The dominant global state in the asymptotic analysis of 2D systems

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Abstract

The dominant state plays an essential role in the asyntotic analysis of dynamical systems. The global state in a 2D system consists in a sequence, and the existence of a dominant global state means that the free evolution of the global states tends to approximate this sequence, up to the multiplication by a normalizing factor. In this contribution the existence of a global state is proved under the hypothesis that the initial global state is the Fuorier Stieltjes transform of a bounded variation function.

1 Introduction

The asymptotic analysis of dynamical systems is extremely important both in control theory and in signal processing. When describing the asymptotic behavior, the first issue to be considered is the stability of the system, which amounts to decide whether its trajectories converge, remain bounded, or diverge. In some specific situations, however, this is not enough and it may be relevant to obtain further information on the asymptotic feature of the signal. This is exactly the information provided by the dominant eigenvector of a linear state space system. Indeed, it can be shown that, under some weak hypotheses, the state of a linear state space system, when normalized in a suitable way, converges to one of its eigenvectors, which is called the dominant eigenvector.

Two-dimensional (2D) systems involve input and output signals as well as state dynamics evolving on a two-dimensional time set and play an important role in image processing [1] and in control of repetitive or learning systems [6]. The stability analysis of this class of systems is well-known to be quite difficult [4]. In particular, very little is known about the possibility of performing an asymptotic analysis, based on the concept of dominant eigenvector for this class of systems [3, 5].

In this contribution we propose a preliminary solution to this problem, based on the theory of Fourier-Stieltijs series [8]. It turns out that, if the initial global state of a 2D system is represented by (the coefficients of) a Fourier-Stieltjes series having a non zero component on the dominant frequency of the 2D system, then the dominant state analysis can be performed and a clear limiting behavior can be distinguished. More precisely, it is shown that in this the dominant state, up to the multiplication by a normalizing factor, tends to approximate a sinusoidal sequence of frequency equal to the dominant frequency.

The main advantage of this approach is that it seems to be easily exendable from the scalar case to the general vector case.

2 Mathematical preliminaries

Let $F(\omega)$ be a complex valued function of bounded variation, defined on the closed interval $-\pi \leq \omega \leq \pi$, and consider the sequence $(\hat{F}_k)_{k\in\mathbb{Z}}$ whose elements are given by the following Fourier-Stieltjes integrals

$$\hat{F}_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-jk\omega} dF(\omega), \qquad k \in \mathbb{Z}.$$

The sequence $\hat{F}$ is called the *Fourier Stieltjes transform of* F and

$$\sum_{k=-\infty}^{+\infty} \hat{F}_k e^{jk\omega}, \tag{1}$$

is the *Fourier-Stieltjes series of* F.
Notice that an equivalent definition of the Fourier Stieltjes transform $\hat{F}_k$ is based of the fact that, given a function F of bounded variation on $[-\pi, \pi]$, there exists a uniquely determined complex Borel measure μ_F on $[-\pi, \pi]$, called the Lebesgue-Stieltjes measure corresponding to F, allowing to express the Fourier-Stieltjes integral as a Lebesgue integral with respect to μ_F. We therefore have

$$\hat{F}_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-j\omega} d\mu_F, \qquad k \in \mathbb{Z}.$$

Since complex Borel measures constitute a subspace of the space of distributions, the series (1) converges to μ_F in the topology of distrubutions.

If $\mathrm{Var}(F)$ denotes the total variation of F, its Fourier-Stieltjes coefficients satisfy

$$|\hat{F}_k| \leq \mathrm{Var}(F), \quad \forall k \in \mathbb{Z},$$

showing that the Fourier- Stieltjes transform $(\hat{F}_k)_{k\in\mathbb{Z}}$ of a function of bounded variation is always in ℓ^∞. However, not every sequence in ℓ^∞ is the Fourier-Stieltjes transform of a suitable bounded variation function. A characterization of the space $\mathcal{S} \subset l^\infty$ of Fourier-Stieltjes transforms is provided by the following theorem.

