nD polynomial matrices with applications to multidimensional signal analysis

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Abstract

In this paper, different primeness definitions and factorization properties, arising in the context of nD Laurent polynomial matrices, are investigated and applied to a detailed analysis of nD finite support signal families produced by linear multidimensional systems. As these families are closed w.r.t. linear combinations and shifts along the coordinate axes, they are naturally viewed as Laurent polynomial modules or, in a system theoretic framework, as nD finite behaviors. Correspondingly, inclusion relations and maximality conditions for finite behaviors of a given rank are expressed in terms of the polynomial matrices involved in the algebraic module descriptions.

Internal properties of a behavior, like local detectability and various notions of extendability, are finally introduced, and characterized in terms of primeness properties of the corresponding generator and parity check matrices.

1 Introduction

Since the sixties (1D) polynomial matrices have constituted a fundamental tool for investigating the dynamics of a linear system and for designing feedback control laws [1]. The behavioral approach to system analysis, introduced by J.C.Willems during the eighties [2], has fruitfully resorted to the algebra of Laurent polynomial matrices, and actually almost every notion of Willems theory has proved to mirror into a particular algebraic property of the Laurent polynomial matrices adopted for the behavior description.
Behaviors viewpoint is presently adopted in convolutional coding, where the interest is in the code produced by an encoder rather than in the machinery that underlies its generation. Moreover, it turns out to be of great relevance in failure detection, where the output signals are the only information available to check whether a system operates correctly. As a consequence, many questions connected with the realization of decoders and residual generators can be correctly answered if codewords and system signals are represented as behavior trajectories and hence as images or kernels of polynomial matrices.

In a 1D context, the polynomial matrix algebra one applies for solving the aforementioned problems is rather simple, and efficient algorithms, based on elementary transformations, allow for a complete analysis of the system dynamics.

The use of polynomial matrices in 2D system analysis and control began in the late seventies [3, ?, 5], while more recently P.Rocha and J.C.Willems [6] resorted to polynomial matrices in two variables for introducing 2D behaviors. As expected, the richer structure a family of trajectories on $\mathbb{Z} \times \mathbb{Z}$ is endowed with has a natural counterpart in the higher complexity that 2D polynomial rings and matrices exhibit in comparison with their 1D analogs.

Somehow unexpectedly, however, the transition from 2D to nD still deserves a conspicuous interest. From a mathematical point of view, when n is greater than 2 new phaenomena arise involving the primeness definitions of polynomial matrices (as pointed out by D.C.Youla in [7]), and many decomposition techniques of the 1D and 2D cases are no longer effective. In fact, at least four not equivalent notions of polynomial primeness are worth to be considered: zero-primeness, variety-primeness, minor-primeness and factor-primeness. This constitutes an important warning that nD behaviors should admit a finer description based on new internal features, which make their appearance only for $n > 2$.
Moreover, up to now no algebraic algorithm is available to check factor-primeness, thus making this property rather elusive. Last but not least, the complexity of virtually all nD algorithms represents a serious drawback when trying to get an intuition on possible solutions of open problems.

The aim of this comunication is twofold: to investigate the primeness properties and factorization results holding for nD (Laurent) polynomial matrices, and to relate the structure of nD trajectories with the algebra of the polynomial matrices involved in their generation. For sake of simplicity, we confine ourselves to families of finite support nD trajectories. This subject, indeed, seems quite interesting on its own, and provides nontrivial examples of how the hierarchy among the primeness notions reflects into a similar hierarchy among extendability properties, which express the possibility of interpolating nD signals under different operating conditions.

The paper is organized as follows. In the next section we analyse different primeness notions in a general nD context, and provide some examples showing that they are not equivalent, (in particular, that factor prime matrices are not always minor prime). Section 3 introduces the basic definitions for finite support behaviors and reduces their analysis to the theory of Laurent polynomial modules. Several factorization results for nD polynomial matrices find a natural counterpart in terms of behavior inclusions and maximality conditions.

In sections 4 and 5 trajectories recognition and signal extension problems are afforded. As it is immediately apparent, the possibility of recognizing behavior trajectories by means of a finite set of compact support parity checks is quite appealing. In algebraic terms, this happens if and only if the behavior can be described as the kernel of a polynomial matrix, a property which has been investigated in the previous sections. When a signal is not completely know or corrupted by noise, and hence satisfies the parity checks of the behavior only in certain regions of $\mathbb{Z}^n$, it is natural to look for legal trajectories that interpolate the available data on these regions. The conditions guaranteeing the existence of such interpolating trajectories depend both on the shape of the regions where parity checks are fulfilled and on the algebraic structure of the generator matrices.

2 Primeness properties of nD L-polynomial matrices

Let $\mathbb{F}$ be a field and denote by $\mathbf{z}$ the n-tuple $(z_1, z_2, ..., z_n)$ and by z_i^c the $(n-1)$-tuple $(z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_n)$, so that $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ and $\mathbb{F}[z_i^c, (z_i^{-1})^c]$ are shorthand notations for the Laurent polynomial (*L-polynomial*) rings in the indeterminates $z_1, ..., z_n$ and $z_1, ..., z_{i-1}, z_{i+1}, ..., z_n$, respectively, and $\mathbb{F}(\mathbf{z})$ denotes the field of rational functions with coefficients in $\mathbb{F}$.

A matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$ has *rank* r if it has a nonzero r-th order minor, whereas all its higher orders minors are zero. The rank of a matrix coincides with the dimensions of the $\mathbb{F}(\mathbf{z})$-spaces generated either by its rows or by its columns. By referring to the maximal order minors, we can introduce the following right-primeness notions. The analogous definitions of left-primeness (ℓP) are obvious.

Definition An L-polynomial matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$ is
• *right minor prime* (rMP) if $p \geq m$ and all the L-polynomials in the ideal $\mathcal{I}_G$, generated by its maximal order minors, are devoid of (nontrivial) common factors;
• *right variety prime* (rVP) if $p \geq m$ and the ideal $\mathcal{I}_G$ includes (nonzero) L-polynomials in $\mathbb{F}[z_i, z_i^{-1}]$, for every $i = 1, 2, ..., n$;
• *right zero prime* (rZP) if $p \geq m$ and the ideal $\mathcal{I}_G$ is the ring $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ itself. In particular, when $p = m$, a right (and hence left) zero prime matrix is called *unimodular*.

Clearly
$$\text{rZP} \quad \Rightarrow \quad \text{rVP} \quad \Rightarrow \quad \text{rMP}.$$

Moreover, when G is rMP, in every factorization $G = \bar{G}T$, with $\bar{G} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ and $T \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m\times m}$, T has to be unimodular. So, a rMP matrix is *right factor prime* (rFP), as its unique right (matrix) factors are the trivial ones.
It is well-known that in the 1D case all properties of the above definition collapse and are equivalent to right factor primeness [8], whereas for 2D L-polynomial matrices only factor, minor and variety primeness coincide [3]. When n is greater than 2, instead, none of the implications

$$\text{rZP} \;\Rightarrow\; \text{rVP} \;\Rightarrow\; \text{rMP} \;\Rightarrow\; \text{rFP}$$

can be reversed [7].

Example 1 It is a matter of simple computation to show that the matrix

$$G_1 = \begin{bmatrix} (z_1+1)(z_2+1) & 0 \\ 0 & z_2+1 \\ z_3+2 & z_1+1 \end{bmatrix} \in \mathbb{R}[z_1, z_2, z_3, z_1^{-1}, z_2^{-1}, z_3^{-1}]^{3\times 2}$$

is factor prime, as it has no right L-polynomial factors, except for the unimodular ones. However, G_1 is not minor prime as the g.c.d. of its maximal order minors is $z_2 + 1$.
Also,

$$G_2 = \begin{bmatrix} (z_1^{-1}+1)(z_2-2) \\ (z_1^{-1}+1)(z_3^2+1+z_1z_2) \\ (z_2-2)(z_2^{-2}z_1+2 \end{bmatrix} \in \mathbb{R}[z_1, z_2, z_3, z_1^{-1}, z_2^{-1}, z_3^{-1}]^{3\times 1}$$

is rMP but not rVP. Finally,

$$G_3 = \begin{bmatrix} z_1+1 \\ z_2^{-1}+1 \\ z_3+2 \end{bmatrix} \in \mathbb{R}[z_1, z_2, z_3, z_1^{-1}, z_2^{-1}, z_3^{-1}]^{3\times 1}$$

is rVP but not rZP.

When n is greater than 2, factor primeness does not reduce to a condition on the ideal $\mathcal{I}_G$ of the maximal order minors and, to our knowledge, a general algorithm for recognizing nD factor prime matrices is not available, yet.
For the other primeness properties several characterizations, based on Bézout equations, matrix completions and input/output primeness relations are available. As we shall see, rMP and rVP properties reduce to zero primeness on suitable extensions of the original Laurent polynomial ring, and therefore it seems more convenient to discuss first zero primeness and then exploit the results for the analysis of the other properties.

