RECENT DEVELOPMENTS IN 2D POSITIVE SYSTEM THEORY

ETTORE FORNASINI AND MARIA ELENA VALCHER *

Abstract. Two-dimensional (2D) positive systems are 2D state space models whose variables take only nonnegative values and, hence, are described by a family (A, B, M, N, C, D) of nonnegative matrices. In the paper the notions of asymptotic and simple stability, corresponding to arbitrary set of nonnegative initial conditions, are introduced and related to the spectral properties of the matrix sum $A+B$. Some results concerning the positive realization problem for 2D rational functions are also presented.

2D compartmental models are introduced as 2D positive systems which obey some conservation law, and consequently are characterized by the property that the matrix pair (A, B), responsible for their state-updating, has substochastic sum. A canonical form, to which every 2D compartmental model can be reduced, is here derived, thus leading to obtain interesting results about stability and positive realizability problems. The relevance of these models is illustrated by means of a couple of examples.

Keywords. 2D positive systems, 2D compartmental models, stability, positive realization theory

1. Introduction

The interest in 2D systems goes back to the early seventies [?, ?, ?], and was initially motivated by the relevance of these models in seismology applications, X-ray image enhancement, image deblurring, digital picture processing, etc. More recently, some contributions dealing with river pollution modelling [?] and the discretization of PDE's which describe gas absorption and water stream heating [?], naturally introduced a nonnegativity constraint in 2D system equations. Also, two-dimensional models involving only nonnegative variables were succesfully adopted for describing the diffusion process of a tracer into a blood vessel [?].

This kind of instances stimulated, in the last few years, a systematic analysis of *2D positive systems*, i.e. 2D state-space models whose input, state and output variables take positive (or at least nonnegative) values, where the results presented in [?, ?, ?] could be naturally framed. Research efforts in this context were first oriented to extend "positive matrix theory" to pairs of matrices. As a consequence, Perron-Frobenius theorem [?] and the notions of irreducibility [?] and primitivity [?], as well as some interesting interpretation of these notions in terms of graphs, are now available also for nonnegative matrix pairs.

Although these results allow for a satisfactory analysis of the free state evolution of 2D positive systems and for a complete characterization of their asymptotic stability

*The authors are with the Dipartimento di Elettronica ed Informatica dell'Università di Padova, via Gradenigo 6a, 35131 Padova, Italy - phones: + 39-49-827-7605 (7795) - fax: + 39-49-827-7699 - e-mails:fornasini@dei.unipd.it,meme@dei.unipd.it

[**?**, **?**, **?**, **?**, **?**], a number of interesting issues remains still unexplored, and will be addressed in this paper.

Our objective is twofold. First, we aim to supply a unified discussion of several topics that can be grouped around the concepts of internal and external stability of 2D systems and the related notion of stable realization. Second, the results we present are intended to serve as a motivation for the study of 2D compartmental systems, the central theme of this contribution.

During the last decades compartmental modelling techniques have been increasingly applied to the analysis of biological and chemical processes, and, more generally, for investigating dynamical systems to which the law of conservation of matter (of energy, etc.) applies [**?**]. As a rule, compartmental models consist of a finite number of compartments with specified interconnections among them that either represent fluxes of materials from one site to another or chemical transformations or both. Consequently, their behavior is described by a finite set of ordinary differential equations or, in the discrete case, by one-dimensional (1D) difference equations.

There are situations, however, where the physics of the phenomenon one aims to model has an intrinsic multidimensional nature, as both time and spatial coordinates are involved. Actually, if the propagation time cannot be neglected, lumped parameter models are inadequate to describe the system behavior, and we have to resort to partial differential equations or to multidimensional (nD) discrete systems.

In this paper we start introducing 2D compartmental models by means of some simple physical examples (section 4). The structure of the resulting equations induces quite naturally a definition of a *2D compartmental model* as a 2D positive system with the property that its state updating matrices have a substochastic sum. This constraint, although rather weak, entails far rich consequences on the stability properties of the system. Moreover, it allows to derive a canonical form for 2D compartmental models which gives deep insights in their asymptotic behavior and, in particular, on the asymptotic contents of the various compartments, when no external input is applied (section 5).

Finally, some preliminary results on the realization problem by means of 2D compartmental models are presented.

Before proceeding, it is convenient to introduce some notation. In order not to digress too far, the notions of cone, polyhedral cone, positive matrix and directed graph are only briefly reviewed for notational purposes: adequate information can be found e.g. in [**?**, **?**, **?**]. Also, in the attempt to gain the basic information on the subject as economically as possible, no detailed account is included on the basics of 2D system theory and of classical complex analysis; the interested reader is referred, for instance, to [**?**, **?**, **?**] and [**?**], respectively.

Throughout the paper we let $\mathbb{R}^{\nu}_{+}$ denote the nonnegative orthant, namely the set of all nonnegative vectors in the ν-dimensional Euclidean space $\mathbb{R}^{\nu}$. A set $K \subset \mathbb{R}^{\nu}$ is said to be a *cone* if $\alpha K \subset K$ for all $\alpha \geq 0$; a cone is *convex* if it contains, with any two points, the line segment between them.

A convex cone K in $\mathbb{R}^\nu$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of *generating vectors*. This amounts to saying that a positive integer ℓ and an $\nu \times \ell$ matrix K can be found, such that K coincides with the set of nonnegative combinations of the columns of K. When so, we adopt the notation $K := \text{Cone}(K)$.

If $M = [m_{ij}]$ is a matrix (in particular, a vector), we write $M \gg 0$ (M *strictly positive*), if $m_{ij} > 0$ for all i, j; $M > 0$ (M *positive*), if $m_{ij} \geq 0$ for all i, j, and $m_{hk} > 0$ for at least one pair (h, k), and $M \geq 0$ (M *nonnegative*), if $m_{ij} \geq 0$ for all i, j. The *spectral radius* of a matrix M is the maximal among the moduli of its eigenvalues and is denoted by $\rho(M)$, while its *index* [**?**] is the smallest nonnegative integer k for which $\ker(\rho(M)I - M)^k = \ker(\rho(M)I - M)^{k+1}$.

To every $\nu \times \nu$ positive matrix M we make it correspond [**?**] a *directed graph* (*digraph*), $D(M)$, of *order* ν, with vertices indexed by $1, 2, \ldots, \nu$. There is an *arc* from vertex i to vertex j if and only if $m_{ij} > 0$.
We say that vertex j is *accessible* from i if there exists a positive integer h such that the (i, j)th entry of M^h, $[M^h]_{ij}$, is positive. Vertices i and j are said to *communicate* if each is accessible from the other. The concept of communicating vertices allows to partition the totality of ν vertices in $D(M)$ into *communicating classes*, such that each vertex within a class communicates with every other vertex in the class, and with no other vertex. The *spectral radius of a class* C is the spectral radius of the submatrix of M whose rows and columns are indexed by the vertices in C.

A *chain* of classes in $D(M)$ is a collection of classes such that each class in the collection has access to, or from, every other class in the collection. The *length* of a chain is the number of classes in the chain whose spectral radius coincides with $\rho(M)$. In the paper, we indicate by

$$P_r := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < r, |z_2| < r\}$$

the *open polydisc of radius* r and by $\bar{P}_r$ its closure. Given a polynomial $d(z_1, z_2) \in \mathbb{R}[z_1, z_2]$, the *variety of* d, denoted by $V(d)$, is the set of all points (α, β) of $\mathbb{C}^2$ such that $d(\alpha, \beta) = 0$.

For a pair of $\nu \times \nu$ matrices, (A, B), the *characteristic polynomial* is defined as $\Delta_{A,B}(z_1, z_2) := \det(I_\nu - Az_1 - Bz_2)$, and the *Hurwitz products*, $A^h \sqcup^k B$, are inductively defined as

$$A^h \sqcup^0 B = A^h, \quad h \geq 0, \qquad \text{and} \qquad A^0 \sqcup^k B = B^k, \quad k \geq 0, \tag{1.1}$$

and, when h and k are both positive,

$$A^h \sqcup^k B = A(A^{h-1} \sqcup^k B) + B(A^h \sqcup^{k-1} B). \tag{1.2}$$

It is easily seen that $A^h \sqcup^k B$ is the sum of all matrix products that include the factors A and B, h and k times, respectively.