Theorem 1 (see [8, vol. 1, page 136] and [2, vol.2, page 81]) *Let* $(a_k)_{k\in\mathbb{Z}}$ *be a given sequence, and put*

$$\sigma_N(\omega) := \sum_{|k|\leq N} \left(1 - \frac{|k|}{N+1}\right) a_k e^{jk\omega}.$$

In order that $(a_k) \in \mathcal{S}$*, it is necessary and sufficient that*

$$\limsup_{N\to\infty} ||\sigma_N||_1 < \infty$$

where $||\cdot||_1$ *means the* L^1 *norm in* $[-\pi, \pi]$.

As a consequence of the above Theorem, every ℓ^1 sequence is the Fourier-Stieltjes transform of some bounded variation function. On the other hand, ℓ^1 does not include all Fourier Stieltjes transforms. For instance, the transform of

$$F(\omega) = \begin{cases} -\pi & \text{if } \omega \leq 0 \\ \pi & \text{if } \omega > 0 \end{cases}$$

is the constant sequence $\hat{F}_k = 1, \forall k \in \mathbb{Z}$, that is not summable. Using Theorem 1, it is also easy to show that the ℓ^∞ sequence

$$a_k = \begin{cases} -1 & \text{if } k < 0 \\ 0 & \text{if } k = 0 \\ 1 & \text{if } k > 0 \end{cases}$$

does not belong to $\mathcal{S}$. In fact,

$$\sigma_N(\omega) = \frac{1}{N+1}\left[\frac{\sin\frac{1}{2}(N+1)\omega}{\sin\frac{1}{2}\omega}\right]^2.$$

Therefore L^1 norm of $\sigma_N(\omega)$ coincides with the square of the L^2 norm of

$$\frac{\sin\frac{1}{2}(N+1)\omega}{\sin\frac{1}{2}\omega} = \sum_{k=N}^{N} e^{j\omega k}$$

which, using Parseval identity, is $2\pi(2N+1)$. Therefore,

$$\sigma_N(\omega) = \frac{4\pi^2(2N+1)^2}{N+1}$$

which diverges as N goes to ∞. We therefore have that $\mathcal{S}$ satisfies the strict inclusions

$$\ell^1 \quad\subset\quad \mathcal{S} \quad\subset\quad \ell^\infty.$$

A classical representation theorem by F.Riesz states that, given any continuous linear functional ψ on the space $C(-\pi,\pi)$ of continuous functions, there exists a function F_ψ of bounded variation such that

$$\psi(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)dF_\psi(\omega), \quad \forall f \in C(-\pi,\pi) \tag{2}$$

and, vice versa, all functions of bounded variation induce, via (2), a continuous linear functional on $C(-\pi,\pi)$.
The function F_ψ is uniquely determined by ψ, up to the values it assumes on a denumerable set of points. On the other hand, ψ is uniquely determined by the values it assumes on the exponential functions $e^{jk\omega}$, $k \in \mathbb{Z}$, as the set of the trigonometric polynomials is dense in $C(-\pi,\pi)$. Consequently, the Fourier-Stieltjes transform $\hat{F}$ of a bounded variation function $F(\omega)$ allows to recover F (except on a denumerable set).

Since a real valued function of bounded variation is the difference of two increasing functions, the Lebesgue theorem on the derivative of monotonic functions guarantees that any (real or complex valued) function $F(\omega)$ of bounded variation on $[-\pi,\pi]$ is differentiable a.e. Moreover, if $\{\omega_1,\omega_2,\dots\}$ is the (countable) set of its discontinuities in $(-\pi,\pi)$ and we define $\Delta F(\omega_i) = F(\omega_i^+) - F(\omega_i^-)$, $F(\omega)$ can be decomposed into the sum of three functions

$$F(\omega) = J(\omega) + A(\omega) + S(\omega),$$

where $J(\omega)$ is a *jump function*, defined by

$$J(\omega) = \begin{cases} 0 & \text{if } \omega = -\pi \\ F(-\pi^+) - F(-\pi) + \sum_{\omega_i<\omega}\Delta F(\omega_i) + F(\omega) - F(\omega^-) & \text{if } \omega > -\pi \end{cases}$$