Proposition 2.1 *Let $G = [g_{ij}] \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$, $p > m$. The following statements are equivalent:*

i) *G is right zero prime;*

ii) *G can be column-bordered into a $p \times p$ unimodular matrix $[G \mid C]$;*

iii) *there exists a* $p \times p$ *unimodular matrix* U *such that*

$$U\ G = \begin{bmatrix} I_m \\ 0 \end{bmatrix}; \tag{2.1}$$

iv) G *has an L-polynomial left inverse, or, equivalently, satisfies the Bézout identity*

$$X\ G = I_m,$$

for some matrix $X \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m \times p}$*;*

v) *for every rZP vector* $\mathbf{u} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m$, $G\mathbf{u}$ *is rZP, too.*

Proof i) $\Rightarrow$ ii) is the Quillen-Suslin theorem [9].
ii) $\Rightarrow$ iii) follows immediately upon setting $U := [G \mid C]^{-1}$.
iii) $\Rightarrow$ iv) Obvious.
iv) $\Rightarrow$ i) follows from the Binet-Cauchy formula [10].
i) $\Rightarrow$ v) Assume that both G and $\mathbf{u}$ are rZP matrices. As i) and iv) are equivalent, there exist $X \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m \times p}$ and $\mathbf{r} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m$ such that $XG = I_m$ and $\mathbf{r}^T\mathbf{u} = 1$, and hence $(\mathbf{r}^T X)(G\mathbf{u}) = 1$. This proves that $G\mathbf{u}$ is rZP.
v) $\Rightarrow$ i) Assume, by contradiction, that $\mathcal{I}_G \neq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$. Then $\mathcal{I}_G$ is included in some maximal ideal $\mathcal{M}$, and $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]/\mathcal{M}$ is a field [11]. The matrix $\bar{G} \in (\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]/\mathcal{M})^{p \times m}$, whose (i,j)-th element is $g_{ij} + \mathcal{M}$, has not full column rank, and therefore there exists a nonzero vector $\bar{\mathbf{u}} \in (\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]/\mathcal{M})^m$ such that $\bar{G}\bar{\mathbf{u}} = 0$. It is not restrictive to assume that in

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 + \mathcal{M} \\ u_2 + \mathcal{M} \\ \vdots \\ u_m + \mathcal{M} \end{bmatrix}$$

one of the u_i's, say u_k, is 1. Consequently, the G-image of the zero prime vector $\mathbf{u} := [u_1\ u_2\ \dots\ 1\ \dots\ u_m]^T$ has all entries in $\mathcal{M}$, and hence is not zero prime. ■

Lemma 2.2 *Let* G *be a matrix in* $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$.

i) G *is rMP if and only if it is right (zero) prime in every ring* $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$, $i = 1, 2, ..., n$*;*

ii) G *is rVP if and only if it is rZP in every ring* $\mathbb{F}(z_i)[z_i^c, (z_i^{-1})^c]$, $i = 1, 2, ..., n$.

Proof i) Assume that G is a rMP matrix in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$, and denote by $pp_i(\mu_\ell)$ the primitive part of the ℓ-th maximal order minor of G, μ_ℓ, w.r.t. the ring $\mathbb{F}[z_i^c, (z_i^c)^{-1}][z_i, z_i^{-1}]$. Notice that in this ring the only common divisors of all $pp_i(\mu_\ell)$ are units. On the other hand, set $h = \text{g.c.d.}_{\mathbb{F}(z_i^c)[z_i, z_i^{-1}]}\{pp_i(\mu_\ell); \ell = 1, 2, ..., \binom{p}{m}\}$ and denote by d the product of all denominators of the coefficients of h. Then, by applying Theorem 2.69 of [12] to the ring $\mathbb{F}[z_i^c,(z_i^c)^{-1}][z_i,z_i^{-1}]$, we have that $pp_i(hd)$ is a g.c.d. of $pp_i(\mu_\ell), \ell = 1,2,...,\binom{p}{m}$ in $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$, and hence is 1. Therefore, $h \in \mathbb{F}(z_i^c)$, which implies that g.c.d.$_{\mathbb{F}(z_i^c)[z_i,z_i^{-1}]}\{\mu_\ell\}$ is an element of $\mathbb{F}(z_i^c)$. Consequently, G is right prime in $\mathbb{F}(z_i^c)[z_i,z_i^{-1}]$, $i = 1,2,...,n$.

Conversely, assume that G is right prime w.r.t. $\mathbb{F}(z_i^c)[z_i,z_i^{-1}]$, $i = 1,2,...,n$, and suppose that all elements in $\mathcal{I}_G$, and, in particular, all maximal order minors of G, have a nonunit common divisor $g \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$. Then g is not a unit in $\mathbb{F}(z_i^c)[z_i,z_i^{-1}]$ for some i, which contradicts the assumption.

The proof of ii) follows the same lines. ■

Proposition 2.3 *Let $G \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times m}$ be a full column rank matrix. The following facts are equivalent:*

i) *G is right minor prime;*

ii) *for $i = 1,2,...,n$ there exist matrices $C_i \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times(p-m)}$ and nonzero L-polynomials $c_i \in \mathbb{F}[z_i^c,(z_i^{-1})^c]$ such that*

$$\det[\ G \mid C_i\ ] = c_i; \tag{2.2}$$

iii) *for $i = 1,2,...,n$ there exist matrices $U_i \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times p}$ and nonzero L-polynomials $b_i \in \mathbb{F}[z_i^c,(z_i^{-1})^c]$ such that*

$$U_iG = b_i \begin{bmatrix} I_m \\ 0 \end{bmatrix}; \tag{2.3}$$

iv) *for $i = 1,2,...,n$ there exist matrices $H_i \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{m\times p}$ and nonzero L-polynomials $\psi_i \in \mathbb{F}[z_i^c,(z_i^{-1})^c]$ such that*

$$H_iG = \psi_i I_m; \tag{2.4}$$

v) *for every rMP vector $\mathbf{u} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^m$, also $G\mathbf{u}$ is rMP;*

vi) *for every $\mathbf{u} \in \mathbb{F}(z)^m$, $G\mathbf{u} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ implies $\mathbf{u} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^m$.*

Proof Noting that

- $\det[\ G \mid C_i\ ] = c_i \in \mathbb{F}[z_i^c,(z_i^{-1})^c]$, $c_i \neq 0$, means that $[\ G \mid C_i\ ]$ is unimodular in $\mathbb{F}(z_i^c)[z_i,z_i^{-1}]$;
- (2.3) is equivalent to assume that $\tilde{U}_iG = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$, for some $\tilde{U}_i$, unimodular in $\mathbb{F}(z_i^c)[z_i,z_i^{-1}]$;
- (2.4) corresponds to the existence of a left inverse of G in $\mathbb{F}(z_i^c)[z_i,z_i^{-1}]$,

the equivalence of i) ÷ v) follows from Lemma 2.2 and Proposition 2.1.

v) $\Rightarrow$ *vi)* Suppose, by contradiction, that there exists $\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ which is expressed as $\mathbf{w} = G\mathbf{u}$, for some $\mathbf{u} \in \mathbb{F}(\mathbf{z})^m \setminus \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^m$. It entails no loss of generality assuming that $\mathbf{u} = \frac{n}{d}\,\bar{\mathbf{u}}$, for some rFP (and hence, being a vector, rMP) vector $\bar{\mathbf{u}} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^m$ and $n,d \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$ nonzero factor coprime L-polynomials, with d not a unit.

So, $G(n\bar{\mathbf{u}}) = d\mathbf{w}$, and as n and d are coprime, it follows that $\mathbf{w} = n\bar{\mathbf{w}}$ for some L-polynomial vector $\bar{\mathbf{w}}$. Consequently, the matrix G, when applied to the rMP vector $\bar{\mathbf{u}}$, produces a not rMP L-polynomial vector $d\bar{\mathbf{w}}$, thus contradicting assumption v).

vi) $\Rightarrow$ *v)* If there would be a rFP (rMP) vector $\mathbf{u}$ such that $G\mathbf{u} = d\mathbf{w}$, with d a nonunit L-polynomial and $\mathbf{w} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, then we would have $G\frac{1}{d}\mathbf{u} = \mathbf{w}$, which contradicts assumption vi). ∎

By resorting, again, to Proposition 2.1 and Lemma 2.2 and by adopting the same reasonings as in the proof of the above proposition, one gets the following result.

Proposition 2.4 *Let $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ be a full column rank matrix. The following facts are equivalent:*

i) *G is right variety prime;*

ii) *for $i = 1, 2, ..., n$ there exist matrices $H_i \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m\times p}$ and nonzero L-polynomials $\psi_i(z_i) \in \mathbb{F}[z_i, z_i^{-1}]$ such that*
$$H_i G = \psi_i(z_i) I_m; \tag{2.4}$$

iii) *for $i = 1, 2, ..., n$ there exist matrices $U_i \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times p}$ and nonzero L-polynomials $b_i \in \mathbb{F}[z_i, z_i^{-1}]$ such that*
$$U_i G = b_i \begin{bmatrix} I_m \\ 0 \end{bmatrix};$$

iv) *for $i = 1, 2, ..., n$ there exist matrices $C_i \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times(p-m)}$ and nonzero L-polynomials $c_i \in \mathbb{F}[z_i, z_i^{-1}]$ such that*
$$\det[\ G \mid C_i\ ] = c_i;$$

v) *for every $\mathbf{u} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m$ rVP, also $G\mathbf{u}$ is rVP.* ∎

3 Finite support nD behaviors

Given any L-polynomial matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$, the $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$-module generated by its columns can be naturally viewed as a family of finite support nD signals with p components, closed w.r.t. linear combinations and shifts along the coordinate axes. To this end one must associate every L-polynomial vector
$$\mathbf{w} = \sum_{\mathbf{h}\in\mathbb{Z}^n} w(\mathbf{h})\ \mathbf{z}^{\mathbf{h}}, \tag{3.1}$$
where $\mathbf{z}^h$ denotes the term $z_1^{h_1} z_2^{h_2} ... z_n^{h_n}$, with the nD sequence $\{w(\mathbf{h})\}_{\mathbf{h}\in\mathbb{Z}^n}$.
Conversely, every linear and shift-invariant family $\mathcal{B}$ of nD trajectories is a Noetherian $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$-module and hence can be represented as the image of some L-polynomial matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$, i.e.
$$\mathcal{B} = \mathrm{Im}G := \{\mathbf{w} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p : \mathbf{w} = G\mathbf{u}, \mathbf{u} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m\}.$$

We will call $\mathcal{B}$ an nD (finite support) *behavior* and G a *generator matrix* of $\mathcal{B}$.