2. Stability properties of 2D positive systems

A *2D positive system* is defined [**?**, **?**, **?**] as a discrete quarter-plane causal 2D state model [**?**]

$$\begin{aligned} \mathbf{x}(h+1,k+1) &= A\mathbf{x}(h,k+1) + B\mathbf{x}(h+1,k), \\ &\quad + M\mathbf{u}(h,k+1) + N\mathbf{u}(h+1,k) \end{aligned} \tag{2.1}$$

$$\begin{aligned} \mathbf{y}(h,k) &= C\mathbf{x}(h,k) + E\mathbf{u}(h,k) \\ &\quad h,k \in \mathbb{Z},\ h+k \geq 0, \end{aligned} \tag{2.2}$$

$\Sigma = (A, B, M, N, C, E)$ for short, where the doubly indexed *local states* $\mathbf{x}(h,k)$, the *outputs* $\mathbf{y}(h,k)$ and the *inputs* $\mathbf{u}(h,k)$ are elements of $\mathbb{R}^\nu_+$, $\mathbb{R}^p_+$ and $\mathbb{R}^m_+$, respectively, and A, B, M, N, C and E are nonnegative matrices of suitable dimensions. Furthermore, *initial conditions* are given by assigning a sequence $X_0 := \{\mathbf{x}(\ell,-\ell) : \ell \in \mathbb{Z}\}$ of nonnegative local states on the *separation set* $S_0 := \{(\ell,-\ell) : \ell \in \mathbb{Z}\}$.

Stability issues for 2D positive systems are naturally concerned with the unforced state evolutions determined by arbitrary assignements of nonnegative initial conditions on the separation set S_0. In the special case when the initial conditions on S_0 are all zero, except at $(0,0)$, the unforced state evolution at point (h,k) is given by

$$\mathbf{x}(h,k) = (A^h \sqcup^k B)\ \mathbf{x}(0,0), \qquad \forall\ h,k \in \mathbb{N},$$

while for an arbitrary set of initial conditions X_0, the local state in an arbitrary point $(h,k) \in \mathbb{Z}^2$, $h+k \geq 0$, can be obtained by linearity as

$$\mathbf{x}(h,k) = \sum_\ell (A^{h-\ell} \sqcup^{k+\ell} B)\ \mathbf{x}(\ell,-\ell), \tag{2.3}$$

where the Hurwitz product $A^{h-\ell} \sqcup^{k+\ell} B$ is assumed zero when either $h-\ell$ or $k+\ell$ is negative.

Intuitively speaking, a 2D system will be considered positively asymptotically stable if the free state evolution corresponding to an arbitrary set of nonnegative initial conditions uniformly extinguishes on the separation sets $S_t := \{(\ell, t-\ell) : \ell \in \mathbb{Z}\}$, as t goes to infinity, while for positive stability we only require that all free state trajectories generated by nonnegative initial conditions remain bounded as t goes to infinity.

It is clear, however, that a sequence of unbounded initial conditions on S_0 usually determines a free evolution which fulfills neither of these requirements, as local state vectors on each separation set constitute an unbounded sequence, except in the case of finite memory systems. So, it is convenient to restrict the family of admissible initial conditions, by assuming that the initial local states $\mathbf{x}(\ell,-\ell)$ on S_0 satisfy

$$0 \leq \mathbf{x}(\ell,-\ell) \leq \mathbf{v}, \qquad \forall\ \ell \in \mathbb{Z}, \tag{2.4}$$

for some suitable vector $\mathbf{v} \in \mathbb{R}^\nu_+$. Under this assumption, stability definitions are naturally formalized as follows.

Definition 2.1. *The 2D positive system (2.1)÷(2.2), or equivalently the pair* (A, B) *of* $\nu \times \nu$ *nonnegative matrices, is said to be*

- *positively asymptotically stable if every set* X_0 *of bounded nonnegative initial conditions determines a free evolution which asymptotically extinguishes, i.e.*

$$\mathbf{x}(h,k) \to 0 \quad \text{as} \quad h+k \to +\infty;$$

- *positively stable if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any sequence of initial conditions satisfying* $0 \leq \mathbf{x}(\ell, -\ell) < \delta \mathbf{u}_\nu$*, with* $\mathbf{u}_\nu$ *denoting the* ν*-dimensional vector* $[1\ 1\ \dots\ 1]^T$*, determines a free evolution for which*

$$0 \leq \mathbf{x}(h,k) < \varepsilon \mathbf{u}_\nu, \qquad \forall\ h,k \in \mathbb{Z},\ h+k \geq 0.$$

The characterization of asymptotic stability given in the following proposition was first derived in [**?**], while point ii) provides a complete characterization of simple stability which refers to the structure of the digraph $D(A+B)$ associated with the sum of the two (one-step) transition matrices.

Proposition 2.2. *Consider a 2D positive system (2.1)÷(2.2), with state transition matrices* $A, B \in \mathbb{R}^{\nu \times \nu}$.

i) (A, B) *is positively asymptotically stable if and only if* $\rho(A+B) < 1$, *(i.e.* $A+B$ *is the state transition matrix of an asymptotically stable 1D system);*

ii) (A, B) *is positively stable if and only if* $\rho(A+B) \leq 1$ *and* $\rho(A+B) = 1$ *implies that in the directed graph* $D(A+B)$ *there are no chains of length greater than 1, (i.e.* $A+B$ *is the one-step state transition matrix of a stable 1D system).*

The proof depends upon the following two lemmas.

Lemma 2.3. *Consider the 1D system*

$$\mathbf{z}(t+1) = M\mathbf{z}(t), \tag{2.5}$$

with M *a* $\nu \times \nu$ *positive matrix. System (2.5) is*

i) *asymptotically stable if and only if all state evolutions corresponding to nonnegative initial conditions* $\mathbf{z}(0)$ *asymptotically extinguish, and*

ii) *stable if and only if all state evolutions corresponding to nonnegative initial conditions* $\mathbf{z}(0)$ *are bounded.*

Proof. i) and ii) The "only if" parts are obvious. The "if" parts follow from linearity and the fact that every initial condition $\mathbf{z}(0)$ can be expressed as the difference of two nonnegative vectors: $\mathbf{z}(0) = \mathbf{z}_+(0) - \mathbf{z}_-(0)$, for some $\mathbf{z}_+(0), \mathbf{z}_-(0) \geq 0$. □

Lemma 2.4. *Consider a 2D positive system (2.1)÷(2.2), with state transition matrices* $A, B \in \mathbb{R}^{\nu \times \nu}$. *Then* (A, B) *is* *i) positively asymptotically stable if and only if the 1D system*

$$\mathbf{z}(t+1) = (A+B)\mathbf{z}(t) \tag{2.6}$$

is asymptotically stable;

ii) positively stable if and only if system (2.6) is stable.

Proof. i) If (A, B) is asymptotically stable, then for every set of nonnegative initial conditions X_0 the corresponding free dynamics asymptotically goes to zero. In particular, when the initial local states $\mathbf{x}(-\ell, \ell)$ are all equal, namely $\mathbf{x}(-\ell, \ell) = \mathbf{x}_0 \geq 0$ for all $\ell \in \mathbb{Z}$, then $\mathbf{x}(h, t-h) \to 0$ as $t \to \infty$. But $\mathbf{x}(h, t-h) = (A+B)^t \mathbf{x}_0$, and thus stability implies $(A+B)^t \mathbf{x}_0 \to 0$ as t goes to infinity, for every nonnegative $\mathbf{x}_0$. By the previous lemma, this allows to say that (2.6) is asymptotically stable.
Conversely, assume that (2.6) is asymptotically stable. If X_0 is an arbitrary set of initial global conditions satisfying (2.4), for some suitable $\mathbf{v} \in \mathbb{R}^{\nu}_{+}$, then

$$\begin{aligned}
\mathbf{x}(h, t-h) &= \sum_{\ell} (A^{h+\ell} \sqcup\!\sqcup\, ^{t-h-\ell} B)\mathbf{x}(-\ell, \ell) \leq \sum_{\ell} (A^{h+\ell} \sqcup\!\sqcup\, ^{t-h-\ell} B)\mathbf{v} \\
&= (A+B)^t \mathbf{v} \xrightarrow[t \to \infty]{} 0,
\end{aligned}$$

which proves that (A, B) is positively asymptotically stable.
ii) Follows the same lines of part i). □

We are now in a position to prove Proposition 2.2.