$A(\omega)$ is an absolutely continuous function, and $S(\omega)$ is a singular function, i.e. a continuous function whose derivative is zero a.e.. The functions J, A and S are unique to within additive constants.
The Fourier Stieltjes transform of $A(\omega)$ coincides with the Fouries transform of $A'(\omega)$, i.e.

$$\hat{A}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jk\omega}dA(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jk\omega}A'(\omega)d\omega$$

and therefore, by the Riemann-Lebesgue lemma (Edwards I, pg 36; Zygmund, I, pg 45)

$$\lim_{|k|\to\infty} \hat{A}_k = 0.$$

On the other hand, the Fourier Stieltjes transform of a jump function $J(\omega)$ whose jumps in $\omega_1, \omega_2, \ldots$ have amplitudes $J^{(1)}, J^{(2)}, \ldots$ respectively, is the sum of a countable family of (complex) sequences $\hat{J}^{(1)}, \hat{J}^{(2)}, \ldots$ whose values are given by

$$\hat{J}_k^{(\nu)} = \frac{1}{2\pi} J^{(\nu)} e^{-jk\omega_\nu}, \quad \nu = 1, 2, \ldots; \quad k = 0, \pm 1, \pm 2, \ldots$$

In particular, if two jumps with the same real amplitude $\bar{J}$ (with imaginary amplitudes $j\bar{J}$ and $-j\bar{J}$) occur in $-\bar{\omega}$ and in $\bar{\omega}$, the sum of the corresponding sequences is $\frac{\bar{J}}{\pi}\cos(\bar{\omega}k)$ (resp. $\frac{\bar{J}}{\pi}\sin(\bar{\omega}k)$).
An interesting insight into some connections between the jump function of $F(\omega)$ and the partial sums of the associated Fourier Stieltjes series is provided by the following theorem.

Theorem 2 *Let $(\hat{F}_k)_{k\in\mathbb{Z}}$ be the Fouries Stieltjes transform of $F(\omega)$. Then, for all $p \in \mathbb{Z}$,*

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=p-N}^{p+N} \hat{F}_k e^{jk\omega} = \frac{F(\omega^+) - F^(\omega^-)}{2\pi}$$

Proof Note that the sum

$$\sum_{k=p-N}^{p+N} \hat{F}_k e^{jk\omega} \tag{3}$$

can be seen as the pointwise product of the sequence $\hat{F}_k$ and the sequence R_k defined as

$$R_k = \begin{cases} 1 & \text{if} \quad p - N \le k \le p + N \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can argue that the Fuorier transform of (3) coincides with the convolution (in the distributional sense) of the Fourier transforms of the sequences $\hat{F}_k$ and R_k. Notice that the Fourier transforms of R_k is

$$\sum_{k=-\infty}^{+\infty} R_k e^{jk\omega} = \sum_{k=p-N}^{p+N} e^{jk\omega} = e^{j\omega p} \frac{\sin\frac{1}{2}(N+1)\omega}{\sin\frac{1}{2}\omega}.$$

Therefore we can write

$$\frac{1}{2N+1} \sum_{k=p-N}^{p+N} \hat{F}_k e^{jk\omega} = \int_{-\pi}^{\pi} \frac{1}{2N+1} e^{j(\omega-\sigma)p} \frac{\sin\frac{1}{2}(N+1)(\omega-\sigma)}{\sin\frac{1}{2}(\omega-\sigma)} d\mu_F(\sigma).$$

Since

$$\left| \frac{1}{2N+1} \frac{\sin\frac{1}{2}(N+1)(\omega-\sigma)}{\sin\frac{1}{2}(\omega-\sigma)} \right| \le 1$$

na d since the constant function equal to 1 is absolutely integrable with respect to μ_F we are in position to apply the and that the Lesbegue dominated convergence theorem. Observing that

$$\frac{1}{2N+1} \frac{\sin\frac{1}{2}(N+1)(\omega-\sigma)}{\sin\frac{1}{2}(\omega-\sigma)} \xrightarrow[l\to\infty]{} \chi_\omega(\sigma) = \begin{cases} 1 & \text{if } \sigma = \omega \\ 0 & \text{otherwise} \end{cases}$$

we can argue that

$$\frac{1}{2N+1} \sum_{k=p-N}^{p+N} \hat{F}_k e^{jk\omega} \xrightarrow[l\to\infty]{} \int_{-\pi}^{+\pi} e^{j(\omega-\sigma)k} \chi_\omega(\sigma) d\mu_F(\sigma) = \mu(\{\omega_0\}) = \frac{1}{2\pi}\Delta F(\omega_0).$$