From an abstract point of view nD behaviors with p components coincide with the submodules of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, and we can expect that most of the internal features of nD behaviors mirror into specific algebraic properties of the associated generator matrices.

As in this contribution behaviors will always be assumed "finite support", when no confusion arises this attribute will be dropped.

It is easy to realize that generator matrices of a same behavior $\mathcal{B}$ have the same rank over the field $\mathbb{F}(z)$. Actually, if G_1 and G_2 are two such matrices, both $G_1 = G_2P_2$ and $G_2 = G_1P_1$ have to hold true for appropriate P_i, $i = 1, 2$, with entries in $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}] \subset \mathbb{F}(\mathbf{z})$. Consequently, both the inequalities $\text{rank}G_1 \leq \text{rank}G_2$ and $\text{rank}G_2 \leq \text{rank}G_1$ are satisfied, which proves $\text{rank}G_1 = \text{rank}G_2$. As a consequence, it is meaningful to define *rank* of $\mathcal{B}$ the rank of anyone of its generator matrices.

The set of all behaviors $\mathcal{B}$ of rank r in $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, partially ordered w.r.t. the inclusion, will be denoted by $\mathcal{G}(p,r)$. The rank r of a behavior $\mathcal{B}$ somehow represents its complexity index, as r independent trajectories can be found in $\mathcal{B}$, while every $r+1$-tuple of trajectories $(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_{r+1})$ satisfies an autoregressive equation

$$p_1\mathbf{w}_1 + p_2\mathbf{w}_2 + ... + p_{r+1}\mathbf{w}_{r+1} = 0,$$

where not every $p_i \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$ is zero.

A behavior $\mathcal{B}$ is *free* if it can be represented as the image of some full column rank L-polynomial matrix. The set of all free behaviors of rank r in $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, partially ordered w.r.t. the inclusion, will be denoted by $\mathcal{F}(p,r)$.
While in the 1D case every behavior is free [11], nD behaviors, $n \geq 2$, are not necessarily free.

Example 2 Let $\mathcal{B}$ be the behavior of rank 1 generated by the L-polynomial matrix

$$G = \begin{bmatrix} z_1+1 & z_2^{-1}+1 \\ z_1^{-1}-1 & (z_1-1)(z_2^{-1}+1) \end{bmatrix}.$$

If $\mathcal{B}$ were free, there would be a column vector $\bar{\mathbf{g}} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^2$ which generates $\mathcal{B}$ and, in particular, both columns of G. But then, $\bar{\mathbf{g}}$ should differ from $[1 \mid z_1 - 1]^T$ in a nonzero monomial, and hence could not be an element of $\mathcal{B}$.

Proposition 3.1 *Let $\mathcal{B}$ be an element of $\mathcal{G}(p,r)$ and let $G \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times m}$ be any generator matrix of $\mathcal{B}$. $\mathcal{B}$ is free if and only if G factors as*

$$G = \bar{G}\, T, \tag{3.2}$$

*for suitable L-polynomial matrices $\bar{G}$ and T, with $\bar{G}$ full column rank and T ℓZP.
Moreover, $\mathcal{B}$ admits a rFP (rMP/rVP/rZP) generator matrix if and only if in (3.2) $\bar{G}$ is rFP (rMP/rVP/rZP).*

Proof Assume that $G = \bar{G}T$ for some full column rank L-polynomial matrix $\bar{G}$ and some ℓZP matrix $T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{r\times m}$. Obviously, $\text{Im}G \subseteq \text{Im}\bar{G}$. On the other hand, if T^{-1} is a right L-polynomial inverse of T, then $\bar{G} = G\,T^{-1}$, which implies Im$\bar{G} \subseteq$ ImG. So, $\mathcal{B}$ is free, since it is generated by the full column rank matrix $\bar{G}$.

Conversely, if $\mathcal{B}$ is free, it admits a full column rank generator matrix $\bar{G}$. Consequently, there exist two L-polynomial matrices of suitable dimensions, T and $\hat{T}$, such that $\bar{G}T = G$ and $G\hat{T} = \bar{G}$. Therefore $\bar{G} = G\hat{T} = \bar{G}T\hat{T}$. As $\bar{G}$ has full column rank, $T\hat{T} = I_r$, which implies that T is an ℓZP matrix.

The proof of the remaining part follows the same lines. ■

When $\mathcal{B}$ is a free behavior and a full column rank generator matrix is available, it is possible to give a complete parametrization of the generator matrices of $\mathcal{B}$, as shown in the following corollary.

Corollary 3.2 *Let $\mathcal{B}$ be an element of $\mathcal{F}(p,r)$ and let $\bar{G} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times r}$ be a full column rank generator matrix of $\mathcal{B}$. The set of all generator matrices of $\mathcal{B}$ is given by*

$$\{G = \bar{G}T : T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{r\times m}\ell\mathrm{ZP}, m \in \mathbb{N}\}. \qquad \blacksquare$$

Given any element $\mathcal{B}$ in $\mathcal{F}(p,r)$, we may wonder whether $\mathcal{B}$ is a maximal element of $\mathcal{F}(p,r)$ and, in case it is not, what are the maximal elements of $\mathcal{F}(p,r)$ that (properly) include $\mathcal{B}$. The answers to these questions depend upon the structure of the generator matrices of $\mathcal{B}$ as clarified by the following proposition.

Proposition 3.3 *Let $\mathcal{B}$ be an element of $\mathcal{F}(p,r)$ and let G be a full column rank generator matrix of $\mathcal{B}$.*

i) $\mathcal{B}$ is a maximal element of $\mathcal{F}(p,r)$ if and only if G is rFP.

If $\mathcal{B}$ is not maximal

ii) $\bar{\mathcal{B}} =$ Im$\bar{G}$ is a maximal element of $\mathcal{F}(p,r)$ including $\mathcal{B}$ if and only if $\bar{G}$ is rFP and $G = \bar{G}T$ for some nonsingular L-polynomial matrix T;

iii) if in

$$G = \bar{G}T, \qquad \bar{G}\ \mathrm{rFP} \qquad T \text{ nonsingular L} - \text{polynomial matrix}, \tag{3.3}$$

$\bar{G}$ is rMP, then the factorization (3.3) is essentially unique, i.e. all the $\bar{G}$'s satisfying (3.3) differ in a right unimodular factor, and the maximal element of $\mathcal{F}(p,r)$ including $\mathcal{B}$ is unique.

Proof i) Assume that G is rFP, and suppose that $\hat{\mathcal{B}} :=$ Im$\hat{G}$, $\hat{G} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times r}$ a full column rank matrix, is a free behavior including $\mathcal{B}$. Then $G = \hat{G}T$, for some nonsingular L-polynomial matrix $T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{r\times r}$. As G is a rFP matrix, T has to be unimodular and hence $\hat{\mathcal{B}} =$ Im$\hat{G} \equiv$ Im$G = \mathcal{B}$. This implies that $\mathcal{B}$ is maximal.

Conversely, suppose that $\mathcal{B}$ is a maximal element of $\mathcal{F}(p,r)$. If G were not rFP, it would factor as $G = \bar{G}T$, for suitable L-polynomial matrices $\bar{G}$ and T, with $T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{r\times r}$ not unimodular, and therefore $\bar{\mathcal{B}} :=$ Im$\bar{G}$ would properly include $\mathcal{B}$, a contradiction.

ii) If $\bar{\mathcal{B}}$ has a rFP generator matrix $\bar{G}$, by i) it is a maximal behavior in $\mathcal{F}(p,r)$. Moreover as $\bar{G}$ satisfies $G = \bar{G}T$, $\bar{\mathcal{B}}$ includes $\mathcal{B}$. The converse can be proved in the same way.

iii) Suppose that in (3.3) $\bar{G}$ is a rMP matrix, and let $G = \bar{G}_1 T_1$, be another factorization of G like (3.3). As $\bar{G}_1 = \bar{G}(TT_1^{-1})$, the rMP property of $\bar{G}$ and the fact that all columns of $\bar{G}_1$ are L-polynomial vectors imply (cfr. Proposition 2.3) that $U := TT_1^{-1}$ is an L-polynomial matrix. But as $\bar{G}_1$ is a rFP matrix, U has to be unimodular, which proves the uniqueness of the factorization. The uniqueness of the maximal behavior of $\mathcal{F}(p,r)$ including $\mathcal{B}$ follows immediately. ■

Example 3 The 3×2 L-polynomial matrix

$$G = \begin{bmatrix} (z_1+1)(z_2+1)^2 & -(z_1+1)^2(z_2+1) \\ 0 & (z_2+1)(z_3+2) \\ (z_2+1)(z_3+2) & 0 \end{bmatrix}$$

exhibits two different factorizations

$$G = \bar{G}_1 T_1 = \bar{G}_2 T_2, \tag{3.4}$$

where

$$\bar{G}_1 = \begin{bmatrix} (z_1+1)(z_2+1) & 0 \\ 0 & z_2+1 \\ z_3+2 & z_1+1 \end{bmatrix} \quad \text{and} \quad \bar{G}_2 = \begin{bmatrix} (z_1+1)(z_2+1) & -(z_1+1)^2 \\ 0 & (z_3+2) \\ (z_3+2) & 0 \end{bmatrix},$$

are rFP matrices and

$$T_1 = \begin{bmatrix} z_2+1 & -(z_1+1) \\ 0 & z_3+2 \end{bmatrix} \qquad T_2 = \begin{bmatrix} z_2+1 & 0 \\ 0 & z_2+1 \end{bmatrix},$$

are full row rank L-polynomial matrices. $\bar{G}_1$ and $\bar{G}_2$ do not differ in a unimodular right factor, and (3.4) is the only way they can be related.
As a consequence, $\mathcal{B}$ is included in two different maximal free behaviors of rank 2, i.e. $\bar{\mathcal{B}}_1 = \mathrm{Im}\bar{G}_1$ and $\bar{\mathcal{B}}_2 = \mathrm{Im}\bar{G}_2$.