Proof. Part i) follows immediately from the previous lemma. As far as part ii) is concerned, by the previous lemma (A, B) is positively stable if and only if $A+B$ is stable, but this amounts to saying that $\rho(A+B) \leq 1$ and if $\rho(A+B) = 1$ then $A+B$ has unitary index. By a result due to Rothblum [**?**], the index of a nonnegative matrix coincides with the length of the longest chain in the associated digraph, and this proves the result. □

3. The positive realization problem for 2D rational functions

Since the publication of the celebrated paper by Maeda and Kodama [**?**] in 1981, the positive realization problem for (1D) proper rational functions has been the object of a wide-spread interest in the literature: just to cite some fundamental contributions on this subject, we shall mention [**?**, **?**, **?**, **?**, **?**]. The problem statement is a very simple one, namely that of finding, for a given transfer function, a state equation in which the state and the output variables take nonnegative values whenever the initial states and the inputs are nonnegative. Despite its simplicity, it was only recently that Anderson, Farina et al. [**?**, **?**] gave a fundamental contribution to the solution of this problem, by providing an iterative algorithm for testing the positive realizability of a given rational function $w(z)$, based on the analysis of the spectral properties of a family of functions suitably derived from $w(z)$.

Apart from its theoretical importance, the interest for this problem was largely motivated by its possible applications, as pointed out in several contributions [**?**, **?**, **?**, **?**]. As 2D positive systems are also adopted for modelling physical systems in the context of biology, bioengineering, chemistry, etc., when the variables involved are functions of a pair of independent variables (generally time and space or two spatial coordinates), the relevance of the realization problem in the context of 2D rational functions is immediately apparent.

In this section, we will restrict our attention to *strictly proper* 2D transfer functions, namely rational functions $w(z_1, z_2) \in \mathbb{R}(z_1, z_2)$ satisfying $w(0,0) = 0$. These functions are those and those only that can be realized by means of a 2D state-space model with $E = 0$. The analogous results for proper rational functions follow immediately, upon expressing each function $w(z_1, z_2)$ as the sum of its strictly proper part and of $E := w(0,0)$. Also, dealing with strictly proper SISO systems, we will adopt the special notation $\Sigma = (A, B, m, n, c^T)$.

The first step toward the solution of the realization problem for 2D rational functions is given by the following proposition, that strictly reminds of a well-know result of Maeda and Kodama [**?**] (see, also, [**?**]) for the 1D case.

Proposition 3.1. *Let $w(z_1, z_2) \in \mathbb{R}(z_1, z_2)$ be a strictly proper 2D rational transfer function. A necessary and sufficient condition for the existence of a nonnegative realization of $w(z_1, z_2)$ is that there exist a realization $\Sigma = (A, B, m, n, c^T)$ of $w(z_1, z_2)$ and a polyhedral cone K such that the following conditions hold true:*

i) $AK \subseteq K$ and $BK \subseteq K$;

ii) the reachability cone $R(\Sigma) = \mathrm{Cone}(m, n, Am, An + Bm, Bn, ...)$, generated by the vector coefficients of the power series expansion of $(I - Az_1 - Bz_2)^{-1}(mz_1 + nz_2)$, is included in K;

*iii) c belongs to the dual cone of K [**?**], i.e. $c^T\mathbf{v} \geq 0$ for every $\mathbf{v} \in K$.*

Proof. [Necessity] If there exists a positive realization of $w(z_1, z_2)$, $\bar{\Sigma} = (\bar{A}, \bar{B}, \bar{m}, \bar{n}, \bar{c}^T)$, and ν denotes its dimension, then conditions i)÷iii) hold true for $\Sigma := \bar{\Sigma}$ and $K = \mathbb{R}^{\nu}_{+}$.
[Sufficiency] Assume that there exist both a realization $\Sigma = (A, B, m, n, c^T)$ and a polyhedral cone K such that i)÷iii) hold true. If ν is the dimension of Σ and K is an $\nu \times \ell$ matrix generating the cone, i.e., $K = \mathrm{Cone}(K)$, then condition i) guarantees that nonnegative matrices $\bar{A}$ and $\bar{B}$ can be found such that $AK = K\bar{A}$ and $BK = K\bar{B}$. On the other hand, ii) implies, in particular, $m, n \in K$, and hence both $m = K\bar{m}$ and $n = K\bar{n}$ hold true for suitable vectors $\bar{m}, \bar{n} \geq 0$. Finally, condition iii) leads to $\bar{c}^T := c^T K \geq 0$.

We aim to prove that the nonnegative 2D state-space model $\bar{\Sigma} = (\bar{A}, \bar{B}, \bar{m}, \bar{n}, \bar{c}^T)$ realizes $w(z_1, z_2)$. This amounts to saying that $\bar{c}^T(I - \bar{A}z_1 - \bar{B}z_2)^{-1}(\bar{m}z_1 + \bar{n}z_2) \equiv c^T(I - Az_1 - Bz_2)^{-1}(mz_1 + nz_2)$, or, equivalently, that the power series expansions of the two functions coincide. So, it is sufficient to verify the following identity

$$\bar{c}^T\left[(\bar{A}^{h-1} \sqcup\!\sqcup^k \bar{B})\bar{m} + (\bar{A}^h \sqcup\!\sqcup^{k-1} \bar{B})\bar{n}\right] = c^T\left[(A^{h-1} \sqcup\!\sqcup^k B)m + (A^h \sqcup\!\sqcup^{k-1} B)n\right] \tag{3.1}$$

for every pair of nonnegative integers h and k, $h + k > 0$.
It is easy to show, by induction, that for every pair of nonnegative integers, i and j, we have $K(\bar{A}^i \sqcup\!\sqcup^j \bar{B}) = (A^i \sqcup\!\sqcup^j B)K$. Consequently, one gets

$$\begin{aligned}
&\bar{c}^T\left[(\bar{A}^{h-1} \sqcup\!\sqcup^k \bar{B})\bar{m} + (\bar{A}^h \sqcup\!\sqcup^{k-1} \bar{B})\bar{n}\right] = c^T K\left[(\bar{A}^{h-1} \sqcup\!\sqcup^k \bar{B})\bar{m} + (\bar{A}^h \sqcup\!\sqcup^{k-1} \bar{B})\bar{n}\right] \\
&= c^T\left[(A^{h-1} \sqcup\!\sqcup^k B)K\bar{m} + (\ A^h \sqcup\!\sqcup^{k-1} B)K\bar{n}\right] = c^T\left[(A^{h-1} \sqcup\!\sqcup^k B)m + (A^h \sqcup\!\sqcup^{k-1} B)n\right],
\end{aligned}$$

thus proving (3.1). $\square$

The above proposition deserves some comments. Although it may appear just as the two-dimensional analogue of the result presented in [**?**], it turns out to be a weaker characterization of positively realizable 2D functions. Indeed, given a positively realizable function $f(z)$ and anyone of its state-space realizations (F, g, h^T) (for instance, a minimal one), a polyhedral cone can be found satisfying the 1D analogue of conditions i)÷iii). In the 2D case, instead, not every 2D state-space realization of a positively realizable function admits a polyhedral cone K for which conditions i)÷iii) hold true. This is quite unpleasant, as it rules out the possibility of solving the realization problem by analysing one of its realizations, but it is absolutely natural once we think of how the set of all realizations of a 2D proper rational functions is organized.