■

3 Asymptotic behavior of the 2D global state

The unforced evolution of many physical and biological processes can be modelled by means of linear, discrete, quarter-plane causal 2D state models described by the equation

$$\mathbf{x}(h+1,k+1) = A_0\mathbf{x}(h,k+1) + A_1\mathbf{x}(h+1,k) \tag{4}$$

$h,k \in \mathbb{Z},\ h+k \geq 0$. The *local states* $\mathbf{x}(h,k)$ are elements of $\mathbb{R}^n_+$ and A_0, A_1 are suitable square matrices. Initial conditions are usually given by assigning a sequence

$$\mathcal{X}^{(0)} := \{\mathbf{x}(\ell,-\ell) : \ell \in \mathbb{Z}\} \tag{5}$$

of local states on the separation set $\mathcal{C}^{(0)} := \{(\ell,-\ell) : \ell \in \mathbb{Z}\}$. The sequence $\mathcal{X}^{(0)}$, as well as the sequences

$$\mathcal{X}^{(t)} := \{\mathbf{x}(\ell+t,-\ell), \ell \in \mathbb{Z}\}, \quad t > 0 \tag{6}$$

the system reaches according to the updating equation (4) are called the *global states*.

The asymptotic analysis for 2D state models refers to the motion determined by an assignement of initial conditions on the separation set $\mathcal{C}^{(0)}$. More precisely, we assume that the initial global state constitutes an ℓ^∞ sequence, and consider the behavior of the global states $\mathcal{X}^{(t)}$ as $t \to +\infty$. We expect that, under mild assumptions on the initial conditions and the structure of the system matrices, the long term dynamics exhibits interesting features on each separation set, such as a periodic character, the alignment of all local state vectors, etc.

In this communication we shall restrict our attention to scalar local states, and assume that the initial global state is the Fourier Stieltjes transform of a function of bounded variation. The advantage in using the space $\mathcal{S}$ of the Fourier-Stieltjes transforms is that for this class of signals the concept of frequency component can be defined in a coherent way. Actually, a Fourier-Stieltjes series $(\hat{F}_k)_{k\in\mathbb{Z}}$ defined from a bounded variation function F has a component at a frequency $\omega \in [-\pi,\pi]$ of amplitude Δ if F is discontinuous in ω and $\Delta = F(\omega^+) - F(\omega^-)$.
On the other hand, the whole space of bounded sequences ℓ^∞ is too wild a space for applying Fourier analysis, and the subspace of absolutely summable sequences ℓ^1 is too poor, as periodic sequences, which are expected to represent the most interesting asymptotic global states, do not belong to ℓ^1.
We modify, for sake of uniformity, the double index notation in (6) for the global states into

$$\mathcal{X}^{(t)} = \{\mathbf{x}(t+\ell,-\ell)\} = (x^{(t)}_\ell)_{\ell\in\mathbb{Z}}$$

and introduce in $\mathbb{C}^\mathbb{Z}$ the polynomial operator

$$A(\sigma) = A_0 + A_1\sigma : (x_\ell)_{\ell\in\mathbb{Z}} \mapsto (x_\ell)_{\ell\in\mathbb{Z}} = (A_0x_\ell + A_1x_{\ell+1})_{\ell\in\mathbb{Z}}$$

Clearly the global states of (4) update according to the equation $\mathcal{X}^{(t+1)} = (A_0 + A_1\sigma)\mathcal{X}^{(t)}$.
To obtain a more general theory, however, we enlarge our scope to higher order polynomial operators $A(\sigma) = \sum_{i=r}^{R} A_i\sigma^i \in \mathbb{C}[\sigma,\sigma^{-1}]$

$$A(\sigma) : \mathbb{C}^\mathbb{Z} \to \mathbb{C}^\mathbb{Z} : \mathcal{X} \mapsto A(\sigma)\mathcal{X} \tag{7}$$

operating as follows

$$A(\sigma)(x_\ell)_{\ell\in\mathbb{Z}} = \big(\sum_{i=r}^{R} A_i x_{\ell+i}\big)_{\ell\in\mathbb{Z}}.$$

