can be represented via a formal power series

$$\sum_{h_i\in\mathbb{Z}} \mathbf{w}(h_1, h_2, ..., h_n)\, z_1^{h_1} z_2^{h_2} \cdots z_n^{h_n} = \sum_{\mathbf{h}\in\mathbb{Z}^n} \mathbf{w}(\mathbf{h})\, \mathbf{z}^{\mathbf{h}},$$

where $\mathbf{h}$ stands for the n-tuple $(h_1, h_2, ..., h_n)$ and $\mathbf{z}^{\mathbf{h}}$ for the term $z_1^{h_1} z_2^{h_2} ... z_n^{h_n}$.
On the other hand, power series can be viewed as representing vectors with entries in $\mathcal{F}_\infty := \mathbb{F}^{\mathbb{Z}^n}$, thus setting a bijective map between nD sequences taking values in $\mathbb{F}^p$ and formal power series with coefficients in $\mathbb{F}^p$. This allows us to identify nD sequences with the associated power series, in particular, finite support nD signals, with L-polynomial vectors, and to denote both of them with the same symbol $\mathbf{w}$. Sometimes, mostly when a power series $\mathbf{w}$ is obtained as a Cauchy product, it will be useful to denote the coefficient of $\mathbf{z}^{\mathbf{h}}$ in $\mathbf{w}$ as $(\mathbf{w}, \mathbf{z}^{\mathbf{h}})$.

Linear operators on the sequence space are represented by appropriate matrices with elements in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$, whose primeness features have an immediate counterpart in terms of properties of the associated operators. The main primeness notions which arise in the nD context are the following:

Definition *An L-polynomial matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$, $p \geq m$, is*

- unimodular *if $p = m$ and $\det G$ is a unit in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$, i.e. $\det G = c\mathbf{z}^{\mathbf{h}}$ for some nonzero $c \in \mathbb{F}$ and some $\mathbf{h} \in \mathbb{Z}^n$;*

- right factor prime *(rFP) if in every factorization* $G = \bar{G}T$, *with* $\bar{G} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ *and* $T \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m\times m}$, T *is a unimodular matrix;*
- right zero prime *(rZP) if the ideal* I_G, *generated by its maximal order minors, is the ring* $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ *itself.*

The support of a matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ is the union of the supports of its elements.

An nD (finite) *behavior* $\mathcal{B}$ *with* p *components* is a set of finite support signals (trajectories) taking values in $\mathbb{F}^p$ and endowed with the following properties:

(L) [Linearity] *If* $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *belong to* $\mathcal{B}$, *then* $\alpha\mathbf{w}_1 + \beta\mathbf{w}_2 \in \mathcal{B}$, *for all* α, β *in* $\mathbb{F}$*;*

(SI) [Shift-Invariance] $\mathbf{w} \in \mathcal{B}$ *implies* $\mathbf{v} = \mathbf{z}^{\mathbf{h}}\mathbf{w} \in \mathcal{B}$ *for every* $\mathbf{h} \in \mathbb{Z}^n$, *i.e.* $\mathcal{B}$ *is invariant w.r.t. the shifts along the coordinate axes in* $\mathbb{Z}^n$.

As every nD behavior $\mathcal{B}$ can be viewed as an $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$-submodule of $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, which is a Noetherian module [6], $\mathcal{B}$ is finitely generated, i.e. there exists a finite set of column vectors, $\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_m$ in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, such that

$$\mathcal{B} \equiv \left\{ \sum_{i=1}^{m} \mathbf{g}_i u_i : u_i \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}] \right\} = \{\mathbf{w} = G\mathbf{u} : \mathbf{u} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m\} =: \mathrm{Im}G. \tag{2.1}$$

The L-polynomial matrix $G := \mathrm{row}\{\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_m\}$ is a *generator matrix* of $\mathcal{B}$.

$G_1 \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m_1}$ and $G_2 \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m_2}$ are generator matrices of the same behavior if and only if there exist $P_1 \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m_1\times m_2}$ and $P_2 \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m_2\times m_1}$ such that $G_1P_1 = G_2$ and $G_2P_2 = G_1$. Consequently, G_1 and G_2 have the same rank r over the field of rational functions $\mathbb{F}(\mathbf{z})$. Being an invariant w.r.t. all generator matrices of $\mathcal{B}$, r is called the *rank of* $\mathcal{B}$. It somehow represents a complexity index of the behavior, as r independent trajectories can be found in $\mathcal{B}$, while $r+1$ trajectories $(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_{r+1})$ always satisfy an autoregressive equation $\mathbf{w}_1p_1 + \mathbf{w}_2p_2 + ... + \mathbf{w}_{r+1}p_{r+1} = \mathbf{0}$, with $p_i \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ not all zero.
A behavior $\mathcal{B}$ of rank r is *free* if it admits a full column rank generator matrix, that is a generator matrix G with r columns. This amounts to say that each trajectory $\mathbf{w}$ in $\mathcal{B}$ is uniquely expressed as a linear combination $\mathbf{w} = \mathbf{g}_1u_1 + \mathbf{g}_2u_2 + \cdots + \mathbf{g}_ru_r$, $u_i \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$, of the columns of G.

The main properties of a finite behavior $\mathcal{B}$ are connected with certain elementary operations we can perform on the system trajectories. These operations essentially reduce to "paste" pieces of different trajectories into legal elements of $\mathcal{B}$, or to "cut" a set of samples out of a given trajectory, so as to obtain a new behavior sequence.