Actually, the state-space realizations of a rational function $f(z)$ can be thought of as constituting a tree structure, whose root is the (essentially unique) minimal realization, and every other realization can be obtained by the minimal one by suitably increasing the unobservable and/or uncontrollable parts. In the 2D case, the realizations of a given function $w(z_1, z_2)$ are naturally viewed [**?**] as constituting infinitely many different tree structures, each one representing the set of realizations corresponding to a particular noncommutative power series having $w(z_1, z_2)$ as commutative image. Of course, noncommutative power series that exhibit some negative coefficients have no positive realization, and conditions i)÷iii) cannot be fulfilled by anyone of their state-space realizations. As every function $w(z_1, z_2)$ is the commutative image of a noncommutative power series with some negative coefficients, the testing procedure suggested in the above proposition necessarily fails for some realizations of $w(z_1, z_2)$. More precisely, if we consider the realizations of the same noncommutative version of $w(z_1, z_2)$, either for all of them or for none of them polyhedral cones K can be found satisfying the three requirements. This situation is better understood by means of the following simple example.

EXAMPLE Consider the strictly proper rational function $w(z_1, z_2) = z_1 z_2$. It is immediately seen that

$$(A, B, m, n, c^T) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right)$$

is a positive realization of $w(z_1, z_2)$. On the other hand, once we think of $w(z_1, z_2)$ as the commutative image of the noncommutative power series $\tau = 2\xi_1\xi_2 - \xi_2\xi_1$, by applying a modified version of the Ho's algorithm [**?**] we easily get the following realization of τ and hence of $w(z_1, z_2)$:

$$(F_1, F_2, g_1, g_2, h^T) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \right).$$

We aim to show that no polyhedral cone in R^3 can be found satisfying points i)÷iii) of Proposition 3.1, w.r.t. the realization $(F_1, F_2, g_1, g_2, h^T)$. Suppose, by contradiction, that such a polyhedral cone K exists, and let

$$K = \begin{bmatrix} k_1^T \\ k_2^T \\ k_3^T \end{bmatrix}, \qquad k_i \in \mathbb{R}^\ell, i = 1, 2, 3,$$

be a generating matrix of K. Condition iii) guarantees $k_1 \geq 0$, while condition i) implies that both k_2 and k_3 must be nonnegative, and therefore K must be included in the positive orthant $\mathbb{R}^3_+$. But then, as g_1 is not in $\mathbb{R}^3_+$, condition ii) cannot be satisfied.

As a consequence of the situation now described, the problem of determining when a given 2D rational function $w(z_1, z_2)$ admits a positive realization is much more complicated than the analogous one-dimensional. Interestingly enough, however, when $w(z_1, z_2)$ is positively realizable, then, in particular, a positive realization (A, B, m, n, c^T) can be found for which the variety of the characteristic polynomial $\Delta_{A,B}(z_1, z_2)$ satisfies the following special constraint: if $n(z_1, z_2)/d(z_1, z_2)$ is an irreducible representation of $w(z_1, z_2)$, and hence $V(d)$ represents the set of all singularities of w, then

$$\min\{r \in \mathbb{R}_+ : \bar{P}_r \cap V(d) \neq \emptyset\} \equiv \min\{r \in \mathbb{R}_+ : \bar{P}_r \cap V(\Delta_{A,B}) \neq \emptyset\} \equiv (R, R), \quad R \in \mathbb{R}_+. \tag{3.2}$$

This is a quite interesting result, as it extends to the 2D case a theorem due to Anderson et al. [**?**], saying that any positively realizable function $f(z)$ has a real positive pole r with maximum modulus, and it admits a positive realization (F, g, h^T) with $\rho(F) = r$. The proof of this proposition depends upon a couple of technical lemmas.

LEMMA 3.2. *Let $f(z)$ be a rational transfer function, whose power series expansion $\sum_{i=0}^{+\infty} f_i z^i$ has nonnegative coefficients. If R is the radius of convergence [**?**] of the series, then R is a pole of $f(z)$.*

Proof. As $f(z)$ is a rational function, its power series expansion $\sum_{i=0}^{+\infty} f_i z^i$ converges (absolutely and locally uniformly) in every open disc centered in the origin, $B(0,r)$, whose radius r satisfies $r < \min\{|p| : p \text{ a pole of } f(z)\}$. Consequently,

$$R = \min\{|p| : p \text{ a pole of } f(z)\} = |p_0|,$$

for some possibly complex pole p_0 of $f(z)$. We aim to show that $|p_0|$ is a pole of $f(z)$, too. For every real α satisfying $0 < \alpha < 1$, by exploiting the nonnegative assumption on the f_i's, one gets

$$|f(\alpha p_0)| = |\sum_{i=0}^{+\infty} f_i \alpha^i p_0^i| \leq \sum_{i=0}^{+\infty} f_i \alpha^i |p_0|^i = f(\alpha |p_0|). \tag{3.3}$$

As the leftthand side of (3.3) diverges, as α approaches 1, then $f(\alpha|p_0|) \xrightarrow[\alpha\to 1]{} \infty$, and, consequently, $|p_0|$ is a pole of $f(z)$. □

Lemma 3.3. *Let $w(z_1,z_2) \in \mathbb{R}(z_1,z_2)$ be a proper rational 2D transfer function, $n(z_1,z_2)/d(z_1,z_2)$ an irreducible representation of $w(z_1,z_2)$ and $\sum_{h,k=0}^{+\infty} w_{hk} z_1^h z_2^k$ a power series expansion of $w(z_1,z_2)$ within a suitable open polydisc, centered in the origin. If all coefficients w_{hk} of the power series expansion are nonnegative and $R := \min\{r \in \mathbb{R}_+ : \bar{P}_r \cap V(d) \neq \emptyset\}$, then*

i) $f(z) := w(z,z)$ has a pole of minimum modulus in $z = R$;

ii) $w(z_1,z_2)$ has a (nonessential) singularity in (R,R).

Proof. i) Observe, first, that the power series expansion $\sum_{h,k=0}^{+\infty} w_{hk} z_1^h z_2^k$ is absolutely convergent in every point (z,z), with $|z| \leq r < R$. Consequently, the 1D power series $\sum_{\nu=0}^{+\infty}\Big(\sum_{h+k=\nu} w_{hk}\Big) z^\nu$ converges for every z with $|z| < R$ and hence $f(z)$ is analytic in the open disc $B(0,R)$. If $f(z)$ had not a pole at $z = R$, then, by Lemma 3.2, it would be devoid of singularities within the closed disk $\bar{B}(0,R)$, and there would be some $\varepsilon > 0$ such that $R+\varepsilon$ is the radius of convergence of $f(z)$. In this case, we would have

$$\sum_{\nu=0}^{+\infty}\Big(\sum_{h+k=\nu} w_{hk}\Big)\Big(R+\frac{\varepsilon}{2}\Big)^\nu < \infty, \tag{3.4}$$

and hence

$$\sum_{h,k=0}^{+\infty} w_{hk}\Big(R+\frac{\varepsilon}{2}\Big)^h\Big(R+\frac{\varepsilon}{2}\Big)^k < \infty. \tag{3.5}$$

This implies that the power series expansion of $w(z_1,z_2)$ is convergent in $P_{R+\frac{\varepsilon}{2}}$, thus contradicting the assumption that R is the radius of convergence of $w(z_1,z_2)$.
ii) As $f(z) = w(z,z)$ has a pole in R, then $w(z_1,z_2)$ has a singularity in (R,R), which is nonessential by the rationality assumption. □

Proposition 3.4. *Let $w(z_1,z_2) \in \mathbb{R}(z_1,z_2)$ be a strictly proper rational 2D transfer function, which is positively realizable, and let $n(z_1,z_2)/d(z_1,z_2)$ be an irreducible representation of $w(z_1,z_2)$. If $R := \min\{r \in \mathbb{R}_+ : \bar{P}_r \cap V(d) \neq \emptyset\}$, there exists a positive realization $\Sigma = (A,B,m,n,c^T)$ with $\rho(A+B) = 1/R$, and when so*

$$R \equiv \min\{r \in \mathbb{R}_+ : \bar{P}_r \cap V(\Delta_{A,B}) \neq \emptyset\}. \tag{3.6}$$