It is not difficult to verify that the eigenfunctions for this class of operators coincide with the so called exponential polynomial sequences, that are sequences like

$$\mathcal{X}(k) = k^N e^{\lambda k},$$

where $N \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. It is quite clear that the eigenfunctions of $A(\sigma)$ will play a central role in understanding what happens if we apply infinitely many times the operator $A(\sigma)$ to a initial sequence $\mathcal{X}$. However, in order to apply Fourier analysis we need to restrict the space of sequences. The largest subspace of $\mathbb{C}^{\mathbb{Z}}$ in which Fourier series make sense consists in sequences which grow at most polynomially [7]. The sum in this case is a distribution over $[-\pi, \pi]$. However this set up proved to be not treatable. The situation does not simplify if we take sequences in ℓ^∞. Indeed in this case the series are distributions which are called pseudomeasures. A treatable class of sequences are those whose series converge to a (complex valued) measure. These sequences constitutes the space $\mathcal{S}$ introduced in the previous section. Notice that this space is reach enough to contain a large class of eigenfunctions. Actually, all imaginary exponential sequences

$$\mathcal{X}(k) = e^{j\omega k}$$

belong to $\mathcal{S}$ for all frequencies $\omega \in \mathbb{R}$.

Consider now a polynomial operator $A(\sigma)$. By substituting σ with $e^{j\omega}$, we obtain $A(e^{j\omega})$, which is a function mapping each $\omega \in [-\pi, \pi]$ to the complex number $A(e^{j\omega})$.

Definition Given a polynomial operator $A(\sigma)$, we say that $\omega_0 \in [-\pi, \pi]$ is a dominant frequency for $A(\sigma)$ if

$$|A(e^{j\omega_0})| \geq |A(e^{j\omega})| \quad \text{for all} \quad \omega \in [-\pi, \pi].$$

In this set up we can present the main result of this contribution.

Theorem 3 *Let $A(\sigma)$ be a polynomial operator and assume that there exists a unique dominant frequency $\omega_0 \in [-\pi, \pi]$ for $A(\sigma)$. Assume moreover that the initial global state $\mathcal{X} \in \mathcal{S}$ and that ω_0 belongs to the jump set of $\mathcal{X}$. If we denote*

$$\mathcal{X}^{(l)} := (A(\sigma))^l \mathcal{X},$$

then for each $k \in \mathbb{Z}$ we have that

$$\frac{\mathcal{X}^{(l)}(k)}{A(e^{j\omega_0})^l} \xrightarrow[l \to \infty]{} \frac{1}{2\pi} \Delta F(\omega_0) e^{j\omega_0 k}$$

Proof Let μ be the complex valued measure over $[-\pi, \pi]$ such that

$$\mathcal{X}(k) = \int_{-\pi}^{+\pi} e^{j\omega k} d\mu.$$

Observe that

$$\begin{aligned} (A(\sigma)\mathcal{X})(k) &= \sum_{i=r}^{R} \int_{-\pi}^{+\pi} e^{j\omega(k+i)} d\mu = \int_{-\pi}^{+\pi} \sum_{i=r}^{R} e^{j\omega(k+i)} d\mu = \\ &= \int_{-\pi}^{+\pi} e^{j\omega k} \sum_{i=r}^{R} e^{j\omega i} d\mu = \int_{-\pi}^{+\pi} e^{j\omega k} A(e^{j\omega}) d\mu. \end{aligned}$$

By iterating this we obtain that

$$\mathcal{X}^{(l)}(k) = \int_{-\pi}^{+\pi} e^{j\omega k} A(e^{j\omega})^l d\mu$$

and hence

$$\frac{\mathcal{X}^{(l)}(k)}{A(e^{j\omega_0})^l} = \int_{-\pi}^{+\pi} e^{j\omega k} \left[\frac{A(e^{j\omega})}{A(e^{j\omega_0})} \right]^l d\mu.$$