One of the pillars of Willems behavior theory is the notion of (external) controllability. For 1D controllable behaviors the past has no lasting implications about the far future [13], which means that the restriction of a 1D trajectory to $(-\infty, t]$ does not provide any information on the values the trajectory takes on $[t + \delta, +\infty)$, when $\delta > 0$ is properly chosen. In a multidimensional context the notions of "past" and "future" are quite elusive and, in many cases, unsuitable for classifying and processing the available data. What seems more reasonable, instead, is to investigate to what extent the values a trajectory $\mathbf{w}$ assumes on a subset $\mathcal{S}_1 \subset \mathbb{Z}^n$ influence the values on the subset $\mathcal{S}_2$, disjoint from $\mathcal{S}_1$, and to check if there exists a lower bound on the distance

$$d(\mathcal{S}_1, \mathcal{S}_2) := \min \left\{ \sum_{i=1}^{n} |h_i - k_i|, (h_1, h_2, ..., h_n) \in \mathcal{S}_1, (k_1, k_2, ..., k_n) \in \mathcal{S}_2 \right\}, \tag{2.2}$$

which guarantees that $\mathbf{w}|\mathcal{S}_2$, the *restriction to* $\mathcal{S}_2$ *of the sequence* $\mathbf{w}$, is independent of $\mathbf{w}|\mathcal{S}_1$. This point of view led to the following definition [8].

(C_1) [Controllability] *A finite behavior $\mathcal{B}$ is* controllable *if there exists an integer $\delta > 0$ such that, for any pair of nonempty subsets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathbb{Z}^n$, with $d(\mathcal{S}_1, \mathcal{S}_2) \geq \delta$, and any pair of trajectories $\mathbf{w}_1$ and $\mathbf{w}_2 \in \mathcal{B}$, there exists $\mathbf{v} \in \mathcal{B}$ such that*

$$\mathbf{v}|\mathcal{S}_1 = \mathbf{w}_1|\mathcal{S}_1 \qquad \text{and} \qquad \mathbf{v}|\mathcal{S}_2 = \mathbf{w}_2|\mathcal{S}_2. \tag{2.3}$$

While definition (C_1) requires to paste together different signals into a new one, the following statement refers to the possibility of finding a legal extension for every portion $\mathbf{w}|\mathcal{S}$ of a behavior trajectory $\mathbf{w}$, by adjusting the sample values in a small area surrounding $\mathcal{S}$. More precisely, once introduced for $\varepsilon \geq 0$ the *ε-extension* of the set $\mathcal{S}$

$$\mathcal{S}^\varepsilon := \{(h_1, h_2, ..., h_n) \in \mathbb{Z}^n : d((h_1, h_2, ..., h_n), \mathcal{S}) \leq \varepsilon\},$$

one can give the following definition.

(C_2) [Zero-controllability] *A finite behavior $\mathcal{B}$ is* zero-controllable *if there exists an integer $\varepsilon > 0$ such that, for any nonempty set $\mathcal{S}$ of $\mathbb{Z}^n$ and any $\mathbf{w} \in \mathcal{B}$, there exists $\mathbf{v} \in \mathcal{B}$ satisfying*

$$\mathbf{v}|\mathcal{S} = \mathbf{w}|\mathcal{S} \qquad \text{and} \qquad \mathrm{supp}(\mathbf{v}) \subseteq \mathcal{S}^\varepsilon. \tag{2.4}$$

Properties (C_1) and (C_2) make sense both for finite and infinite support behaviors, and the proof of their equivalence given below holds for both of them. However, while conditions (C_1) and (C_2) are always met by a finite behavior $\mathcal{B}$, and essentially follow from the module structure $\mathcal{B}$ is endowed with, for an infinite behavior controllability constitutes an additional constraint w.r.t. linearity and shift invariance [8, 9].

Proposition 2.1 *Controllability and zero controllability are equivalent.*

PROOF (C_1) $\Rightarrow$ (C_2) Assume that $\mathcal{B}$ meets condition (C_1). Given $\mathbf{w} \in \mathcal{B}$ and $\mathcal{S} \subset \mathbb{Z}^n$, take in ($C_1$) $\mathbf{w}_1 = \mathbf{w}$, $\mathbf{w}_2 = \mathbf{0}$, $\mathcal{S}_1 = \mathcal{S}$, $\mathcal{S}_2 = \mathcal{C}\mathcal{S}^\delta$, where $\mathcal{C}\mathcal{S}$ denotes the complementary set of $\mathcal{S}$. Then the trajectory $\mathbf{v}$ which fulfills (2.3), satisfies (2.4) with $\varepsilon = \delta$.
(C_2) $\Rightarrow$ (C_1) Assume that $\mathcal{B}$ satisfies condition (C_2). Given $\mathbf{w}_1$ and $\mathbf{w}_2 \in \mathcal{B}$ and $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{Z}^n$, with $d(\mathcal{S}_1, \mathcal{S}_2) > \varepsilon$, by ($C_2$) there exist $\mathbf{v}_1$ and $\mathbf{v}_2 \in \mathcal{B}$ such that

$$\mathbf{v}_i|\mathcal{S}_i = \mathbf{w}_i|\mathcal{S}_i, \qquad \mathrm{supp}(\mathbf{v}_i) \subset \mathcal{S}_i^\varepsilon, \qquad i = 1, 2.$$

Thus $\mathbf{v} := \mathbf{v}_1 + \mathbf{v}_2 \in \mathcal{B}$ satisfies $\mathbf{v}|\mathcal{S}_i = \mathbf{w}_i|\mathcal{S}_i$, $i = 1, 2$, and (C_1) holds for $\delta = \varepsilon$. ■

Proposition 2.2 *Any finite behavior $\mathcal{B}$ is controllable.*

PROOF Suppose that $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$ is a generator matrix of $\mathcal{B}$ and let η be a positive integer such that $B(\mathbf{0}, \eta)$, the ball of radius η and center in the origin, includes supp(G). Consider any set $\mathcal{S} \subset \mathbb{Z}^n$ and $\mathbf{w} = G\mathbf{u} \in \mathcal{B}$. If $\bar{\mathbf{u}}$ is the sequence which coincides with $\mathbf{u}$ on $\mathcal{S}$ and is zero elsewhere, the trajectory $\mathbf{v} := G\bar{\mathbf{u}}$ satisfies $\mathbf{v}|\mathcal{S} = \mathbf{w}|\mathcal{S}$, and has support which does not exceed $\mathcal{S}^{2\eta}$. So (C_2) is met with $\varepsilon = 2\eta$. ■