Proof. Let $\bar{\Sigma} := (\bar{A},\bar{B},\bar{m},\bar{n},\bar{c}^T)$ be a positive realization of $w(z_1,z_2)$. If the spectral radius of $\bar{A}+\bar{B}$ does not coincide with $1/R$, it must be $\rho(\bar{A}+\bar{B}) > 1/R$. Clearly, $\bar{\Sigma}_1 := (\bar{A}+\bar{B},\bar{m}+\bar{n},\bar{c}^T)$ is a positive realization of the 1D rational function $f(z) := w(z,z)$, whose minimal modulus pole is located in R, as a result of the previous lemma. From the inequality $\rho(\bar{A}+\bar{B}) > 1/R$, it follows that the eigenvalue $\rho(\bar{A}+\bar{B})$ belongs either to the unreachable or to the unobservable part of $\bar{\Sigma}_1$. Suppose, for instance, that $\rho(\bar{A}+\bar{B})$ is not observable. Then, there exists a nonnegative eigenvector $\mathbf{v}$ of $\bar{A}+\bar{B}$, corresponding to $\rho(\bar{A}+\bar{B})$, such that $H\mathbf{v} = 0$. Without loss of generality, we can reorder the entries of the state vector of $\bar{\Sigma}_1$ so that

$$\begin{aligned} \bar{c}^T &= [\bar{c}_1^T \quad 0 \quad 0] \\ \mathbf{v}^T &= [0 \quad 0 \quad \mathbf{v}_3^T] \end{aligned}$$

with $\bar{c}_1^T$ and $\mathbf{v}_3$ strictly positive vectors. Let

$$\bar{A}+\bar{B} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} \\ A_{31}+B_{31} & A_{32}+B_{32} & A_{33}+B_{33} \end{bmatrix}.$$

Because $(\bar{A}+\bar{B})\mathbf{v} = \rho(\bar{A}+\bar{B})\mathbf{v}$, the zeros in $\mathbf{v}$ force $A_{13}+B_{13} = 0$ and $A_{23}+B_{23} = 0$, and therefore all matrices A_{13}, A_{23}, B_{13} and B_{23} are zero. But then the zero blocks in $\bar{c}^T$ and $\bar{A}+\bar{B}$ mean that an unobservable part is displayed, and a lower dimension, but still positive realization of $f(z)$ is provided by

$$\left(\begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}, \begin{bmatrix} m_1+n_1 \\ m_2+n_2 \end{bmatrix}, [\bar{c}_1^T \quad 0] \right).$$

Correspondingly,

$$\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, [\bar{c}_1^T \quad 0] \right)$$

constitutes a lower dimension positive realization of $w(z_1,z_2)$. Similarly, if $\rho(\bar{A}+\bar{B})$ belongs to the unreachable part, we can obtain lower dimension positive realizations both of $f(z)$ and of $w(z_1,z_2)$.

So, starting with an arbitrary positive realization of $w(z_1,z_2)$, and hence of $f(z)$, we can reduce it until obtaining a positive realization of $f(z)$, $\Sigma_1 = (A+B,m+n,c^T)$ with $\rho(A+B)$ coinciding with $1/R$. Consequently, $\Sigma = (A,B,m,n,c^T)$ will be the desired positive realization of $w(z_1,z_2)$.

We are remained to show that when $\rho(A+B) = 1/R$, then (3.6) holds true. The result has already been proved in [**?**] for the case of $A+B$ irreducible. So, suppose now that $A+B$ is a reducible matrix, and reduce it to (upper triangular) Frobenius normal form [**?**]

$$P^T(A+B)P = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \dots & A_{1h}+B_{1h} \\ & A_{22}+B_{22} & \dots & A_{2h}+B_{2h} \\ & & \ddots & \\ & & & A_{hh}+B_{hh} \end{bmatrix}, \tag{3.7}$$

with $A_{ii}+B_{ii}$ irreducible blocks, by means of a suitable permutation matrix P. Clearly, there exists some index k such that $\rho(A_{kk}+B_{kk}) = 1/R$, and hence, by the irreducibility of $A_{kk}+B_{kk}$,

$$R \equiv \min\{r \in \mathbb{R}_+ : \bar{P}_r \cap V(\Delta_{A_{kk},B_{kk}}) \neq \emptyset\}.$$

On the other hand, one has

$$\min\{r \in \mathbb{R}_+ : \bar{P}_r \cap V(\Delta_{A_{kk},B_{kk}}) = \min\{r \in \mathbb{R}_+ : \bar{P}_r \cap V(\Delta_{A,B}) \neq \emptyset\},$$

which proves the result. □

By combining together the results of Proposition 2.2 and Proposition 3.4, we get the following result, which provides a necessary and a sufficient condition for the existence of a positively (asymptotically) stable positive realization.

Corollary 3.5. *Let $w(z_1, z_2) \in \mathbb{R}(z_1, z_2)$ be a positively realizable rational function and let $n(z_1, z_2)/d(z_1, z_2)$ be one of its irreducible representations.*

i) If $V(d) \cap \bar{P}_1 = \emptyset$, then there exists a positively asymptotically stable positive realization of $w(z_1, z_2)$;

ii) if $V(d) \cap P_1 \neq \emptyset$, then no stable realization of $w(z_1, z_2)$ can be found and hence, in particular, there exist no positively stable positive realizations.

Proof. i) If $V(d) \cap \bar{P}_1 = \emptyset$ and $w(z_1, z_2)$ is positively realizable, then, by Proposition 3.4, there exists a positive realization (A, B, m, n, c^T) with $\rho(A+B) < 1$, which is positively asymptotically stable, as a consequence of Proposition 2.2.
ii) Follows immediately from the fact that for every realization (A, B, m, n, c^T) of $w(z_1, z_2)$, $V(d) \subseteq V(\Delta_{A,B})$. □

4. 2D compartmental systems

4.1. Some examples of 2D compartmental systems

2D compartmental models are 2D positive systems satisfying additional constraints which represent the mathematical formalization of some conservation laws. Before explicitly investigating the properties of this class of systems, it may be useful to have a couple of physical applications in mind, as examples of the sort of phenomena we aim to model. In both cases, the derivation involves making many simplifying assumptions and 2D difference equations we end up provide only crude descriptions. We will concentrate, instead, on some aspects that illustrate how these examples can be viewed as paradigms of a broad class of dynamical behaviors, that can be potentially investigated by applying 2D compartmental systems techniques.

EXAMPLE 1 [SINGLE-CARRIAGEWAY TRAFFIC FLOW] Our aim is to represent, by means of a discrete model, the traffic flow along one carriageway of a motorway. To this end we introduce the following assumptions:

a) The road is partitioned into elementary stretches of length L and the time into elementary intervals of duration T.

b) At time instant $tT, t \in \mathbb{Z}$, the set of cars inside the stretch $[\ell L, (\ell+1)L), \ell \in \mathbb{Z}$, is partitioned into groups of equal speed span, say V km/h. This amounts to say that the first group consists of all cars whose speed belongs to the interval $(0, V]$, in the second group there are all cars with speed in $(V, 2V]$, and so on. Also, one more group is considered, which includes all cars that at time tT are temporarily stopping at a gas station, or in a parking place etc. The groups are sequentially indexed from 0 through ν, with 0 denoting the class of stopping cars, 1 the lowest speed group and ν the highest. If $v_i(\cdot,\cdot)$ represents the number of cars belonging to the i-th group, then the "state" at time tT of the ℓ-th stretch, $[\ell L, (\ell+1)L)$, is given by the vector

$$\mathbf{v}(\ell,t) = \begin{bmatrix} v_0(\ell,t) \\ v_1(\ell,t) \\ \vdots \\ v_\nu(\ell,t) \end{bmatrix}.$$

c) The number of vehicles is large enough to allow for assuming that the v_i's are continuous, rather than integer, variables.

d) Inputs and outputs at motorway intersections are modelled apart. Typically, only some stretches exhibit an intersection, and it is obvious that the output levels in $[tT, (t+1)T)$ cannot exceed the number of cars running through those stretches in that time interval.

e) Car drivers belonging to the i-th group at time tT exhibit a propension (probability) p_{ji} to istantaneously move to the j-th speed class at the beginning of the next time interval, and to drive at that speed during $(tT, (t+1)T]$. Clearly, $\sum_{j=0}^{\nu} p_{ji} = 1$.

f) The length L of a road stretch satisfies $L > \nu VT$. Consequently, every car that belongs to the ℓ-th stretch at time tT, at time $(t+1)T$ belongs either to the same stretch or to the $(\ell+1)$-th. If we assume that there are r cars moving within the i-th speed class during the time interval $[tT, (t+1)T)$, and that at time tT they are uniformly distributed along the stretch $[\ell L, (\ell+1)L)$, then, only $g_i r$ of them, with

$$g_i := \frac{(2i-1)VT}{2L},$$

reach the following stretch before $(t+1)T$. The remaining $(1-g_i)r$ cars are still in the original stretch at time $(t+1)T$.