Notice that, for all $l \in \mathbb{N}$, we have

$$\left| e^{j\omega k} \left[\frac{A(e^{j\omega})}{A(e^{j\omega_0})} \right]^l \right| \leq 1$$

and that

$$\left[\frac{A(e^{j\omega})}{A(e^{j\omega_0})} \right]^l \xrightarrow[l \to \infty]{} \chi_{\omega_0}(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Since the constant function equal to 1 is absolutely integrable with respect to μ, we can apply the Lesbegue dominated convergence theorem from which we can argue that

$$\frac{\mathcal{X}^{(l)}(k)}{A(e^{j\omega_0})^l} \xrightarrow[l \to \infty]{} \int_{-\pi}^{+\pi} e^{j\omega k} \chi_{\omega_0}(\omega) d\mu = \mu(\{\omega_0\}) e^{j\omega_0 k} = \frac{1}{2\pi} \Delta F(\omega_0) e^{j\omega_0 k}$$

■

The previous result allows to deduce the asymptotic behavior of $\mathcal{X}^{(l)} := (A(\sigma))^l \mathcal{X}$ for large l. Consider the following polar representation of the complex numbers $A(e^{j\omega_0})$ and $\Delta F(\omega_0)$

$$A(e^{j\omega_0}) = |A(e^{j\omega_0})| e^{j \arg(A(e^{j\omega_0}))}, \qquad \Delta F(\omega_0) = |\Delta F(\omega_0)| e^{j \arg(\Delta F(\omega_0))}.$$

We can argue that, as l tends to ∞, the global state

$$\mathcal{X}^{(l)}(k) \sim \frac{1}{2\pi} |\Delta F(\omega_0)| |A(e^{j\omega_0})|^l e^{j(\omega_0 k + \arg(A(e^{j\omega_0}))l + \arg(\Delta F(\omega_0))}.$$

We consider now the real case in which $A(\sigma) \in \mathbb{R}[\sigma, \sigma^{-1}]$ and $\mathcal{X} \in \mathbb{R}^{\mathbb{Z}}$. In this situation the hypothesis requiring uniqueness of the dominant frequency is too restrictive. Actually, in this case

$$A(e^{-j\omega}) = A(e^{j\omega})^*,$$

where $*$ means complex conjugate, and so, if ω_0 is a dominant frequency, then also $-\omega_0$ is a dominant frequency.

Assume now that $\pm\omega_0$ are the only dominant frequencies of the polynomial operator $A(\sigma)$. Considering the fact that $\mathcal{X}^{(l)}(k)$ is a real number and using the fact that any complex valued measure μ admits the decomposition $\mu = \mu_R + j\mu_I$, where $\mu_R, \ \mu_I$ are signed real valued measures, we can argue that

$$\begin{aligned}
\mathcal{X}^{(l)}(k) &= \Re\left\{ \mathcal{X}^{(l)}(k) \right\} = \Re\left\{ \int_{-\pi}^{+\pi} e^{j\omega k} A(e^{j\omega})^l d\mu \right\} = \\
&= \Re\left\{ \int_{-\pi}^{+\pi} e^{j\omega k} A(e^{j\omega})^l d\mu_R + j \int_{-\pi}^{+\pi} e^{j\omega k} A(e^{j\omega})^l d\mu_I \right\} = \\
&= \Re\left\{ \int_{-\pi}^{+\pi} e^{j\omega k} A(e^{j\omega})^l d\mu_R \right\} - \Im\left\{ \int_{-\pi}^{+\pi} e^{j\omega k} A(e^{j\omega})^l d\mu_I \right\} = \\
&= \int_{-\pi}^{+\pi} \cos(\omega k) \Re\left\{ A(e^{j\omega})^l \right\} d\mu_R - \int_{-\pi}^{+\pi} \sin(\omega k) \Im\left\{ A(e^{j\omega})^l \right\} d\mu_R + \\
&- \int_{-\pi}^{+\pi} \cos(\omega k) \Im\left\{ A(e^{j\omega})^l \right\} d\mu_I - \int_{-\pi}^{+\pi} \sin(\omega k) \Re\left\{ A(e^{j\omega})^l \right\} d\mu_I = \\
&= \Re\left\{ A(e^{j\omega_0})^l \right\} \left[\int_{-\pi}^{+\pi} \cos(\omega k) \frac{\Re\left\{ A(e^{j\omega})^l \right\}}{\Re\left\{ A(e^{j\omega_0})^l \right\}} d\mu_R - \int_{-\pi}^{+\pi} \sin(\omega k) \frac{\Re\left\{ A(e^{j\omega})^l \right\}}{\Re\left\{ A(e^{j\omega_0})^l \right\}} d\mu_I \right] + \\
&+ \Im\left\{ A(e^{j\omega_0})^l \right\} \left[- \int_{-\pi}^{+\pi} \cos(\omega k) \frac{\Im\left\{ A(e^{j\omega})^l \right\}}{\Im\left\{ A(e^{j\omega_0})^l \right\}} d\mu_I - \int_{-\pi}^{+\pi} \sin(\omega k) \frac{\Im\left\{ A(e^{j\omega})^l \right\}}{\Im\left\{ A(e^{j\omega_0})^l \right\}} d\mu_R \right].
\end{aligned}$$