Remark Generally, when $\mathcal{B}$ is an element of $\mathcal{F}(p,r)$ which is not maximal, there is more than one maximal free behavior in $\mathcal{F}(p,r)$ including $\mathcal{B}$. This situation does not arise when the number n of the indeterminates is less than three, as in this case factor primeness and minor primeness are equivalent properties, and therefore every full column rank L-polynomial matrix has an essentially unique factorization (3.3).

The general problem of embedding into a maximal free behavior of rank r an arbitrary (not necessarily free) behavior $\mathcal{B} \in \mathcal{G}(p,r)$ is always solvable for $n \leq 2$. Actually, every $p \times m$ L-polynomial matrix G of rank r, in one or two indeterminates, can be expressed in essentially a unique way as

$$G = \bar{G}T,$$

where $\bar{G}$ is a $p \times r$ rMP matrix, while T is an $r \times m$ L-polynomial matrix with full row rank. The procedure for obtaining such a factorization is the following: select in G a $p \times r$ submatrix $\hat{G}$, with full column rank, and write G as

$$G = \hat{G}Q,$$

Q an $r \times m$ rational matrix (of rank r). By extracting a greatest right factor of $\hat{G}$, represent G as

$$G = \bar{G}T, \tag{3.7}$$

with $\bar{G}$ rFP, and hence rMP, and T a rational matrix. By Proposition 2.3, T is L-polynomial. The uniqueness of the factorization can be proved as in Proposition 3.3, part iii).

As a consequence of this factorization, every 1D or 2D behavior can be embedded into a unique maximal free behavior of the same rank, and all maximal behaviors in $\mathcal{G}(p, r)$ are free. When n is greater than 2, however, examples can be given of matrices $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$ of rank $r < m$ which cannot factor as $G = \bar{G}T$, with $\bar{G} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times r}$ and $T \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{r \times m}$. This makes it clear that there are behaviors in $\mathcal{G}(p, r) \setminus \mathcal{F}(p, r)$ which cannot be embedded into any free behavior of rank r.

Example 4 Let $\mathcal{B}$ be the behavior generated by the 3×2 L-polynomial matrix

$$\begin{aligned} G &= \begin{bmatrix} (z_1+1)^2 & (z_1+1)(z_2+1) & 0 \\ -(z_3+2) & 0 & z_2+1 \\ 0 & z_3+2 & z_1+1 \end{bmatrix} \\ &= \begin{bmatrix} (z_1+1)(z_2+1) & 0 \\ 0 & z_2+1 \\ z_3+2 & z_1+1 \end{bmatrix} \begin{bmatrix} (z_1+1)/(z_2+1) & 1 & 0 \\ (-z_3-2)/(z_2+1) & 0 & 1 \end{bmatrix} =: G_1T_1. \end{aligned}$$

As already seen in Example 1, G_1 is rFP but not rMP, since the g.c.d. of its maximal order minors is z_2+1. If G could be expressed as $G = \bar{G}T$, for suitable matrices $\bar{G} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{3 \times 2}$ and $T \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{2 \times 3}$, $\bar{G}$ should be rMP, as the g.c.d. of the 2nd order minors of G is 1. But $G_1T_1 = \bar{G}T$ would imply $G_1 = \bar{G}(TT_1^{-1})$, T_1^{-1} a rational right inverse of T_1. Since $\bar{G}$ is right minor prime, point vi) of Proposition 2.3 implies that TT_1^{-1} is L-polynomial, and hence unimodular, otherwise G_1 would have a nontrivial right factor. Thus G_1 should be rMP, a contradiction.
This shows that $\mathcal{B}$ cannot be embedded into a free behavior of rank 2.

Once we extend our investigation to the whole set $\mathcal{G}(p, r)$, and look for the maximal elements of $\mathcal{G}(p, r)$ which include a given behavior $\mathcal{B}$ of rank r, we obtain very neat results, as every $\mathcal{B}$ can be embedded into a unique maximal behavior. In order to characterize the maximal elements of $\mathcal{G}(p, r)$, we introduce the notions of orthogonal $\mathcal{B}^\perp$ and of rational envelope $\mathcal{B}_{\text{rat}}$ of a behavior $\mathcal{B}$.

Definition Given $\mathcal{B} \subset \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ its *orthogonal behavior* is the module

$$\mathcal{B}^\perp := \{\mathbf{p} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p : \mathbf{p}^T\mathbf{w} = \mathbf{0}, \forall \mathbf{w} \in \mathcal{B}\}. \tag{3.5}$$

As a submodule of the Noetherian module $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, $\mathcal{B}^\perp$ can be represented as the image of a matrix $H \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times q}$, for some $q \in \mathbb{N}$, namely $\mathcal{B}^\perp = \mathrm{Im}H$. Condition $\mathbf{p}^T\mathbf{w} = 0$, $\forall\ \mathbf{p} \in \mathcal{B}^\perp$, however, does not imply that $\mathbf{w}$ belongs to $\mathcal{B}$, and in general, the module

$$\mathcal{B}^{\perp\perp} := \{\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p : \mathbf{p}^T\mathbf{w} = 0, \forall\ \mathbf{p} \in \mathcal{B}^\perp\}$$

properly includes $\mathcal{B}$.

Definition Given a behavior $\mathcal{B} \subset \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ its *rational envelope* is the subspace of $\mathbb{F}(\mathbf{z})^p$

$$\mathcal{B}_{\mathrm{rat}} := \Big\{\sum_{i=1}^{r} \mathbf{w}_i a_i : \mathbf{w}_i \in \mathcal{B},\ a_i \in \mathbb{F}(\mathbf{z}),\ r \in \mathbb{N}\Big\}. \tag{3.6}$$

We are now in a position to provide a complete characterization of the maximal elements of $\mathcal{G}(p,r)$.

Proposition 3.4 *Let $\mathcal{B}$ be an element of $\mathcal{G}(p,r)$. The following statements are equivalent:*
(1) $\mathcal{B}$ is a maximal element of $\mathcal{G}(p,r)$;
(2) $s\mathbf{w} \in \mathcal{B}\ \Rightarrow\ \mathbf{w} \in \mathcal{B}$, for every $\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ and every nonzero $s \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$;
(3) $\mathcal{B} = \mathcal{B}^{\perp\perp}$;
(4) $\mathcal{B} = \ker H^T$, for some $H^T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{q\times p}$;
(5) $\mathcal{B} \equiv \mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$.

Proof (1) $\Rightarrow$ (2) Suppose $s\mathbf{w} \in \mathcal{B}$, $s \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$. The behavior $\mathcal{B}'$ generated by $\mathcal{B}$ and $\mathbf{w}$ has the same rank of $\mathcal{B}$, and hence, by the maximality assumption, coincides with $\mathcal{B}$.
(2) $\Rightarrow$ (3) As $\mathcal{B}$ and $\mathcal{B}^{\perp\perp}$ have the same rank r and $\mathcal{B}^{\perp\perp} \supseteq \mathcal{B}$, both behaviors generate the same $\mathbb{F}(\mathbf{z})$-subspace of $\mathbb{F}(\mathbf{z})^p$. In particular, $\mathbf{w} \in \mathcal{B}^{\perp\perp}$ implies $\mathbf{w} \in (\mathcal{B}^{\perp\perp})_{\mathrm{rat}} = \mathcal{B}_{\mathrm{rat}}$. So, there exist $p_i, s_i \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$ and $\mathbf{w}_i \in \mathcal{B}$, such that $\mathbf{w} = \sum_{i=1}^r \mathbf{w}_i\ p_i/s_i$, which implies $s\mathbf{w} \in \mathcal{B}$, $s = \ell.\mathrm{c.m.}\{s_i\}$. By assumption (2), also $\mathbf{w}$ is in $\mathcal{B}$.
(3) $\Rightarrow$ (4) As $\mathcal{B}^\perp = \mathrm{Im}H$ for some $H \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times q}$, we have $\mathcal{B} \equiv \mathcal{B}^{\perp\perp} := (\mathcal{B}^\perp)^\perp = \ker H^T$.
(4) $\Rightarrow$ (5) The inclusion $\mathcal{B} \subseteq \mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ is obvious. To show the converse inclusion, let $\mathbf{w} = \sum_i a_i\mathbf{w}_i$, with $a_i \in \mathbb{F}(\mathbf{z})$ and $\mathbf{w}_i \in \mathcal{B}$, be an element of $\mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$. Clearly $H^T\mathbf{w} = \sum_i a_i(H^T\mathbf{w}_i) = 0$, which shows that $\mathbf{w}$ is in $\mathcal{B}$.
(5) $\Rightarrow$ (1) If $\mathcal{B}' \supseteq \mathcal{B}$ and $\mathrm{rank}\mathcal{B}' = \mathrm{rank}\mathcal{B}$, it is clear that $\mathcal{B}$ and $\mathcal{B}'$ generate the same $\mathbb{F}(\mathbf{z})$-subspace of $\mathbb{F}(\mathbf{z})^p$ and, consequently, $\mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p = \mathcal{B}'_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$. So, the inclusions chain $\mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p \supseteq \mathcal{B}' \supseteq \mathcal{B}$ and assumption (5) together imply $\mathcal{B}' = \mathcal{B}$, which means that $\mathcal{B}$ is maximal. ∎