As a consequence of the above assumptions, and disregarding outflows and inflows at the interconnections, we get the following model:

$$\mathbf{v}(\ell+1,t+1) = GP\mathbf{v}(\ell,t) + (I_{\nu+1} - G)P\mathbf{v}(\ell+1,t), \tag{4.1}$$

where $G = \text{diag}\{0, g_1, g_2, \dots, g_\nu\}$, $P = [p_{ij}]$ and $I_{\nu+1}$ is the identity matrix of size $\nu+1$. Finally, by resorting to the following transformation

$$T : \mathbb{Z}^2 \to \mathbb{Z}^2 : (\ell,t) \mapsto (h,k) = (\ell, t-\ell)$$

and assuming

$$\mathbf{x}(h,k) := \mathbf{v}\Big(T^{-1}(h,k)\Big) = \mathbf{v}(h,h+k),$$

we can rewrite (4.1) as

$$\mathbf{x}(h+1,k+1) = GP\mathbf{x}(h,k+1) + (I_{\nu+1} - G)P\mathbf{x}(h+1,k). \tag{4.2}$$

Example 2 [Streeter-Phelps discrete model for river pollution] [**?**] In modelling the self-purification process of a polluted river, we introduce the following assumptions:

a) The variety of pollutants dissolved in the river can be reduced to one class of oxidizable substances, whose concentration is measured by the amount of oxygen (BOD = biological oxygen demand) needed for their complete biochemical oxidation.

b) The selfpurification process is essentially due to dissolved oxygen (DO) which oxidizes polluting materials and eventually convert them into abiotic substances and heat.

c) As the variations of BOD and DO concentrations on river cross sections can be reasonably considered less significative than the longitudinal ones, we assume for the river a (spatially) one-dimensional model. Moreover, hydrological variables and, in particular, the stream velocity V, are supposed constant all over the river.

d) The river is divided into elementary reaches of length L. The time step T and the elementary reach L are connected through the stream velocity V by the equation

$$T = \frac{L}{V},$$

so that the water element centered in ℓL at time tT will be centered in $(\ell+1)L$ at time $(t+1)T$.
We denote by $\beta(\ell,t)$ and $\delta(\ell,t)$ the concentration of BOD and the deficit of DO w.r.t. the saturation level, respectively, in the elementary reach centered in ℓL at time tT.

BOD and DO values at $\Big((\ell+1)L,(t+1)T\Big)$ are obtained on the basis of a discretized balance equation accounting for different contributions. In fact:

• *Diffusion* is modelled by assuming that the BOD content of the elementary water volume, centered in ℓL at time tT, undergoes in $[tT,(t+1)T)$ a variation proportional to the differences $\beta(\ell-1,t)-\beta(\ell,t)$ and $\beta(\ell+1,t)-\beta(\ell,t)$. Same assumption is made for the DO diffusion process.

• *Self-purification*: in the time interval $[tT,(t+1)T)$ the BOD concentration in the ℓ-th river reach is decreased by the same amount

$$a_1 T\beta(\ell,t)$$

the DO deficit is increased.

• *Reaeration* takes place at the water-atmosphere interface. We assume that in $[tT,(t+1)T)$ the DO deficit is reduced of an amount given by

$$a_2 T\delta(\ell,t).$$

• *BOD sources*: effluents, local run-off, etc., modifying the BOD concentration, determine an exogenous input to the system, which is denoted by $\mathbf{u}_\beta(\cdot,\cdot)$.

By making the above assumptions, we obtain the following model:

$$\begin{bmatrix}\beta(\ell+1,t+1)\\ \delta(\ell+1,t+1)\end{bmatrix}=S\begin{bmatrix}\beta(\ell,t)\\ \delta(\ell,t)\end{bmatrix}+D\begin{bmatrix}\beta(\ell-1,t)\\ \delta(\ell-1,t)\end{bmatrix}+D\begin{bmatrix}\beta(\ell+1,t)\\ \delta(\ell+1,t)\end{bmatrix}+\begin{bmatrix}\tilde{M}\\ 0\end{bmatrix}\mathbf{u}_\beta(\ell,t), \tag{4.3}$$

where

$$S=[s_{ij}]=\begin{bmatrix}1-a_1T-2D_\beta T & 0\\ a_1T & 1-a_2T-2D_\delta T\end{bmatrix}\qquad D=[d_{ij}]=\begin{bmatrix}D_\beta T & 0\\ 0 & D_\delta T\end{bmatrix}.$$

Notice that, as $\tilde{M},a_1,a_2,D_\beta$ and D_δ are positive and T is small, all matrices in the above equation are positive.
The model (4.3) can be reduced to an equivalent one having the structure of (4.2). Actually, upon defining

$$\mathbf{z}(\ell,t):=\left[\begin{array}{c}\beta(2\ell,t)\\ \beta(2\ell+1,t)\\ \hline \delta(2\ell,t)\\ \delta(2\ell+1,t)\end{array}\right]\qquad\text{and}\qquad\tilde{\mathbf{u}}(\ell,t):=\begin{bmatrix}\mathbf{u}(2\ell,t)\\ \mathbf{u}(2\ell+1,t)\end{bmatrix},$$

we get

$$\mathbf{z}(\ell+1,t+1)=A\mathbf{z}(\ell,t)+B\mathbf{z}(\ell+1,t)+M\tilde{\mathbf{u}}(\ell,t), \tag{4.4}$$

where

$$A:=\left[\begin{array}{cc|cc}d_{11}&s_{11}&0&0\\ 0&d_{11}&0&0\\ \hline 0&s_{21}&d_{22}&s_{22}\\ 0&0&0&d_{22}\end{array}\right]\quad B:=\left[\begin{array}{cc|cc}d_{11}&0&0&0\\ s_{11}&d_{11}&0&0\\ \hline 0&0&d_{22}&0\\ s_{21}&0&s_{22}&d_{22}\end{array}\right]\quad M:=\left[\begin{array}{cc}\tilde{M}&0\\ 0&\tilde{M}\\ \hline 0&0\\ 0&0\end{array}\right].$$

Finally, by applying the same coordinate transformation T as in Example 1, and letting

$$\mathbf{x}(h,k) := \mathbf{z}\Big(T^{-1}(h,k)\Big) = \mathbf{z}(h,h+k)$$
$$\mathbf{u}(h,k) := \tilde{\mathbf{u}}\Big(T^{-1}(h,k)\Big) = \tilde{\mathbf{u}}(h,h+k),$$

we get the following equation

$$\mathbf{x}(h+1,k+1) = A\mathbf{x}(h,k+1) + B\mathbf{x}(h+1,k) + M\mathbf{u}(h,k+1). \tag{4.5}$$

4.2. Structure of 2D compartmental systems

Both processes analysed in the previous section have been modelled by means of a 2D positive system, described as in equation (2.1)÷(2.2). Models (4.2) and (4.5), moreover, exhibit an additional property: the sums of the state transition matrices, namely $GP + (I-G)P$ in the first example and $A+B$ in the second, are (column) substochastic, i.e. the sum of the entries in each column of $GP + (I-G)P$ and of $A+B$ does not exceed one. This property represents the mathematical formalization of the fact that the number of cars as well as the amounts of chemical components cannot increase unless external inputs are applied. More precisely, the i-th component, $x_j(h,k)$, of the state $\mathbf{x}(h,k)$ influences only the states in $(h+1,k)$ and $(h,k+1)$, and its contributions, $a_{ij}x_j(h,k)$ in $(h+1,k)$ and $b_{ij}x_j(h,k)$ in $(h,k+1)$, $i = 1,2,\dots,\nu$, cannot sum up to a quantity greater than the original $x_j(h,k)$. A complete conservation corresponds to a stochastic matrix sum, whereas leakages or losses motivate the fact that some columns in the matrix sum are not stochastic.