Observe now that the absolute value of the arguments of the four integrals are bounded by the constant function equal to 1 and that this function is absolutely integrable with respect to both μ_R and μ_I. Notice moreover that

$$\frac{\Re\left\{A(e^{j\omega})^l\right\}}{\Re\left\{A(e^{j\omega_0})^l\right\}} \xrightarrow[l\to\infty]{} \chi_{\omega_0}(\omega)+\chi_{-\omega_0}(\omega)=\begin{cases} 1 & \text{if } \omega=\omega_0 \\ 1 & \text{if } \omega=-\omega_0 \\ 0 & \text{otherwise}\end{cases}$$

and that

$$\frac{\Im\left\{A(e^{j\omega})^l\right\}}{\Im\left\{A(e^{j\omega_0})^l\right\}} \xrightarrow[l\to\infty]{} \chi_{\omega_0}(\omega)-\chi_{-\omega_0}(\omega)=\begin{cases} 1 & \text{if } \omega=\omega_0 \\ -1 & \text{if } \omega=-\omega_0 \\ 0 & \text{otherwise}\end{cases}$$

Consequently, we can apply the Lesbegue dominated convergence theorem. Observe that, since the sequence $\mathcal{X}$ is real, then μ_R is an even measure and μ_R is an odd measure. From this we can argue that, as l tends to ∞, the global state converges

$$\begin{aligned}\mathcal{X}^{(l)}(k) &\sim \Re\left\{A(e^{j\omega_0})^l\right\}\left[2\mu_R(\{\omega_0\})\cos(\omega_0 k)-2\mu_I(\{\omega_0\})\sin(\omega_0 k)\right]+\\ &+ \Im\left\{A(e^{j\omega_0})^l\right\}\left[-2\mu_I(\{\omega_0\})\cos(\omega_0 k)-2\mu_R(\{\omega_0\})\sin(\omega_0 k)\right]=\\ &= \Re\left\{A(e^{j\omega_0})^l e^{j\omega_0 k}\mu(\{\omega_0\})\right\}=\\ &= |\mu(\{\omega_0\})||A(e^{j\omega_0})|^l\cos(\omega_0 k+\arg(A(e^{j\omega_0}))l+\arg(\mu(\{\omega_0\})))\end{aligned}$$

4 Example

In this last section we illustrate the power and the limitations of the results we presented by a simple example. Consider the polynomial operator

$$A(\sigma)=1+\sigma.$$

The curve on the complex plane representing $A(e^{j\omega})=1+e^{j\omega}$ as ω varies in $[-\pi,\pi]$ is shown in figure 1.

Figure 1: The curve on the complex plane representing $A(e^{j\omega})=1+e^{j\omega}$ as ω varies in $[-\pi,\pi]$.