Given two disjoint sets $\mathcal{S}_1$ and $\mathcal{S}_2$ which are far enough apart, controllability expresses the possibility of steering any behavioral sequence, known in $\mathcal{S}_1$, into another element of $\mathcal{B}$ assigned on $\mathcal{S}_2$, meanwhile producing a legal trajectory. Like controllability, also observability will be introduced without reference to the concept of state, according to some recent works of Forney et al. [4, 7]. Observability formalizes the possibility of pasting into a legal sequence any pair of trajectories that take the same values on a sufficiently large subset of $\mathbb{Z}^n$. This is equivalent to say that, however chosen a sequence $\mathbf{w} \in \mathcal{B}$ and a subset $\mathcal{S} \subset \mathbb{Z}^n$, the possible extensions of $\mathbf{w}|\mathcal{S}$ only depend on the values of $\mathbf{w}$ on a boundary region of $\mathcal{S}$.

Under this viewpoint, observability endows a behavior with a "separation property" that allows to take into account only a small amount of data in order to extend a portion of behavior sequence. Furthermore, once we think of the samples in $\mathcal{S}$ as the information about the past dynamics of the system, observability enables us to design the "future" evolution by considering only the most "recent" data (those on the boundary), somehow reminding of the notion of state.

(O_1) [Observability] *A finite behavior $\mathcal{B}$ is* observable *if there exists an integer $\delta > 0$ such that, for any pair of nonempty subsets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathbb{Z}^n$, with $d(\mathcal{S}_1, \mathcal{S}_2) \geq \delta$, and any pair of trajectories $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathcal{B}$, satisfying $\mathbf{w}_1|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathbf{w}_2|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2)$, the trajectory*

$$\mathbf{v}(\mathbf{h}) = \begin{cases} \mathbf{w}_1(\mathbf{h}) & \mathbf{h} \in \mathcal{S}_1 \\ \mathbf{w}_1(\mathbf{h}) = \mathbf{w}_2(\mathbf{h}) & \mathbf{h} \in \mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) \\ \mathbf{w}_2(\mathbf{h}) & \mathbf{h} \in \mathcal{S}_2 \end{cases} \tag{2.5}$$

is an element of $\mathcal{B}$.

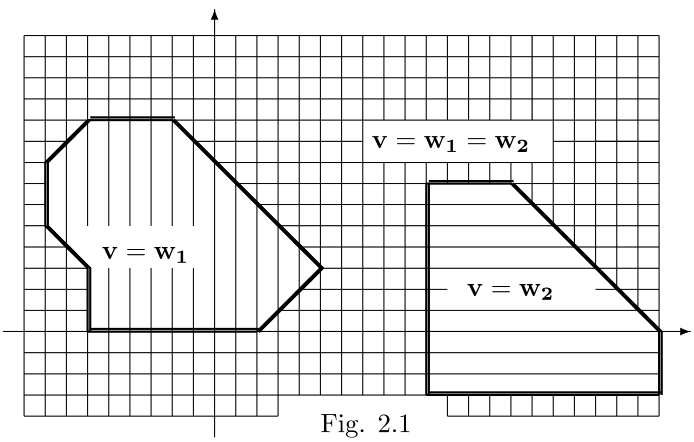

Fig. 2.1

Observability can be equivalently restated as follows: if the support of a behavior sequence $\mathbf{w}$ can be partitioned into a pair of disjoint subsets, which are far enough apart, the restrictions of $\mathbf{w}$ to each subset represent legal trajectories.

(O_2) [Zero-observability] *A finite behavior $\mathcal{B}$ is* zero-observable *if there exists an integer $\varepsilon > 0$ such that for any $\mathbf{w} \in \mathcal{B}$ satisfying $\mathbf{w}|(\mathcal{S}^\varepsilon \setminus \mathcal{S}) = \mathbf{0}$, $\mathcal{S}$ a nonempty set in $\mathbb{Z}^n$, the sequence*

$$\mathbf{v}(\mathbf{h}) = \begin{cases} \mathbf{w}(\mathbf{h}) & \mathbf{h} \in \mathcal{S} \\ 0 & \textit{elsewhere} \end{cases} \tag{2.6}$$

belongs to $\mathcal{B}$.

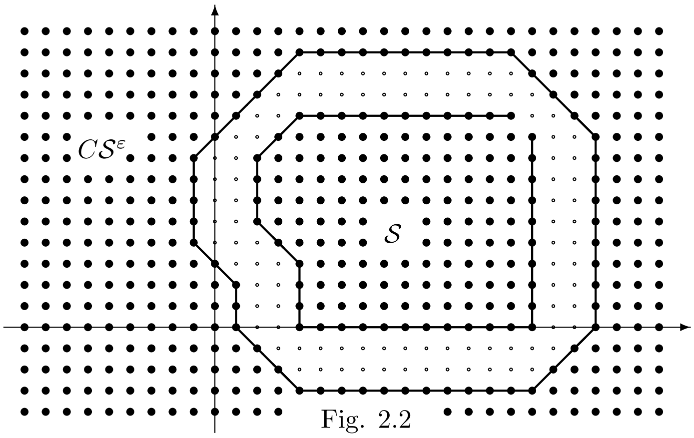

Fig. 2.2

Proposition 2.3 *Observability and zero observability are equivalent.*

PROOF $(\mathrm{O}_1) \Rightarrow (\mathrm{O}_2)$ Assume that $\mathcal{B}$ fulfills condition (O_1). Given $\mathcal{S} \subset \mathbb{Z}^n$ and $\mathbf{w} \in \mathcal{B}$ such that $\mathbf{w}|(\mathcal{S}^\delta \setminus \mathcal{S}) = \mathbf{0}$, take in (O_1) $\mathbf{w}_1 = \mathbf{w}$, $\mathbf{w}_2 = \mathbf{0}$, $\mathcal{S}_1 = \mathcal{S}$, $\mathcal{S}_2 = \mathcal{C}\mathcal{S}^\delta$. The trajectory $\mathbf{v} \in \mathcal{B}$ satisfying (2.5), satisfies also (2.6) with $\varepsilon = \delta$.