It is clear that this kind of systems represent the two-dimensional analogue of discrete time 1D compartmental models, thus motivating the following definition.

Definition 4.1. *A 2D compartmental system is a 2D positive system (2.1)÷(2.2) with $A+B$ substochastic.*

Although this requirement on $A+B$ does not give any information on the zero-patterns of A and B, it introduces, however, strong constraints on the spectral properties of the pair (A,B) we aim now to investigate. To this end, it is convenient to make the (not restrictive) assumption that the matrix sum $A+B$ is in Frobenius normal form

$$A+B = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1r} \\ & M_{22} & & M_{2r} \\ & & \ddots & \vdots \\ & & & M_{rr} \end{bmatrix}, \tag{4.6}$$

with irreducible diagonal blocks $M_{ii}, i = 1,2,\dots,r$.

Proposition 4.2. *Let $A+B \in \mathbb{R}_+^{\nu\times\nu}$ be a substochastic matrix, with the block-triangular structure given in (4.6). Then*

i) $\rho(M_{ii}) \le 1$ *for every* $i \in \{1,2,\dots,r\}$ *and* $\rho(A+B) \le 1$*;*

ii) if $\rho(M_{ii}) = 1$, then M_{ii} is stochastic, $M_{ji} = 0$ for every $j \neq i$, and the maximal modulus eigenvalues of $A + B$ are simple roots of the minimal polynomial of $A + B$.

Proof. i) If M is any substochastic matrix, there exists a nonnegative matrix Δ such that $M + \Delta$ is stochastic, and hence $\rho(M)$, the spectral radius of M, satisfies $\rho(M) \leq \rho(M + \Delta) = 1$. Since $A + B$ is substochastic, and this property is inherited by all diagonal blocks M_{ii}, then $\rho(A + B) \leq 1$ and $\rho(M_{ii}) \leq 1$ for all $i \in \{1, 2, \ldots, r\}$.
ii) Assume $\rho(M_{ii}) = 1$, and suppose, by contradiction, that M_{ii} is not stochastic. Then there exists a nonnegative matrix $\Delta \neq 0$ such that $M_{ii} + \Delta$ is stochastic, and the irreducibility of M_{ii} guarantees [**?**] that $\rho(M_{ii}) < \rho(M_{ii} + \Delta) = 1$, a contradiction. So, as each column of M_{ii} has already a unitary sum, all entries in the blocks M_{ji}, $j \neq i$, must be zero. As a consequence, by applying a suitable cogredience transformation [**?**], we can always assume that $A + B$ has the following structure

$$
(4.7)\quad \left[\begin{array}{cccc|cccc}
M_{11} & & & & & & & \\
& M_{22} & & & & * * * & & \\
& & \ddots & & & & & \\
& & & M_{ss} & & & & \\
\hline
& & & & M_{s+1s+1} & M_{s+1s+2} & \ldots & M_{s+1r} \\
& & & & & M_{s+2s+2} & & M_{s+2r} \\
& \mathbf{0} & & & & & \ddots & \vdots \\
& & & & & & & M_{rr}
\end{array}\right]
= \left[\begin{array}{c|c}
A_{11} + B_{11} & A_{12} + B_{12} \\
\hline
0 & A_{22} + B_{22}
\end{array}\right]
$$

where the M_{ii}'s, $i = 1, 2, \ldots, s$, are irreducible stochastic matrices, while the M_{ii}'s, $i = s + 1, s + 2, \ldots, r$, are irreducible substochastic matrices with $\rho(M_{ii}) < 1$.
Finally, in order to prove that every unitary modulus eigenvalue $e^{j\theta}$ of $A+B$ is a simple root of the minimal polynomial, it is sufficient to show that

$$\ker(e^{j\theta} I - A - B) \equiv \ker(e^{j\theta} I - A - B)^2.$$

Clearly, as $(e^{j\theta} I - A_{22} - B_{22})$ is a nonsingular matrix, all vectors in $\ker(e^{j\theta} I - A - B)^2$, and consequently in $\ker(e^{j\theta} I - A - B)$, have the second block of entries, namely the one corresponding to $(e^{j\theta} I - A_{22} - B_{22})$, identically zero. On the other hand, since all blocks M_{ii}, $i = 1, 2, \ldots, s$, are irreducible and stochastic, then $\ker(e^{j\theta} I - M_{ii}) = \ker(e^{j\theta} I - M_{ii})^2$, which proves the result. $\square$

A 2D compartmental system (2.1)÷(2.2) described by a pair (A, B) whose sum has the structure and the properties of matrix (4.7) is said to be in *canonical form*. This form suggests some interesting remarks that further motivate the name of compartmental models for 2D positive systems with $A + B$ substochastic.
Consider, first, the 1D compartmental system associated with the matrix sum $A + B$, block partitioned as in (4.7),

$$(4.8)\qquad \mathbf{z}(t + 1) = (A + B)\mathbf{z}(t).$$

Each class of compartments corresponding to some irreducible stochastic block M_{ii}, $i \in \{1, 2, \ldots, s\}$, presents no losses, by this meaning that the total content of the compartments in that class cannot decrease as time goes by. On the other hand, the contents of the remaining compartments decrease to zero, partly due to losses and partly due to transfers to lossless compartments.
As a consequence, for every initial assignement $\mathbf{z}(0)$ of the compartment contents, only the components corresponding to stochastic blocks can be nonzero in the state vector $\mathbf{z}(t)$ as t goes to infinity.

When considering 2D models, it is convenient to think of local states on the same separation set $S_t := \{(\ell, t-\ell), \ell \in \mathbb{Z}\}$ as representing the contents at time t of compartments $x_1, x_2, \ldots, x_\nu$ at the different space locations $\ell \in \mathbb{Z}$. The content $x_i(\ell, t-\ell)$ of the i-th compartment at time t and location ℓ distributes at time $t+1$, possibly with losses, among the compartments at locations ℓ and $\ell+1$, with rates given by the i-th column of B and A, respectively. By recursively applying this reasoning, it is easy to see that $x_i(\ell, t-\ell)$ at time $t+N$ distributes (with losses) among the compartments at locations $\ell, \ell+1, \ldots, \ell+N$, and its total contribution to the contents of these compartments is expressed by

$$(A+B)^N \mathbf{e}_i x_i(\ell, t-\ell),$$

where $\mathbf{e}_i$ denotes the i-th canonical vector in $\mathbb{R}^\nu$. Again, as t goes to infinity, all compartments corresponding to nonstochastic blocks are progressively emptied, whereas those corresponding to stochastic blocks accumulate the whole content, apart from losses, of $x_i(\ell, t-\ell)$.
Similar results hold true, by linearity, when taking into account the simultaneous contribution of all local states on S_t, thus making clear in what sense the conservation laws hold true when spatial diffusion processes have to be taken into account. As we can expect, the conservation laws which govern the state updating of 2D compartmental models entail interesting consequences in terms of stability properties.

Corollary 4.3. *A 2D compartmental system with state transition matrices A and B is always positively stable, and is positively asymptotically stable if and only if $\rho(A+B) < 1$.*

Proof. Since $A+B$ is substochastic, its spectral radius never exceeds 1. Moreover, as $A+B$ is cogredient to the Frobenius normal form (4.7), there cannot be chains of length greater than 1. So, both conditions of point ii) in Proposition 2.2 are met, and all 2D compartmental systems are stable. The second statement of the corollary has already been proved in Proposition 2.2. □

To conclude, we aim to solve the following problem: suppose that $w(z_1, z_2)$ is a positively realizable function, under what conditions $w(z_1, z_2)$ can be realized also by means of a 2D compartmental model? Obviously, as a consequence of Corollary 3.5 and Corollary 4.3, the variety of the singularities of $w(z_1, z_2)$ must not intersect the open unitary polydisc. This condition, however, is by no means sufficient. For instance, the rational function $w(z_1, z_2) = \frac{(1-z_1)(z_1+z_2)+z_2^2}{(1-z_1)^2}$ admits the positive realization

$$\Sigma = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right),$$

and the variety of its singularities does not intersect the open polydisc P_1. However,

$$f(z) = w(z, z) = \frac{(1-z)2z + z^2}{(1-z)^2} = \frac{2z - z^2}{(1-z)^2}$$

has a pole of multiplicity 2 at $z = 1$, and hence cannot be realized by any stable 1D system. Consequently, $w(z_1, z_2)$ cannot be realized by means of a 2D compartmental model (which should be positively stable).
The following proposition provides a sufficient condition for problem solvability.