By the above proposition, every behavior $\mathcal{B}$ in $\mathcal{G}(p,r)$ is included in a behavior, $\mathcal{B}^{\perp\perp}$, which is maximal in $\mathcal{G}(p,r)$ and is uniquely determined by $\mathcal{B}$. If G denotes any generator matrix of $\mathcal{B}$, $\mathcal{B}^{\perp\perp}$ can be represented as the kernel of any L-polynomial matrix H of rank $p-r$ satisfying $H^TG = 0$, or, equivalently, as $\mathcal{B}^{\perp\perp} = \mathrm{Im}_{\mathbb{F}(\mathbf{z})}G \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$.
Maximal elements of $\mathcal{G}(p,r)$ which are free are identified in the following corollary.

Corollary 3.5 *Let $\mathcal{B} \in \mathcal{G}(p,r)$ be a free behavior. $\mathcal{B}$ is a maximal element of $\mathcal{G}(p,r)$ if and only if $\mathcal{B} = \mathrm{Im}G$ for some rMP matrix G.*

Proof Let G be a full column rank generator matrix of $\mathcal{B}$. By the above proposition, $\mathcal{B}$ is maximal if and only if

$$\mathrm{Im}_{\mathbb{F}(\mathbf{z})} G \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p = \mathrm{Im} G, \tag{3.7}$$

and, by the equivalence i) $\Leftrightarrow$ vi) of Proposition 2.3, (3.7) holds if and only if G is rMP. ■

The main features of maximal nD behaviors in $\mathcal{G}(p, r)$ can be summarized as follows

- a maximal element of $\mathcal{G}(p, r)$ is free if and only if it admits a rMP generator matrix;
- if $n \leq 2$ all maximal elements of $\mathcal{G}(p, r)$ are free and hence have rMP generator matrices;
- if $n > 2$ maximal elements of $\mathcal{G}(p, r)$ can be found which are not free.

4 Parity checks and trajectories recognition

As mentioned in the Introduction, finite support nD behaviors can be viewed as families of trajectories of multidimensional systems or, in a communication context, as convolutional codes produced by some finite state linear sequential machine. Recognizing legal trajectories/codewords is a basic issue in failure detection and error detection. This problem can be managed by resorting to a linear filter (residual generator or syndrome former) that produces an identically zero output when the input is an admissible trajectory of the behavior $\mathcal{B}$.

From a mathematical standpoint, this requires to find a set of sequences endowed with the property that their convolution with the elements of $\mathcal{B}$ (and those only) is zero. Such a set obviously exists, as, for instance, the algebraic dual $\mathcal{B}^*$ of $\mathcal{B}$ always satisfies the above mentioned conditions. When resorting to $\mathcal{B}^*$, however, we generally have to use also infinite support sequences, which are not convenient from an algorithmic point of view. So, it is natural to look for conditions guaranteeing that an unambiguous decision on a given trajectory can be taken by using only parity checks represented by compact support sequences.

For a given behavior $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, a *compact support parity check* is a vector $\mathbf{p} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ that satisfies $\mathbf{p}^T \mathbf{w} = 0$, $\forall \mathbf{w} \in \mathcal{B}$. So, if G is any $p \times m$ generator matrix of $\mathcal{B}$, the set of all the compact support parity checks (in algebraic terms, the module of the syzygies corresponding to the row module of G) is $\mathcal{B}^{\perp}$.
It is clear that the the trajectories of $\mathcal{B}$ can be recognized by means of a finite set of (compact support) parity checks if and only if $\mathcal{B}$ coincides with $\mathcal{B}^{\perp\perp}$ or, equivalently, $\mathcal{B}$ is the kernel of some L-polynomial matrix H^T, whose rows constitute, therefore, a complete set of parity checks for $\mathcal{B}$.

As we will see, the kernel representation corresponds to the possibility of giving a bound on the size of the windows one has to look at when deciding whether a signal belongs to a given behavior $\mathcal{B}$, and it expresses a sort of "localization" of the system laws.

Denoting by $\mathcal{B}|\mathcal{S} := \{\mathbf{w}|\mathcal{S} : \mathbf{w} \in \mathcal{B}\}$ the set of all restrictions to $\mathcal{S} \subset \mathbb{Z}^n$ of behavior trajectories, the above localization property can be formalized as follows:

(LD) [Local-detectability] *A finite behavior $\mathcal{B}$ is* locally-detectable *if there is an integer $\nu > 0$ such that every signal $\mathbf{w}$ satisfying $\mathbf{w}|\mathcal{S} \in \mathcal{B}|\mathcal{S}$ for every $\mathcal{S} \subset \mathbb{Z}^n$ with $\mathrm{diam}(\mathcal{S}) \leq \nu$, is in $\mathcal{B}$.*

The equivalence between local detectability and kernel description is formally stated in the following proposition.

Proposition 4.1 *Let $\mathcal{B}$ be an element of $\mathcal{G}(p,r)$. $\mathcal{B}$ is locally detectable if and only if there exist $h \in \mathbb{N}$ and an L-polynomial matrix $H^T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{h\times p}$ s.t.*

$$\mathcal{B} = \ker H^T := \{\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p : H^T\mathbf{w} = 0\}. \tag{4.1}$$

Proof Let $\mathcal{B} = \mathrm{Im} G, G \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times m}$, be a locally detectable behavior and assume, by contradiction, that it is not maximal in $\mathcal{G}(p,r)$. Then there exist a behavior $\mathcal{B}'$, maximal in $\mathcal{G}(p,r)$, which properly includes $\mathcal{B}$, and a sequence $\mathbf{w}' \in \mathcal{B}' \setminus \mathcal{B}$. By Proposition 3.4, $\mathbf{w}'$ can be expressed as $\mathbf{w}' = G\mathbf{q}$, for some rational sequence, whereas no L-polynomial sequence $\mathbf{v}$ satisfies $\mathbf{w}' = G\mathbf{v}$.
Let $\delta > 0$ be the radius of a ball, $B(0,\delta)$, centered in the origin, such that $B(0,\delta) \supset \mathrm{supp}(G)$. For any finite window $\mathcal{S}$ the (finite) sequence $\mathbf{u}^{(\mathcal{S})}$, obtained by considering those coefficients of a power series expansion of $\mathbf{q}$ that correspond to the terms in

$$\mathcal{S}^\delta := \{\mathbf{h} \in \mathbb{Z}^n : d(\mathbf{h},\mathcal{S}) \leq \delta\}, \tag{4.2}$$

satisfies

$$\mathbf{w}'|\mathcal{S} = (G\mathbf{q})|\mathcal{S} \equiv (G\mathbf{u}^{(\mathcal{S})})|\mathcal{S} \ \in \mathcal{B}|\mathcal{S},$$

and therefore, by the local detectability assumption, $\mathbf{w}'$ should be in $\mathcal{B}$, a contradiction.
Assume $\mathcal{B} = \ker\, H^T$, for some L-polynomial matrix H^T and let ν be the diameter of a ball, centered in the origin, which includes the support of H^T. It is clear that $\mathbf{w} \in \mathcal{B}$ if and only if the coefficient of $\mathbf{z}^{\mathbf{i}}$ in $H^T\mathbf{w}$, $(H^T\mathbf{w}, \mathbf{z}^{\mathbf{i}})$, is zero for all $\mathbf{i} \in \mathbb{Z}^n$ and hence if and only if

$$\mathbf{w} \mid \mathbf{i} - \mathrm{supp}(H^T) \in \mathcal{B} \mid \mathbf{i} - \mathrm{supp}(H^T), \qquad \forall\, \mathbf{i} \in \mathbb{Z}^n,$$

where $\mathbf{i} - \mathrm{supp}(H^T) := \{\mathbf{i} - \mathbf{j} : \mathbf{j} \in \mathrm{supp}(H^T)\}$. This way the local detectability of $\mathcal{B}$ is proved. ■

The above proposition identifies locally detectable behaviors as kernels of L-polynomial matrices. If we consider the image representations, however, and try to distinguish locally detectable behaviors from their generator matrices, we can obtain a complete characterization only in the case of free behaviors.

Proposition 4.2 *Let $\mathcal{B}$ be an element of $\mathcal{G}(p,r)$. If $\mathcal{B}$ is locally detectable, the r-th order minors of any generator matrix G of $\mathcal{B}$ are devoid of common factors. In particular, if $\mathcal{B}$ is free, $\mathcal{B}$ is locally detectable if and only if it admits a rMP generator matrix.*

Proof Let $\mathcal{B}$ be a locally detectable behavior and let $G \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times m}$ be any generator matrix of $\mathcal{B}$. Suppose that $U_i \in \mathbb{F}(z_i^c)[z_i, z_i^{-1}]^{p\times p}$ is a unimodular matrix in the P.I.D.