$(\mathrm{O}_2) \Rightarrow (\mathrm{O}_1)$ Assume that $\mathcal{B}$ fulfills condition (O_2). Given $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{Z}^n$, with $d(\mathcal{S}_1, \mathcal{S}_2) > \varepsilon$, and $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathcal{B}$ satisfying $\mathbf{w}_1|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathbf{w}_2|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2)$, the sequence $\mathbf{w}_1 - \mathbf{w}_2 \in \mathcal{B}$ satisfies $(\mathbf{w}_1 - \mathbf{w}_2)|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathbf{0}$. As a consequence, the signal $\mathbf{w}$ given by

$$\mathbf{w}(\mathbf{h}) = \begin{cases} \mathbf{w}_1(\mathbf{h}) - \mathbf{w}_2(\mathbf{h}) & \mathbf{h} \in \mathcal{S}_1 \\ 0 & \text{elsewhere} \end{cases}$$

is in $\mathcal{B}$, and $\mathbf{v} := \mathbf{w} + \mathbf{w}_2 \in \mathcal{B}$ fulfills (2.5). So, (O_1) holds for $\delta = \varepsilon$. ■

3 Trajectories recognition and error detection

Underlying the definition of controllability is the idea of driving a portion of trajectory into another one, provided that there is room enough for adjustements. In rough terms, the objective one has in mind is that of manipulating the control variables in order to make the system behave in $\mathcal{S}_2$ in a more desirable manner than it is expected by watching the system trajectory on $\mathcal{S}_1$. So, controllability is naturally connected with the generation of $\mathcal{B}$ as the image of some matrix G, acting on the input space.

Observability is somehow related with the "dual" issue of recognizing whether a given sequence $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ is an element of $\mathcal{B}$. This problem, that tipically arises in fault detection and convolutional encoding contexts, can be managed by resorting to a linear filter (residual generator or syndrome former) that produces an identically zero output signal when the input is an admissible trajectory of $\mathcal{B}$. From a mathematical point of view, this requires to find a set of sequences (parity checks) endowed with the property that their convolution with the elements of $\mathcal{B}$ is zero.

So, for a given behavior $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, a (finite) *parity check* is a column vector $\mathbf{s} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ that satisfies $\mathbf{s}^T\mathbf{w} = \mathbf{0}$, for all $\mathbf{w} \in \mathcal{B}$. The set $\mathcal{B}^\perp$ of all finite parity checks of $\mathcal{B}$ is the *orthogonal behavior*, and as a submodule of $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, it is generated by the columns of some matrix $H \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times q}$, that is

$$\mathcal{B}^\perp = \{\mathbf{s} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p : \mathbf{s} = H\mathbf{x}, \mathbf{x} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^q\}. \tag{3.1}$$

Condition $\mathbf{s}^T\mathbf{w} = \mathbf{0}$, $\forall\ \mathbf{s} \in \mathcal{B}^\perp$, however, needs not imply $\mathbf{w} \in \mathcal{B}$. In general

$$\mathcal{B}^{\perp\perp} := \{\mathbf{w} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p : \mathbf{s}^T\mathbf{w} = \mathbf{0}, \forall\ \mathbf{s} \in \mathcal{B}^\perp\} \tag{3.2}$$

properly includes $\mathcal{B}$, and is the set of all L-polynomial vectors obtained by combining the columns of G over the field of rational functions $\mathbb{F}(\mathbf{z})$. It is clear that $\mathcal{B}$ can be identified via a finite set of parity checks if and only if $\mathcal{B} = \mathcal{B}^{\perp\perp}$ or, equivalently,

$$\mathcal{B} \equiv \ker H^T := \{\mathbf{w} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p : H^T\mathbf{w} = \mathbf{0}\}. \tag{3.3}$$

In this setting, observability finds a somewhat more substantial interpretation. Actually, if $\mathcal{B} = \ker H^T$, the restriction of a trajectory to some set $\mathcal{S}$ still provides a legal signal whenever the distance between $\mathcal{S}$ and the remaining support of the trajectory exceeds the range of action of the parity check matrix H.

Proposition 3.1 below shows that kernel representations exactly correspond, as it can be expected, to observable behaviors, and makes it clear that observability induces further constraints on the structure of $\mathcal{B}$, in addition to linearity and shift invariance.

Proposition 3.1 *A behavior $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ is observable if and only if there exist an integer $h > 0$ and an L-polynomial matrix $H^T \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{h\times p}$ such that $\mathcal{B} = \ker H^T$.*