Notice that the dominant frequency is in this case $\omega_0=0$ and that $A(e^{j\omega_0})=2$. Therefore, if the initial global state $\mathcal{X}$ is a Fourier-Stieltjes sequence such that

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{i=-N}^{N}\mathcal{X}(k)=\Delta,$$

where Δ is a nonzero real number, then by the previous results we can argue that

$$\mathcal{X}^{(l)}(k)\sim 2^l\Delta$$

which is a constant function in k which grows with l as 2^l. Assume now that the hypothesis required by Theorems 3 according which $\omega_0=0$ belongs to the jump set of $\mathcal{X}$ does not hold. Take for instance a sequence

$$\mathcal{X}(k)=\delta(k)+e^{j\theta k},$$

where $\theta\neq 0$ and where $\delta(k)$ is the discrete Dirac delta function. In this case

$$\mathcal{X}^{(l)}(k)=\binom{l}{k}+A(e^{j\theta})^l e^{j\theta k}, \tag{8}$$

where $\binom{l}{k}$ is the binomial coefficient which is assumed to be 0 when $k < 0$ or $k > l$. Observe now that the second term of the sum grows as $|A(e^{j\theta})|^l$. The first term of the sum has its maximum when $l = 2k$ and this, by Stirling formula, can be estimated as

$$\binom{2k}{k} = \frac{(2k)!}{2(k!)} \sim \frac{\sqrt{2\pi 2k}(2k)^{2k}e^{-2k}}{[\sqrt{2\pi k}k^k e^{-k}]^2} = \frac{1}{\pi k}2^{2k} = \sqrt{\frac{2}{\pi l}}2^l.$$

This grows less then 2^l, as we expected, but more than $|A(e^{j\theta})|^l$, since for $\theta \neq 0$ we have $|A(e^{j\theta})| < 2$. We can argue that in this case the second term of the sum (8) does not dominate the first term, showing in this way that the hypothesis requiring that dominant frequency belongs to the jump set of the initial global state is essential.

Consider now another example in which

$$A(\sigma) = -\sigma^{-1} + 2 + 2\sigma + 3\sigma^2.$$

The curve on the complex plane representing $A(e^{j\omega}) = -e^{-j\omega} + 2 + 2e^{j\omega} + 3e^{j2\omega}$ as ω varies in $[-\pi, \pi]$ is shown in figure 2.

Figure 2: The curve on the complex plane representing $A(e^{j\omega}) = -e^{-j\omega} + 2 + 2e^{j\omega} + 3e^{j2\omega}$ as ω varies in $[-\pi, \pi]$.

It can be shown that the dominant frequencies are $\pm\omega_0$ where $\omega_0 = 0.3346$ and that for these frequencies we have that $|A(e^{j\omega})| = 6.0137$. We did a simulation using Matlab. We assigned a initial condition

$$\mathcal{X}^{(0)}(k) = 0.2\cos(\omega_0 k) + n(k)$$

where $n(k)$ is a i.i.d. sequence of random variable uniformly distributed on the interval $[0, 1]$. Therefore the initial state has frequency components both in $\omega = 0$ and in $\pm\omega_0$. In Figures 3, 4 and 5 the normalized global state $|A(e^{j\omega})|^{-l}\mathcal{X}^{(l)}(k)$ is compared with the computed limit global state for $l = 0, 100, 600$. Notice that, justified by Figure 2, we have little attenuation for low frequencies and so the convergence is slower for low frequencies.

Figure 3: The normalized global state $|A(e^{j\omega})|^{-l}\mathcal{X}^{(l)}(k)$ for $l = 0$.

Figure 4: The normalized global state $|A(e^{j\omega})|^{-l}\mathcal{X}^{(l)}(k)$ for $l = 100$.

5 Conclusions

In this paper an asymptotic analysis of 2D systems is proposed. The main limitation concerns the requirement that the initial global state has to be a Fourier-Stieltjes sequence. Under this hypothesis a rather complete analysis has been done in the scalar case. It is our strong opinion that the main advantage of this approach is that it seems to be very suitable for dealing with the general vector case. This possible extension is the subject of our present investigation.

Figure 5: The normalized global state $|A(e^{j\omega})|^{-l}\mathcal{X}^{(l)}(k)$ for $l = 600$.

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