$\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$ that reduces G to its Hermite form w.r.t. $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$, that is

$$U_i\ G = \begin{bmatrix} \Delta_i \\ 0 \end{bmatrix} \begin{matrix} \}r \\ \}p-r \end{matrix}\ ,$$

with $\Delta_i \in \mathbb{F}(z_i^c)[z_i, z_i^{-1}]^{r\times m}$ a full row rank matrix. There is no losss of generality assuming that U_i has an L-polynomial inverse V_i. So, we get the identity

$$G = V_i \begin{bmatrix} \Delta_i \\ 0 \end{bmatrix} = \tilde{V}_i\ \Delta_i,$$

where $\tilde{V}_i$ is an L-polynomial matrix, right prime in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$, and hence the g.c.d. of its maximal order minors is a unit in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$.
As Δ_i has full row rank, it has a (rational) right inverse Δ_i^{-1}, and one has

$$G\Delta_i^{-1} = \tilde{V}_i. \tag{4.3}$$

As $\mathcal{B} = \text{Im } G$ is locally detectable, $\mathcal{B} = \mathcal{B}_{\text{rat}} \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, and therefore an L-polynomial matrix Φ_i can be found such that

$$G\ \Phi_i = \tilde{V}_i.$$

Clearly, the g.c.d. of the $r\times r$ minors of G divides the g.c.d. of the maximal order minors of $\tilde{V}_i$, and hence is a unit in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$. But this holds true for every $i = 1, 2, ..., n$, and therefore the g.c.d. of the $r \times r$ minors of G is a unit in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$.
The second part follows immediately from Corollary 3.5. ■

Remark The implication in the first part of Proposition 4.2 cannot be reversed. Actually, consider a behavior $\mathcal{B}$ in $\mathcal{G}(p, r)$ generated by some $p\times m$ matrix G, that factors as $G = \bar{G}T$, with $\bar{G}$ a rMP matrix and T a ℓMP matrix, which is not ℓZP. Clearly, the r-th order minors of G are devoid of common factors. However, ImG is not locally detectable, as it is properly included in Im $\bar{G}$ and hence is not maximal in $\mathcal{G}(p, r)$.

5 Signal extension

In many situations the available data represent just a portion of the complete trajectory and are corrupted by noise and system failures. As a consequence, the parity checks, in general, give a positive answer only on some region $\mathcal{S}$ of $\mathbb{Z}^n$, and it is natural to look for a legal signal (i.e. a trajectory in $\mathcal{B}$) that fits on $\mathcal{S}$ the available data.

A thorough discussion of this problem is based on the definition of what we precisely mean by " satisfying the parity checks" on a set $\mathcal{S} \subset \mathbb{Z}^n$.

Definition Let $\mathcal{B} = \ker H^T$ be a locally detectable behavior. A sequence $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ satisfies the parity checks of $\mathcal{B}$ in $\mathbf{h} \in \mathbb{Z}^n$ if

$$\left(H^T\mathbf{v}, \mathbf{z}^{\mathbf{i}}\right) = 0, \qquad \forall\ \mathbf{i} \in \mathbf{h} + \text{supp}(H^T). \tag{5.1}$$

In general, if $\mathcal{S}$ is any subset of $\mathbb{Z}^n$, $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}$ on $\mathcal{S}$ if satisfies them in every point of $\mathcal{S}$.

Letting $H^T := \sum_{\mathbf{j}} H_{\mathbf{j}}^T \mathbf{z}^{\mathbf{j}}$, the above condition reduces to the following system of linear equations

$$\sum_{\mathbf{j} \,\in\, \mathrm{supp}(H^T)} H_{\mathbf{j}}^T \mathbf{v}(\mathbf{i}-\mathbf{j}) = 0, \qquad \forall\ \mathbf{i} \in \mathcal{S} + \mathrm{supp}(H^T), \tag{5.2}$$

and hence to a system of all difference equations which involve the sample $\mathbf{v}(\mathbf{h})$.

Fig. 5.1 below describes the two-dimensional case; each dashed polygon intersecting $\mathcal{S}$ represents the coordinates $(i_1 - j_1, i_2 - j_2)$ of the samples which appear in a system like (5.2). As it is suggested by Fig. 5.1, and clearly implied by the convolutional nature of the system laws expressed by condition $H^T\mathbf{v} = \mathbf{0}$, knowing the data on a finite window $\mathcal{W}$ allows to check the signal only on those subsets $\mathcal{S}$ of $\mathcal{W}$ satisfying the inclusion $\mathcal{S}^\nu \subseteq \mathcal{W}$, $\nu > 0$ being an integer selected according to the size of the support of H^T.

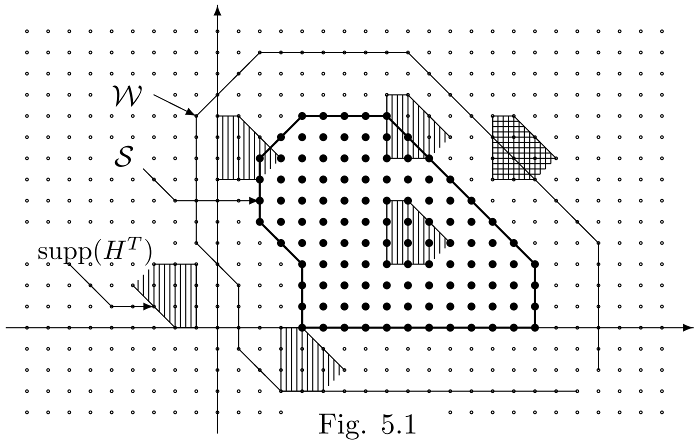

Fig. 5.1

Even when the parity checks have been successfully performed on a sequence $\mathbf{v}$ in a subset $\mathcal{S}$ which fulfills the above inclusion, in general the data on $\mathcal{S}$ cannot be extended into a legal trajectory, namely no signal in $\mathcal{B}$ fits on $\mathcal{S}$ the available data. When the local detectability hypothesis on $\mathcal{B}$ is properly strengthened, however, an integer $\varepsilon > 0$ can be found, such that a positive check on $\mathcal{S}^\varepsilon$ guarantees the existence of some $\mathbf{w} \in \mathcal{B}$ which coincides with $\mathbf{v}$ in $\mathcal{S}$. Note that the amount of data we need may far exceed the part of them we interpolate. Actually, checking $\mathbf{v}$ on $\mathcal{S}^\varepsilon$ requires to know the samples of $\mathbf{v}$ on a superset, say $\mathcal{S}^{\nu+\varepsilon}$, of $\mathcal{S}^\varepsilon$, whereas the data we are able to fit are those belonging to the set $\mathcal{S}$.
The formal definition of extendability property is the following.

(E) [Extendability] *A locally detectable behavior* $\mathcal{B} = \ker H^T$ *is* extendable *if there is an integer* $\varepsilon > 0$ *such that, for every subset* $\mathcal{S} \subset \mathbb{Z}^n$ *and every* $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, *which satisfies on* $\mathcal{S}^\varepsilon$ *the parity checks of* $\mathcal{B}$, *a trajectory* $\mathbf{w} \in \mathcal{B}$ *can be found s.t.* $\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S}$.

An alternative definition of extendability refers to pairs of sequences and pairs of sets.

(TE) [Twin-extendability] *A locally detectable behavior $\mathcal{B} = \ker H^T$ is* twin-extendable *if there exists an integer $\delta > 0$ such that, for every pair of sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{Z}^n$, with $d(\mathcal{S}_1, \mathcal{S}_2) > \delta$ and every pair of signals $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, which satisfy the parity checks of $\mathcal{B}$ on $\mathcal{S}_1^\delta$ and $\mathcal{S}_2^\delta$, respectively, $\mathbf{w} \in \mathcal{B}$ can be found such that*

$$\mathbf{w}|\mathcal{S}_1 = \mathbf{v}_1|\mathcal{S}_1, \qquad \text{and} \qquad \mathbf{w}|\mathcal{S}_2 = \mathbf{v}_2|\mathcal{S}_2. \tag{4.4}$$

Proposition 5.1 *Extendability and twin-extendability are equivalent properties.*

Proof (E) $\Rightarrow$ (TE) Consider a positive integer ρ such that $\text{supp}(H^T) \subset B(0, \rho)$ and introduce $\delta := \varepsilon + 2\rho$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two sequences that satisfy the parity checks on the sets $\mathcal{S}_1^\delta$ and $\mathcal{S}_2^\delta$, respectively, with $d(\mathcal{S}_1, \mathcal{S}_2) > \delta$. The signal $\tilde{\mathbf{v}}_1$ which coincides with $\mathbf{v}_1$ on $\mathcal{S}_1^{\varepsilon+\rho}$ and is zero on $\mathcal{C}(\mathcal{S}_1^{\varepsilon+\rho})$, the complementary set of $\mathcal{S}_1^{\varepsilon+\rho}$, satisfies the parity checks on $\mathcal{S}_1^\varepsilon \cup \mathcal{C}(\mathcal{S}_1^{\varepsilon+2\rho})$. By (E) there exists $\mathbf{w}_1 \in \mathcal{B}$ such that

$$\begin{aligned} &\mathbf{w}_1|\mathcal{S}_1 = \tilde{\mathbf{v}}_1|\mathcal{S}_1 = \mathbf{v}_1|\mathcal{S}_1 \\ &\mathbf{w}_1|\mathcal{C}(\mathcal{S}_1^\delta) = \tilde{\mathbf{v}}_1|\mathcal{C}(\mathcal{S}_1^\delta) = \mathbf{0}|\mathcal{C}(\mathcal{S}_1^\delta). \end{aligned}$$