PROOF Assume that $\mathcal{B} = \mathrm{Im} G$, $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$, is an observable behavior, and let $\mathcal{B}^{\perp} = \mathrm{Im}\, H$, $H \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times q}$, denote the orthogonal behavior introduced in (3.1). We will show that $\mathcal{B} \equiv \ker H^T$. Since $H^T G = \mathbf{0}$, it is clear that $\ker H^T \supseteq \mathcal{B}$. To prove the converse, express $\mathbf{w} \in \ker H^T$ as $\mathbf{w} = G\mathbf{n}/d(\mathbf{z})$, $d \in \mathbb{F}[\mathbf{z}], \mathbf{n} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m\times 1}$. For every integer $\rho > 0$ there is a suitable polynomial $p(\mathbf{z})$ [2] such that $p(\mathbf{z})d(\mathbf{z}) \in \mathbb{F}[z_1^\rho, \ldots, , z_n^\rho]$. If property ($O_2$) holds w.r.t. $\varepsilon > 0$, and $r > 0$ is an integer such that $\mathrm{supp}(\mathbf{w}) \subseteq B(\mathbf{0}, r)$, we choose $\rho > 2r + \varepsilon$. So, the behavior sequence $p(\mathbf{z})d(\mathbf{z})\mathbf{w} = G\mathbf{n}p(\mathbf{z})$ can be written as

$$\sum_{i_1, i_2, \ldots, i_n} c_{i_1, i_2, \ldots, i_n} z_1^{\rho i_1} z_2^{\rho i_2} \cdots z_n^{\rho i_n}\ \mathbf{w},$$

and thus is the sum of disjoint shifted copies of $\mathbf{w}$, and the distance between two arbitrary copies exceeds ε. So, by (O_2), each copy of $\mathbf{w}$, and hence $\mathbf{w}$ itself, is in $\mathcal{B}$.

Conversely, let $\mathcal{B} = \ker H^T$, and set $\varepsilon = 2s$, with $s > 0$ an integer s.t. $B(\mathbf{0}, s) \supseteq \mathrm{supp}(H^T)$. If $\mathcal{S}$ is a subset of $\mathbb{Z}^n$ and $\mathbf{w} \in \mathcal{B}$ satisfies $\mathbf{w}|(\mathcal{S}^\varepsilon \setminus \mathcal{S}) = \mathbf{0}$, the sequence

$$\mathbf{v}(\mathbf{h}) = \begin{cases} \mathbf{w}(\mathbf{h}) & \mathbf{h} \in \mathcal{S} \\ 0 & \text{elsewhere} \end{cases}$$

is in $\ker H^T$ and hence in $\mathcal{B}$. Consequently, $\mathcal{B}$ is zero-observable. ■

The kernel description given in Proposition 3.1 leads to new insights into the internal structure of an observable behavior. Observability, indeed, expresses a sort of "localization" of the system laws or, equivalently, the existence of a bound on the size of all windows (in $\mathbb{Z}^n$) we have to look at when deciding whether a signal belongs to $\mathcal{B}$. Denoting by $\mathcal{B}|\mathcal{S} := \{\mathbf{w}|\mathcal{S} : \mathbf{w} \in \mathcal{B}\}$ the set of all restrictions to $\mathcal{S}$ of behavior trajectories, the above localization property finds a formal statement as follows:

(O_3) [Local-detectability] *A finite behavior $\mathcal{B}$ is* locally-detectable *if there is an integer $\nu > 0$ such that every signal $\mathbf{w}$ satisfying $\mathbf{w}|\mathcal{S} \in \mathcal{B}|\mathcal{S}$ for every $\mathcal{S} \subset \mathbb{Z}^n$ with $\mathrm{diam}(\mathcal{S}) \leq \nu$, is in $\mathcal{B}$.*

Proposition 3.2 *Local detectability and observability are equivalent.*

PROOF Assume that $\mathcal{B}$ satisfies (O_3) for a certain $\nu > 0$. Given $\mathcal{S} \subset \mathbb{Z}^n$ and $\mathbf{w} \in \mathcal{B}$ such that $\mathbf{w}|(\mathcal{S}^\nu \setminus \mathcal{S}) = \mathbf{0}$, define $\mathbf{v}$ as follows

$$\mathbf{v}(\mathbf{h}) = \begin{cases} \mathbf{w}(\mathbf{h}) & \mathbf{h} \in \mathcal{S}^\nu \\ 0 & \text{elsewhere.} \end{cases} \tag{3.4}$$

Consider any window $\mathcal{W}$, with diam$(\mathcal{W}) \le \nu$. If $\mathcal{W}$ is included in $\mathcal{S}^\nu$, then $\mathbf{v}|\mathcal{W} = \mathbf{w}|\mathcal{W} \in \mathcal{B}|\mathcal{W}$, otherwise we have $\mathcal{W} \cap \mathcal{S} = \emptyset$, and therefore

$$\mathbf{v}|\mathcal{W} = \mathbf{0}|\mathcal{W} \in \mathcal{B}|\mathcal{W}. \tag{3.5}$$

So, by (O_3), $\mathbf{v}$ is a legal trajectory, and (O_2) holds for $\varepsilon = \nu$.

Conversely, assume that $\mathcal{B}$ is observable. By Proposition 3.1, there exists an L-polynomial matrix $H \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times q}$ such that $\mathcal{B} = \ker H^T$. Let $\nu > 0$ be an integer such that supp$(H^T) \subseteq B(\mathbf{0}, \nu)$, and suppose that $\mathbf{v}$ is any signal satisfying $\mathbf{v}|\mathcal{S} \in \mathcal{B}|\mathcal{S}$ for every $\mathcal{S} \subset \mathbb{Z}^n$ with diam$(\mathcal{S}) \le 2\nu$. If $\bar{\mathcal{S}} := -$supp(H^T), the computation of the coefficient of $\mathbf{z}^\mathbf{k}$ in $H^T\mathbf{v}$ involves only samples of $\mathbf{v}$ indexed in

$$\mathbf{k} + \bar{\mathcal{S}} := \{\mathbf{h} \in \mathbb{Z}^n : \mathbf{h} - \mathbf{k} \in \bar{\mathcal{S}}\} = -\text{supp}(\mathbf{z}^\mathbf{k} H^T). \tag{3.6}$$

On the other hand, since diam$(\mathbf{k}+\bar{\mathcal{S}}) \le 2\nu$, there exists $\mathbf{w}_\mathbf{k} \in \mathcal{B}$ which satisfies $\mathbf{v}|(\mathbf{k}+\bar{\mathcal{S}}) = \mathbf{w}_\mathbf{k}|(\mathbf{k}+\bar{\mathcal{S}})$, and this result holds for every $\mathbf{k} \in \mathbb{Z}^n$. So, the coefficient of $\mathbf{z}^\mathbf{k}$ in $H^T\mathbf{v}$ is the same as in $H^T\mathbf{w}_\mathbf{k} \equiv \mathbf{0}$, and hence $\mathbf{v} \in \ker H^T = \mathcal{B}$. ■

We may intuitively view Proposition 3.1 as claiming that the trajectories of an observable behavior are recognized by a finite family of parity checks, which involve only a finite number of samples at every step of the testing procedure. When no a priori information on the support of a trajectory is available, however, a positive outcome of the parity checks, performed on some finite window $\mathcal{S}$, does not guarantee that a behavior sequence can be found interpolating the available data on $\mathcal{S}$. So, in general, the checking procedure should be extended to the whole space $\mathbb{Z}^n$.