Proposition 4.4. *Let $w(z_1, z_2) \in \mathbb{R}(z_1, z_2)$ be a strictly proper rational 2D transfer function, which is positively realizable, and let $n(z_1, z_2)/d(z_1, z_2)$ be an irreducible representation of $w(z_1, z_2)$. If $V(d) \cap P_1 = \emptyset$, then*

- *i) there exists a positive realization $\Sigma = (A, B, m, n, c^T)$ with $\rho(A + B) \leq 1$;*
- *ii) if in the Frobenius normal form of $A + B$*

$$(4.9) \quad M := P^T(A + B)P = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1r} \\ & M_{22} & & M_{2r} \\ & & \ddots & \vdots \\ & & & M_{rr} \end{bmatrix}, \qquad M_{ii} \text{ irreducible},$$

with P a permutation matrix, $\rho(M_{ii}) = 1$ implies $M_{ji} = 0$ for all $j < i$, then $w(z_1, z_2)$ can be realized via a 2D compartmental system.

Also in this case, we need two preliminary lemmas.

Lemma 4.5. *Let M be a positive $\nu \times \nu$ matrix, with $\rho(M) \leq 1$. A necessary and sufficient condition for the existence of a diagonal matrix $D = \mathrm{diag}\{d_1, d_2, \dots, d_\nu\}$, $d_i > 0$, such that $D^{-1}MD$ is substochastic, is that some vector $\mathbf{v} \gg 0$ can be found satisfying $\mathbf{v}^T M \leq \mathbf{v}^T$.*

Proof. Clearly, $D^{-1}MD$ is substochastic, i.e.,

$$[\,1 \quad 1 \quad \dots \quad 1\,](D^{-1}MD) \leq [\,1 \quad 1 \quad \dots \quad 1\,]$$

if and only if

$$[\,\tfrac{1}{d_1} \quad \tfrac{1}{d_2} \quad \dots \quad \tfrac{1}{d_\nu}\,]MD \leq [\,1 \quad 1 \quad \dots \quad 1\,],$$

or, equivalently, $[\,\frac{1}{d_1} \quad \frac{1}{d_2} \quad \dots \quad \frac{1}{d_\nu}\,]M \leq [\,\frac{1}{d_1} \quad \frac{1}{d_2} \quad \dots \quad \frac{1}{d_\nu}\,]$, which proves the result.
□

Lemma 4.6. *Let M be a positive $\nu \times \nu$ matrix, in Frobenius normal form (4.9), with $\rho(M_{ii}) \leq 1$, $i = 1, 2, \dots, r$. A necessary and sufficient condition for the existence of a* *diagonal matrix* $D = \mathrm{diag}\{d_1, d_2, \dots, d_\nu\}$, $d_i > 0$, *such that* $D^{-1}MD$ *is substochastic, is that* $\rho(M_{ii}) = 1$ *implies* $M_{ji} = 0$ *for all* $j < i$.

Proof. Assume that M is similar to a substochastic matrix, by means of a positive diagonal matrix. By the previous lemma, there exists $\mathbf{v} \gg 0$ such that $\mathbf{v}^T M \le \mathbf{v}^T$, and we can express $\mathbf{v}$, according to the block partition of M, as $\mathbf{v}^T = [\,\mathbf{v}_1^T \quad \mathbf{v}_2^T \quad \dots \quad \mathbf{v}_r^T\,]$, $\mathbf{v}_i \gg 0$. Let M_{ii}, $i > 2$, be a diagonal block with $\rho(M_{ii}) = 1$. If there would be an index $j < i$ such that $M_{ji} > 0$, then $\mathbf{v}_j^T M_{ji} > 0$ and hence $\mathbf{v}_i M_{ii} < \mathbf{v}_i^T$. But as M_{ii} is irreducible, this would imply ([**?**], pag.28) $\rho(M_{ii}) < 1$, thus contradicting the original assumption.

Conversely, suppose that corresponding to $\rho(M_{ii}) = 1$ we have $M_{ji} = 0$ for all $j < i$. It is not restrictive to assume that the diagonal blocks of M are ordered in such a way that

$$(4.10) \quad M = \left[\begin{array}{cccc|cccc} M_{11} & & & & M_{1\ell+1} & M_{1\ell+2} & \dots & M_{1r} \\ & M_{22} & & & & M_{2\ell+2} & & M_{2r} \\ & & \ddots & & & & \ddots & \\ & & & M_{\ell\ell} & & & & M_{\ell r} \\ \hline & & & & M_{\ell+1\ell+1} & M_{\ell+1\ell+2} & \dots & M_{\ell+1r} \\ & & & & & M_{\ell+2\ell+2} & & M_{\ell+2r} \\ & & & & & & \ddots & \\ & & & & & & & M_{rr} \end{array}\right],$$

with $\rho(M_{ii})$ unitary if $i = 1, 2, \dots, \ell$, and less than unitary for $i = \ell+1, \ell+2, \dots, r$. We aim to explicitly construct a strictly positive vector $\mathbf{v} = [\,\mathbf{v}_1^T \quad \mathbf{v}_2^T \quad \dots \quad \mathbf{v}_r^T\,]$ satisfying $\mathbf{v}^T M \le \mathbf{v}^T$. For each irreducible block M_{ii}, let $\bar{\mathbf{v}}_i^T \gg 0$ be a left eigenvector of M_{ii} corresponding to the spectral radius $\rho(M_{ii})$. For $i = 1, 2, \dots, \ell$, set $\mathbf{v}_i := \bar{\mathbf{v}}_i$, while for $i \ge \ell + 1$ construct vectors $\mathbf{v}_i$ by iteratively applying the following procedure:

- set $\mathbf{w}_i^T := \sum_{j=1}^{i-1} \bar{\mathbf{v}}_j^T M_{ji}$;
- consider any real number $\alpha_i > 0$ such that $\alpha_i(1 - \rho(M_{ii}))\bar{\mathbf{v}}_i^T \ge \mathbf{w}_i^T$. The existence of such an α_i is guaranteed by the fact that $\bar{\mathbf{v}}_i^T$ is strictly positive;
- assume, now, $\mathbf{v}_i^T := \alpha_i \bar{\mathbf{v}}_i^T$.

It is easy to verify that vector $\mathbf{v}$ obtained in this way, satisfies the desired condition, thus proving that M is similar to a substochastic matrix, via some positive diagonal matrix. □

Proof. i) follows immediately from Proposition 3.4.
ii) If we assume that all blocks M_{ji}, $j \ne i$, in (4.9) are zero when $\rho(M_{ii}) = 1$, then M can be described as in (4.10), with $\rho(M_{ii})$ unitary if $i = 1, 2, \dots, \ell$, and strictly smaller than 1 for $i = \ell+1, \ell+2, \dots, r$. This implies that there exists a nonsingular diagonal matrix $D > 0$ such that $D^{-1}MD$ is substochastic. But then $\big((PD)^{-1}A(PD), (PD)^{-1}B(PD), (PD)^{-1}m, (PD)^{-1}n, c^T(PD)\big)$ is a 2D compartmental model realizing $w(z_1, z_2)$. □

5. Final remarks and conclusions

In this paper, internal and external stability of 2D positive systems have been considered and the related problem of obtaining a positive stable realization for a given BIBO stable rational function analysed. The above issues have been later investigated in the special case of 2D compartmental systems, i.e. 2D positive systems with the property that their state updating matrices have a substochastic sum. A couple of examples has also been considered, enlightening concrete applications of the rich body of 2D theory in this area. A distinguishing feature, with respect to procedures based the discretization of ODEs or PDEs models, is that a first principle derivation of the discrete model is obtained, based on balance equations among different compartments.

Some theoretical results here presented have only been touched upon and deserve further investigation; in particular, a complete characterization of the spectral properties of minimal positive realizations is still lacking. Future researches should also take into account state reconstruction and feedback control, hopefully leading to satisfactory algorithms for monitoring and control of 2D positive systems.