In the same way one can find $\mathbf{w}_2 \in \mathcal{B}$ such that

$$\mathbf{w}_2|\mathcal{S}_2 = \mathbf{v}_2|\mathcal{S}_2 \qquad \text{and} \qquad \mathbf{w}_2|\mathcal{C}(\mathcal{S}_2^\delta) = \mathbf{0}|\mathcal{C}(\mathcal{S}_2^\delta).$$

The trajectory $\mathbf{w} := \mathbf{w}_1 + \mathbf{w}_2 \in \mathcal{B}$ clearly satisfies (4.4).
(TE) $\Rightarrow$ (E) Given $\mathcal{S} \subset \mathbb{Z}^n$, assume that $\mathbf{v}$ satisfies the parity checks on $\mathcal{S}^\delta$. As $\mathbf{0}$ satisfies the parity checks on $\mathcal{CS}$, there exists in $\mathcal{B}$ a trajectory $\mathbf{w}$ such that $\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S}$ (and $\mathbf{w}|\mathcal{C}(\mathcal{S}^\delta) = \mathbf{0}|\mathcal{C}(\mathcal{S}^\delta)$). So (E) holds with $\varepsilon = \delta$. ■

Proposition 5.2 below characterizes extendable behaviors as those described by ℓZP parity check matrices or by rZP generator matrices, thus showing that properties (E) and (TE) are stronger versions of local detectability and that extendability is possible only for free behavior.

Proposition 5.2 *Let $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^P$ be a finite behavior of rank r. The following facts are equivalent*

i) $\mathcal{B}$ is extendable;

ii) $\mathcal{B} = \ker H^T$, for some ℓZP matrix $H^T \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{(p-r)\times p}$;

iii) $\mathcal{B} = \text{Im} G$, for some rZP matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times r}$.

Proof i) $\Leftrightarrow$ ii) Showing that the left zero-primeness of H^T implies property (E) does not depend on the finiteness of the signal supports. Hence the necessity part of the proof mimics that given in [13] for infinite 2D behaviors and will be omitted.
The sufficiency part is quite different, and based on the following technical lemmas.

Lemma 5.3 [13] *Let* H^T *be an element of* $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{q\times p}$. *The map* $H^T : \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p \to \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^q \ : \ \mathbf{w} \mapsto H^T\mathbf{w}$ *is onto if and only if* H^T *is* ℓ*ZP.* ■

Lemma 5.4 [14] *Let* $m(\mathbf{z})$ *be in* $\mathbb{F}[\mathbf{z}]$. *For any integer* $\rho > 0$ *there is* $p(\mathbf{z}) \in \mathbb{F}[\mathbf{z}]$ *such that*

$$m(\mathbf{z})p(\mathbf{z}) \ \in \ \mathbb{F}[z_1^\rho, ..., z_n^\rho]. \qquad ■$$

Suppose that $\mathcal{B}$ satisfies property (E) for some $\varepsilon > 0$. As $\mathcal{B}$ is locally detectable, it can be described as $\mathcal{B} = \ker H^T$, and it is not restrictive assuming that H^T is ℓFP. To prove that H^T is ℓZP, we use the above Lemma 5.3 and show that for all vectors $\mathbf{a}$ in $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^q$, equation $H^T\mathbf{x} = \mathbf{a}$ admits an L-polynomial solution. As H^T has full row rank over $\mathbb{F}(\mathbf{z})$, the equation has a rational solution $\mathbf{v} = \mathbf{n}/d$, $\mathbf{n} \in \mathbb{F}[\mathbf{z}]^p$, $d \in \mathbb{F}[\mathbf{z}]$, and hence

$$H^T\mathbf{n} = d\ \mathbf{a}. \tag{5.4}$$

Set $\rho = 2\delta_h + 4\delta_a$, where δ_h and δ_a are the radii of two balls, with center in the origin, including supp(H^T) and supp$(\mathbf{a})$, respectively. By Lemma 5.4, there exists $p(\mathbf{z}) \in \mathbb{F}[\mathbf{z}]$ such that pd belongs to $\mathbb{F}[z_1^\rho, ..., z_n^\rho]$. Multiplying both members of (5.4) by p, we get

$$H^T p\ \mathbf{n} = pd\,\mathbf{a}, \tag{5.5}$$

where the assumption on ρ implies that the support $\mathcal{Q}$ of $pd\mathbf{a}$ is the disjoint union of finitely many shifted copies $\mathcal{Q}_\mathbf{i}$ of supp$(\mathbf{a})$, whose mutual distance is lower bounded by $2(\varepsilon + \delta_h)$:

$$\mathcal{Q} = \cup_\mathbf{i}\ \mathrm{supp}(c_i\ \mathbf{z}^{\rho\mathbf{i}}\mathbf{a}) = \cup_\mathbf{i}\ \mathcal{Q}_\mathbf{i}. \tag{5.6}$$

From equation (5.5) it follows that the sequence $p\mathbf{n}$ fulfills the parity checks of the behavior on the set $C(\mathcal{Q}^{\delta_h})$. So, by the extendability assumption, a behavior sequence $\mathbf{w}$ can be found, coinciding with $p\mathbf{n}$ on $C(\mathcal{Q}^{\delta_h+\varepsilon})$. As the support of the L-polynomial sequence $\mathbf{y} := \mathbf{w} - p\mathbf{n}$ is included in $\cup_\mathbf{i}\mathcal{Q}_\mathbf{i}^{\delta_h+\varepsilon}$, $\mathbf{y}$ can be rewritten as $\mathbf{y} = \sum_\mathbf{i}\mathbf{y}_\mathbf{i}$, where $\mathbf{y}_\mathbf{i}$ denotes the restriction of $\mathbf{y}$ to $\mathcal{Q}_\mathbf{i}^{\delta_h+\varepsilon}$. Also, by the choice of ρ we made, all the supports of $H^T\mathbf{y}_\mathbf{i}$ are disjoint, and therefore $H^T\mathbf{y} = pd\mathbf{a}$ implies $H^T\mathbf{y}_\mathbf{i} = c_i\ \mathbf{z}^{\rho\mathbf{i}}\mathbf{a}$, thus proving that the original equation has an L-polynomial solution.

ii) $\Leftrightarrow$ iii) Assume first that $\mathcal{B} = \mathrm{Im}G$, for some rZP matrix $G \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times r}$. By Proposition 2.1, then, G can be column-bordered into a square unimodular matrix $U = [\ G\ L\ ]$. If $V := \begin{bmatrix} X \\ H^T \end{bmatrix}$ is the (L-polynomial) inverse of U, we have

$$\begin{bmatrix} X \\ H^T \end{bmatrix} [\, G \quad L\,] = \begin{bmatrix} I_r & 0 \\ 0 & I_{p-r} \end{bmatrix}.$$

Being a submatrix of a unimodular matrix, H^T will be ℓZP, and it is easy to see that $\mathcal{B}^{\perp\perp} = \ker H^T$. As G is rZP, and hence rMP, $\mathcal{B}$ is maximal in $\mathcal{G}(p,r)$ and $\mathcal{B} = \mathcal{B}^{\perp\perp} = \ker H^T$, which proves the result.

Conversely, if $\mathcal{B} = \ker H^T$, for some ℓZP matrix $H^T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{(p-r)\times p}$, by the same reasoning adopted before, one can show that there exists a rZP matrix $G \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times r}$ satisfying $H^TG = 0$.

Clearly, $\mathcal{B} \equiv \mathrm{Im}_{\mathbb{F}(\mathbf{z})} G \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, and hence, as rZP matrices are rMP, $\mathcal{B} \equiv \mathrm{Im}G$, with $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times r}$ rZP. ■

In the definition of extendability no constraints are assumed on the shape and cardinality of the set $\mathcal{S}$ where the parity checks are performed. As a counterpart of adopting this general setting, only behaviors endowed with strong structural properties possess this feature. If we agree to extend into behavior sequences only data sets which fulfill the parity checks on particular subsets of $\mathbb{Z}^n$, we can partly relax the primeness requirements on the generator matrices. The subsets of $\mathbb{Z}^n$ we will refer to are (infinite) cylinders with either one-dimensional or n-1-dimensional bases, enveloping a given finite set $\mathcal{S}$. More precisely, 1-*cylinders* enveloping $\mathcal{S}$ are defined as

$$C_i(\mathcal{S}) := \{\mathbf{h} \in \mathbb{Z}^n : h_i = k_i,\ \exists \mathbf{k} \in \mathcal{S}\}, \qquad i = 1, 2, \dots, n, \tag{5.7}$$

while n-1-*cilinders* are

$$C_{i^c}(\mathcal{S}) := \{\mathbf{h} \in \mathbb{Z}^n : \mathbf{h}_{i^c} = \mathbf{k}_{i^c},\ \exists \mathbf{k} \in \mathcal{S}\}, \qquad i = 1, 2, \dots, n, \tag{5.8}$$

where $\mathbf{h}_i^c$ denotes the n-1-tuple in $(h_1, h_2, \dots, h_n)$ complementary to h_i.