A noteworthy exception is represented by the case when $\mathcal{S}$ is surrounded by a sufficiently large boundary region where the signal is zero. If so, extending the data out of $\mathcal{S}$ via the identically zero sequence leads to a signal which satisfies the parity checks all over $\mathbb{Z}^n$. Clearly, it would be highly desirable if the extension into a legal trajectory could be accomplished without any particular assumption on the data values in the boundary region. A thorough discussion of this problem is based on the definition of what we precisely mean by " satisfying the parity checks" on a set $\mathcal{S} \subset \mathbb{Z}^n$.

Definition *Let $\mathcal{B} = \ker H^T$ be an observable behavior. A sequence $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$* satisfies the parity checks of $\mathcal{B}$ in $\mathbf{h} \in \mathbb{Z}^n$ *if*

$$\left(H^T\mathbf{v}, \mathbf{z}^\mathbf{i}\right) = 0, \quad \forall\, \mathbf{i} \in \mathbf{h} + \text{supp}(H^T), \tag{3.7}$$

where $\mathbf{h} + \text{supp}(H^T) := \{\mathbf{h} + \mathbf{j} : \mathbf{j} \in \text{supp}(H^T)\}$. More generally, if $\mathcal{S}$ is any subset of $\mathbb{Z}^n$, $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}$ on $\mathcal{S}$ if it satisfies them in each point of $\mathcal{S}$, i.e.

$$\left(H^T\mathbf{v}, \mathbf{z}^\mathbf{i}\right) = 0, \qquad \forall\, \mathbf{i} \in \mathcal{S} + \text{supp}(H^T) := \bigcup_{\mathbf{h}\in\mathcal{S}} \left(\mathbf{h} + \text{supp}(H^T)\right). \tag{3.8}$$

Letting $H^T := \sum_\mathbf{j} H_\mathbf{j}^T \mathbf{z}^\mathbf{j}$, (3.7) reduces to the following system of linear equations

$$\sum_{\mathbf{j}\,\in\,\text{supp}(H^T)} H_\mathbf{j}^T \mathbf{v}(\mathbf{i}-\mathbf{j}) = 0, \quad \forall\, \mathbf{i} \in \mathbf{h} + \text{supp}(H^T), \tag{3.9}$$

and hence to the system of all difference equations which involve the sample $\mathbf{v}(\mathbf{h})$.
Analogously, $\mathbf{v}$ meets condition (3.8) if all difference equations involving the samples $\mathbf{v}(\mathbf{h})$, with $\mathbf{h}$ in $\mathcal{S}$, are satisfied. Fig.3.1 below describes the two-dimensional case; each dashed polygon intersecting $\mathcal{S}$ represents the coordinates $(i_1 - j_1, i_2 - j_2)$ of the samples which appear in a system like (3.9). As it is suggested by Fig. 3.1, and clearly implied by the convolutional nature of the system laws expressed by condition $H^T\mathbf{v} = \mathbf{0}$, knowing the data on a finite window $\mathcal{W}$ allows to check the signal only on those

subsets $\mathcal{S}$ of $\mathcal{W}$ satisfying the inclusion $\mathcal{S}^\nu \subseteq \mathcal{W}$, $\nu > 0$ being an integer selected according to the size of the support of H^T.

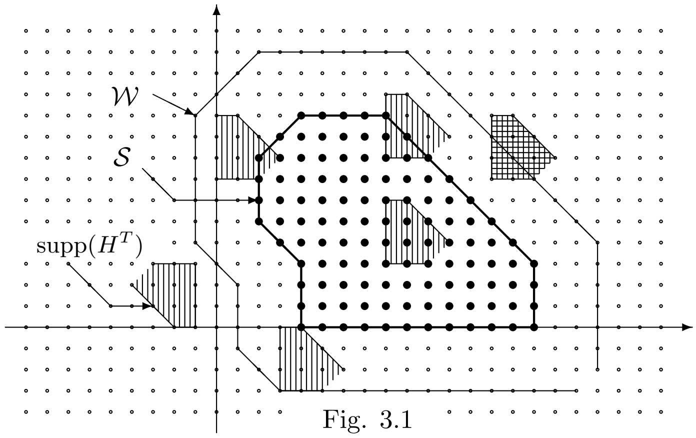

Fig. 3.1

Once the parity checks have been successfully performed on a sequence $\mathbf{v}$ in a subset $\mathcal{S}$ which fulfills the above inclusion, the natural question arises whether the data on $\mathcal{S}$ can be extended into a legal trajectory, namely if there exists a behavior signal that fits on $\mathcal{S}$ the available data. In general, even under the observability assumption, the answer is negative. When the hypotheses on $\mathcal{B}$ are properly strengthened, however, an integer $\varepsilon > 0$ can be found, such that a positive check on $\mathcal{S}^\varepsilon$ guarantees the existence of some $\mathbf{w} \in \mathcal{B}$ which coincides with $\mathbf{v}$ in $\mathcal{S}$. Note that the amount of data we need may far exceed the part of them we interpolate. Actually, checking $\mathbf{v}$ on $\mathcal{S}^\varepsilon$ requires to know the samples of $\mathbf{v}$ on a superset, say $\mathcal{S}^{\nu+\varepsilon}$, of $\mathcal{S}^\varepsilon$, whereas the data we are able to fit are those belonging to the set $\mathcal{S}$. The formal definition of this property is the following.