($\mathbf{E}_1$) [1-Extendability] *A locally detectable behavior $\mathcal{B} = \ker H^T$ is* 1-extendable *if there is an integer $\varepsilon > 0$ such that, for every finite subset $\mathcal{S} \subset \mathbb{Z}^n$ and every $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, if $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}$ on the 1-cylinder $C_i(\mathcal{S}^\varepsilon)$, for some $i \in \{1, 2, \dots, n\}$, a trajectory $\mathbf{w} \in \mathcal{B}$ can be found s.t. $\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S}$.*

($\mathbf{E}_{n-1}$) [$(n-1)$-Extendability] *A locally detectable behavior $\mathcal{B} = \ker H^T$ is $(n-1)$-*extendable *if there is an integer $\varepsilon > 0$ such that, for every finite subset $\mathcal{S} \subset \mathbb{Z}^n$ and every $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, if $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}$ on the $(n-1)$-cylinder $C_{i^c}(\mathcal{S}^\varepsilon)$, for some $i \in \{1, 2, \dots, n\}$, a trajectory $\mathbf{w} \in \mathcal{B}$ can be found s.t. $\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S}$.*

Proposition 5.5 *Let $\mathcal{B} = \mathrm{Im}G$ be a free behavior, $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times r}$ a full column rank generator matrix.*
i) G is rMP $\Leftrightarrow$ $\mathcal{B}$ is 1-extendable;
ii) if G is rVP $\mathcal{B}$ is n-1-extendable.

Proof *i)* By definition, 1-extendability implies local detectability, and therefore, by Proposition 4.2, $\mathcal{B}$ admits a rMP generator matrix.
If G is rMP, by Proposition 4.2 $\mathcal{B}$ is locally detectable. As G is rZP in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$, $i = 1, 2, \dots, n$, the behaviors $\mathcal{B}_i := \mathrm{Im}_{\mathbb{F}(z_i^c)[z_i, z_i^{-1}]} G$, $i = 1, 2, \dots, n$, satisfy definition (E) for suitable $\varepsilon_i > 0$ and hence are extendable (and locally detectable) in a 1D context. Let ε be the maximum of the ε_i. If we represent $\mathcal{B}$ as the kernel (in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$) of some L-polynomial matrix H^T, then $\mathcal{B}_i \equiv \ker_{\mathbb{F}(z_i^c)[z_i, z_i^{-1}]} H^T = \{\mathbf{w} \in \mathbb{F}(z_i^c)[z_i, z_i^{-1}]^p : H^T\mathbf{w} = 0\}$. Consider, now, a finite set $\mathcal{S} \subset \mathbb{Z}^n$ and some $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ which satisfies the parity checks of $\mathcal{B}$ in $C_i(\mathcal{S}^\varepsilon)$, for some $i \in \{1, 2, \dots, n\}$. As an element of $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]^p$, $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}_i$ on the one-dimensional projection I_i^ε of $\mathcal{S}^\varepsilon$ onto the i-th coordinate axis, and hence there is $\tilde{\mathbf{w}} = G\tilde{\mathbf{u}}$, $\tilde{\mathbf{u}} \in \mathbb{F}(z_i^c)[z_i, z_i^{-1}]^m$, in $\mathcal{B}_i$ s.t. $\tilde{\mathbf{w}}|I_i = \mathbf{v}|I_i$.

So, as n-dimensional sequences, $\mathbf{v}$ and $\tilde{\mathbf{w}}$ coincide on $C_i(\mathcal{S})$. If r is the radius of a ball $B(0,r)$ centered in the origin which includes $\mathrm{supp}(G)$, clearly the values of $\tilde{\mathbf{w}}$ in $\mathcal{S}$ depend only on the values of $\tilde{\mathbf{u}}$ in $\mathcal{S}^r$. Thus, the finite sequence $\mathbf{u}$, which coincides with $\tilde{\mathbf{u}}$ on $\mathcal{S}^r$ and is zero elsewhere, produces a behavior sequence $\mathbf{w} = G\mathbf{u}$ which coincides with $\mathbf{v}$ on $\mathcal{S}$.

ii) Following an analogous reasoning one shows that G rVP implies $\mathcal{B}$ n-1-extendable. ■

It is worthwhile to notice that, as a consequence of i) of the above proposition, for free behaviors local detectability and 1-extendability are equivalent properties.
In case of general (not necessarily free) locally detectable behaviors, sufficient conditions for 1 and n-1 extendability are provided by the following proposition.

Proposition 5.6 *Let $\mathcal{B} = \mathrm{ker} H^T$ be a locally detectable behavior.*
i) if H^T is ℓMP then $\mathcal{B}$ is 1-extendable;
ii) if H^T is ℓVP then $\mathcal{B}$ is n-1-extendable.

PROOF *i)* If H^T is ℓMP, by Lemma 2.2, it is ℓZP in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$, $i = 1, 2, \ldots, n$. So, by Proposition 5.2, the behaviors

$$\hat{\mathcal{B}}_i := \mathrm{ker}_{\mathbb{F}(z_i^c)[z_i, z_i^{-1}]} H^T = \{\mathbf{w}_i \in \mathbb{F}(z_i^c)[z_i, z_i^{-1}]^p : H^T\mathbf{w}_i = 0\},$$

with elements in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]^p$, $i = 1, 2, \ldots, n$, can be expressed as $\hat{\mathcal{B}}_i = \mathrm{Im}_{\mathbb{F}(z_i^c)[z_i, z_i^{-1}]} G_i$ for suitable L-polynomial matrices G_i, rZP in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$, and they fulfill the extendability condition for suitable positive integers ε_i. Set $\varepsilon := \max_i \varepsilon_i$.
Consider a finite set $\mathcal{S}$ of $\mathbb{Z}^n$, and assume that $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ satisfies the parity checks of $\mathcal{B}$ on $\mathcal{C}_i(\mathcal{S}^\varepsilon)$ for some i. Then $\mathbf{v}$ can be viewed as an element of $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]^p$, i.e. as a one-dimensional sequence, which satisfies the parity checks of $\hat{\mathcal{B}}_i$ on the projection, I_i^ε, of $\mathcal{C}_i(\mathcal{S}^\varepsilon)$ onto the i-th coordinate axes. By the extendability of $\hat{\mathcal{B}}_i$, there exists $\mathbf{w}_i = G_i\mathbf{u}_i$, $\mathbf{u}_i$ with elements in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]^p$, such that

$$\mathbf{w}_i | I_i = \mathbf{v} | I_i.$$

Thus, if we regard $\mathbf{w}_i$ and $\mathbf{v}$ as n-dimensional sequences, we get

$$\mathbf{w}_i | \mathcal{C}_i(\mathcal{S}) = \mathbf{v} | \mathcal{C}_i(\mathcal{S}).$$

Let r be a positive integer such that $B(0,r) \supseteq \mathrm{supp}(G_i)$, and consider the finite support sequence $\mathbf{u}$ which coincides with a power series expansion of $\mathbf{u}_i$ on $\mathcal{S}^r$ and is zero elsewhere. Then $\mathbf{w} := G_i\mathbf{u}$ is a finite support sequence which satisfies

$$\mathbf{w} | \mathcal{S} = \mathbf{w}_i | \mathcal{S} = \mathbf{v} | \mathcal{S},$$

and belongs to $\mathcal{B}$, as

$$H^T\mathbf{w} = H^T G_i \mathbf{u} = 0.$$

ii) The proof is essentially the same given for i). ■

References

[1] H.H. Rosenbrock, *State-space and multivariable theory*, J.Wiley & Sons, New York, 1970

[2] J.C. Willems, Models for dynamics, in *Dynamics reported*, vol.2, pp.171-269, 1988

[3] M. Morf, B.C. Lévy, S.Y. Kung and T. Kailath, New results in 2D systems theory, Part I and II, in *Proc. of IEEE*, vol 65, no.6, pp. 861-872;945-961, 1977

[4] J.P. Guiver and N.K. Bose, Causal and weakly causal 2-D filters with applications in stabilization, In N.K.Bose, editor, *Multidimensional Systems theory*, D.Reidel Publ. Co., Dordrecht (NL), pp. 52-100, 1985

[5] M. Bisiacco, State and output feedback stabilizability of 2D systems, *IEEE Trans. on Circ. and Sys*, CAS 32, pp. 1246-1249, 1985

[6] P. Rocha and J.C. Willems, Controllability of 2-D Systems, *IEEE Trans. Aut. Contr.*, AC 36, pp. 413-23, 1991

[7] D.C. Youla and G. Gnavi, Notes on n-dimensional system theory, *IEEE Trans. on Circ. and Sys.*, CAS 26, pp. 105-111, 1979

[8] T. Kailath , *Linear Systems*, Prentice Hall, Inc., 1980

[9] D.C. Youla and P.F. Pickel, The Quillen-Suslin theorem, *IEEE Trans. on Circ. and Sys.*, CAS 31, pp. 513-518, 1984

[10] F.R. Gantmacher, *The theory of matrices*, Chelsea Pub. Co., 1960

[11] N. Jacobson, *Basic Algebra, voll. I and II*, Freeman, San Francisco (CA), 1974

[12] T. Becker and V. Weispfenning, *Gröbner basis* , Springer-Verlag, 1993

[13] E. Fornasini and M.E. Valcher, Algebraic aspects of 2D convolutional codes, *IEEE Trans.Inf.Th.*, IT 40, pp. 1210-1225, 1994

[14] E. Fornasini and M.E. Valcher, Multidimensional systems with finite support behaviors: signal structure, generation and detection, submitted, 1995