($\mathbf{E}_1$) [Extendability] *An observable behavior* $\mathcal{B} = \ker H^T$ *is* extendable *if there is an integer* $\varepsilon > 0$ *such that, for every subset* $\mathcal{S} \subset \mathbb{Z}^n$ *and every* $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$*, which satisfies on* $\mathcal{S}^\varepsilon$ *the parity checks of* $\mathcal{B}$*, a trajectory* $\mathbf{w} \in \mathcal{B}$ *can be found s.t.* $\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S}$.

An alternative definition of extendability, provided by (E_2) below, refers to pairs of sequences and pairs of sets. It clarifies in which sense we can view extendability as a strengthening of controllability: indeed, the assumption that the sequences in (C_1) belong to $\mathcal{B}$ is replaced in (E_2) by the weaker condition that they locally fulfill the parity checks. The proof of (E_1) $\Leftrightarrow$ (E_2) strictly reminds that of Proposition 2.3.

($\mathbf{E}_2$) [Twin-extendability] *An observable behavior* $\mathcal{B} = \ker H^T$ *is* twin-extendable *if there exists an integer* $\delta > 0$ *such that, for every pair of sets* $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{Z}^n$ *and every pair of signals* $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$*, which satisfy the parity checks of* $\mathcal{B}$ *on* $\mathcal{S}_1^\delta$ *and* $\mathcal{S}_2^\delta$*, respectively,* $\mathbf{w} \in \mathcal{B}$ *can be found such that* $\mathbf{w}|\mathcal{S}_1 = \mathbf{v}_1|\mathcal{S}_1$, and $\mathbf{w}|\mathcal{S}_2 = \mathbf{v}_2|\mathcal{S}_2$.

Extendable behaviors turn out to be described by ZP parity check matrices. As a consequence, properties (E_1) and (E_2) constitute stronger versions of observability.

Proposition 3.3 [2] *A finite behavior* $\mathcal{B}$ *with* p *components is extendable if and only if* $\mathcal{B} = \ker H^T$*, for some left zero-prime* (ℓ*ZP*) *matrix* H^T. ∎

4 Signals generation

The analysis we carried out in the previous sections essentially focused on the properties of behavior trajectories, without concern for the way they are generated. Once a behavior $\mathcal{B}$ is represented via a finite set of generators $\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_m$, however, it is natural to look at $G := [\mathbf{g}_1\ \mathbf{g}_2\ ...\ \mathbf{g}_m]$ as a transfer matrix, and hence to consider $\mathcal{B}$ as the image of an input-output map acting on $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m$. This point of view seems particularly appropriate when $\mathcal{B}$ is a convolutional code, as it is customary to regard it as the result of an encoding process, and, consequently, its trajectories (codewords) as the outputs of a dynamical encoder. In a wider context, the trajectories of $\mathcal{B}$ are obtained either from certain processing operations applied to multidimensional data, or from different transformations (desired or not) performed on the original signal. In both cases the analysis of the algebraic properties of the generator matrices makes it possible a detailed knowledge of the behavior structure.

When an input/output description is adopted, it is often imperative to associate trajectories of $\mathcal{B}$ and input sequences bijectively. In data transmission the meaning of this requirement is clear, as input signals represent information messages to be retrieved from the received codewords, and an unambiguous decision at the decoding stage is possible when each codeword encodes a unique information sequence. This amounts to say that the *encoder* G induces a 1-1 map.
If $\mathcal{B}$ is a free module, every full column rank generator matrix G has (possibly infinitely many) rational left inverses G^{-1}. Each of them, when applied to a behavior trajectory $\mathbf{w} = G\mathbf{u}$, allows to uniquely retrieve the (finite) input sequence $\mathbf{u}$. However, when G^{-1} is applied to a finite support sequence $\mathbf{v} \notin \mathcal{B}$, coming, for instance, from a noisy measurement of $\mathbf{w}$, we may obtain an infinite support sequence, which differs from $\mathbf{u}$ in infinitely many points. Clearly, this drawback can be avoided when G^{-1} is an L-polynomial matrix, which requires G to be a rZP L-polynomial matrix.

Also, when resorting to a right zero-prime generator matrix G, a uniform bound can be found on the support of the input sequences which correspond to behavior trajectories. Actually, if G^{-1} is an L-polynomial inverse of G, $\mathbf{w} \in \mathcal{B}$ is generated by the input signal $\mathbf{u} = G^{-1}\mathbf{w}$ whose support cannot exceed "too much" that of $\mathbf{w}$. This feature, we will refer to as *wrapping input property*, is quite appealing, as the mere recognition of the support of a trajectory allows to derive a uniformly tight bound on the support of the corresponding input sequence.

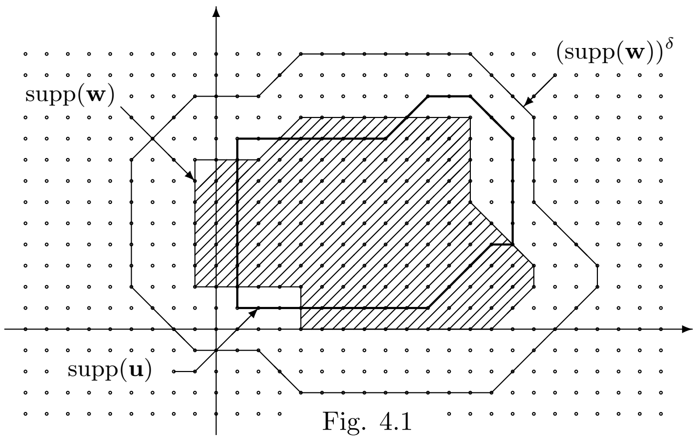

Fig. 4.1

(WI) [Wrapping input property] *A finite behavior $\mathcal{B}$ has the* wrapping input property *if there exist a full column rank generator matrix G and a positive integer δ such that $\mathbf{w} = G\mathbf{u}$ implies*

$$\text{supp}(\mathbf{u}) \subseteq \big(\text{supp}(\mathbf{w})\big)^\delta. \tag{4.1}$$

Property (WI) does not depend on the particular generator matrix of $\mathcal{B}$, provided that it has full column rank. Furthermore, when noninjective generator matrices are considered, and the uniqueness of the input sequence producing a given trajectory is lost, a particular input can be found, however, whose support satisfies (4.1).

Proposition 4.1 *Assume that $\mathcal{B}$ has the (WI) property for some full column rank matrix G and some integer $\delta > 0$. Then for every generator matrix $\bar{G} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times q}$ an integer $\bar{\delta} > 0$ can be found s.t. each trajectory $\mathbf{w} \in \mathcal{B}$ can be expressed as $\mathbf{w} = \bar{G}\bar{\mathbf{u}}$ for some input $\bar{\mathbf{u}}$ with* $\mathrm{supp}(\bar{\mathbf{u}}) \subseteq \big(\mathrm{supp}(\mathbf{w})\big)^{\bar{\delta}}$.

Proof Since G and $\bar{G}$ are generator matrices of the same behavior, there exists a full column rank L-polynomial matrix Q, such that $G = \bar{G}Q$. Let τ be the radius of a ball, with center in the origin, including $\mathrm{supp}(Q)$, and consider $\mathbf{w} \in \mathcal{B}$. By property (WI), there is $\mathbf{u}$ such that $\mathbf{w} = G\mathbf{u}$ and $\mathrm{supp}(\mathbf{u}) \subseteq \big(\mathrm{supp}(\mathbf{w})\big)^{\delta}$. So, $\bar{\mathbf{u}} := Q\mathbf{u}$ satisfies $\mathbf{w} = G\mathbf{u} = \bar{G}Q\mathbf{u} = G\bar{\mathbf{u}}$, and $\mathrm{supp}(\bar{\mathbf{u}}) = \mathrm{supp}(Q\mathbf{u}) \subseteq \big(\mathrm{supp}(\mathbf{u})\big)^{\tau} \subseteq \big(\mathrm{supp}(\mathbf{w})\big)^{\tau+\delta}$. Consequently, the proposition holds for $\bar{\delta} = \tau + \delta$. ■

Interestingly enough, the zero primeness of G is not only sufficient but also necessary for property (WI). So, free behaviors satisfying property (WI) can be identified with behaviors that are generated by ℓZP matrices, and hence are extendable.

Proposition 4.2 [2] *A finite behavior $\mathcal{B}$ has the (WI) property if and only if it admits a right zero-prime generator matrix.*

5 Conclusions

In this paper several internal properties of finite support n-D behaviors have been investigated, and connected with the algebraic structure of the corresponding generator matrices.

Further researches, aiming to clarify certain connections with infinite support behaviors, are currently in progress. Although mathematical tools borrowed from duality theory have already proved to be useful in convolutional coding theory [1, 11], yet an nD extension is far from trivial.

Looking to the future, a satisfactory interpretation of polynomial matrix properties, like factor primeness and column independence, in terms of intrinsic features of the system trajectories, would constitute an important step for subsequent developments of the theory. On the other hand, various topics connected with the specific structure of the ground field remain rather unexplored: in particular, the role played by the field characteristic, and the periodical patterns possibly induced by a finiteness assumption on $\mathbb{F}$ could eventually lead to a better understanding of the structure of nD convolutional codes.

References

[1] E.Fornasini, M.E.Valcher *Algebraic aspects of 2D convolutional codes*, IEEE Trans.Inf.Th., vol.IT-40, no.4, pp.1068-1082, 1994

[2] E.Fornasini, M.E.Valcher *Multidimensional systems with finite support behaviors: signals generation and detection*, submitted, 1995

[3] G.D.Forney *Minimal bases of rational vector spaces, with applications to multivariable linear systems*, SIAM J. of Control, vol.13, n.3, pp.493-521, 1975

[4] G.D.Forney, M.D.Trott *The dynamics of group codes: dual group codes and systems*, submitted, 1994

[5] F.R.Gantmacher *Theory of Matrices*, Chelsea Publishing Co., New York, 1959

[6] S.Lang *Algebra*, Addison-Wesley, Menlo Park, CA, 1993

[7] H.A.Loeliger, G.D.Forney, T.Mittelholzer, M.D.Trott *Minimality and observability of group systems*, to appear, 1994

[8] P.Rocha *Structure and Representation of 2D Systems*, Ph.D.Thesis, Rijksuniversiteit Groningen, 1990

[9] P.Rocha, J.C.Willems *Controllability of 2D systems*, IEEE Trans. Aut. Contr., vol. AC 36, pp.413-23, 1991

[10] S.Sakata *On determining the independent point set for doubly periodic arrays*, IEEE Trans.Inf.Th., vol.IT-27, no.5, pp.556-565, 1981

[11] M.E.Valcher, E.Fornasini *On 2D finite support convolutional codes: an algebraic approach*, Multid. Sys. Sign.Process., 5, pp.231-43, 1994

[12] J.C.Willems *From time series to linear systems, part I: finite dimensional linear time invariant systems*, Automatica, vol.22, pp.561-580, 1986

[13] J.C.Willems *Models for Dynamics*, Dynamics Reported, vol.2, Wiley and Teubner, pp. 171-269, 1989

[14] J.C.Willems *Paradigms and puzzles in the theory of dynamical systems*, IEEE Trans. Aut.Contr., vol.AC-36, pp. 259-294, 1991