**Keywords**:

Introduction

It is a great pleasure for us to contribute to the Festschrift honoring Tadeusz Kaczorek on the occasion of his 70th birthday. Tadeusz is a long-time friend of both of us. The story of our friendship is strictly intertwined with the evolution of the research on 2D systems, which represents the main field of his and ours scientific interests.

The theory of 2D state space models already reached a certain maturity in the mid of 1970's. Subsequent developments, involving a systematic use of polynomial matrix methods, provided new vistas on 2D reachability and observability, allowing for a neat solution to feedback regulators and state observers synthesis problems. In his book, *Two-dimensional linear systems* (1985), Tadeusz provided a first comprehensive and balanced account of the results obtained up to that time by many researchers, working all over the world.

In the early 1990's a major obstacle to the representation of 2D systems was the fact that multidimensional dynamics can exhibit very complicated causal structures, and neither quarter plane nor weakly causal models were able to completely capture their complexity. An essential component in the puzzle of how to represent general 2D systems still appeared to be missing. However, with the maturing of Kaczorek's theory of singular 2D systems and the introduction by Rocha, Willems et other researchers of behavioral methods in multidimensional systems analysis, we can now deal with virtually all classes of 2D systems.

Recent years have seen a growing interest in 2D systems that are subject to positivity constraints on their dynamical variables. There are actually several different motivations for this interest, coming from various domains of science and technology. Positive 2D systems arise, for instance, when discretizing pollution and selfpurification processes along a river stream, or when one tries to construct a discrete model for the traffic flow in a motorway. More generally, the positivity assumption is a natural one when describing, via 2D systems, distributed processes whose variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc. This led the authors of this contribution to embark since 1995 on a new path, where attention is focused essentially on the free evolution of 2D positive systems, and therefore on the investigation of algebraic and combinatorial properties of positive matrix pairs appearing in the state equations. Tadeus Kaczorek's constant concern with emerging research themes led him, almost at the same time, in the field of positive 2D systems, where he took, however, a more eclectic attitude. In fact, his relevant contributions are mainly concerned with various control, estimation and stabilization strategies in a positive environment, and the large number of his papers on the subject certify his constant commitment to analysing system theoretic problems in their different facets.

This paper deals with the circle of ideas that includes the theory of positive matrix pairs and the dynamics of the corresponding 2D state space models. In the next section, the main properties of the characteristic polynomial of a positive matrix pair are summarized, and some special classes of positive pairs, as well as their connections with the dynamics behavior of 2D state models, are investigated. Section 3 is devoted to combinatorial properties of irreducible and primitive positive pairs and to the analysis of 2D systems whose local states updating is governed by such pairs. Section 4 sketches some results on dominant eigenvectors and reports further issues and open problems on matrix pairs arising from current research.

Detailed proofs of the results presented in sections 2 and 3 can be found in [**?**, **?**]. For a more general discussion on (nonnecessarily positive) matrix pairs and 2D systems, we refer the interested reader to [].

Before proceeding, we introduce some notation and recall some basic facts about positive (and nonnegative) vectors and matrices. If $M = [m_{ij}]$ is a matrix (in particular, a vector), we write

*i)* $M \gg 0$ (*$M$ strictly positive*), if $m_{ij} > 0$ for all $i, j$;

*ii*) $M > 0$ (*M positive or strictly nonnegative*), if $m_{ij} \geq 0$ for all $i, j$, and $m_{hk} > 0$ for at least one pair $(h, k)$;

*iii*) $M \geq 0$ (*M nonnegative*), if $m_{ij} \geq 0$ for all $i, j$.

Two matrices $M$ and $N$, with the same dimensions, have the same *zero pattern* if $m_{ij} = 0$ implies $n_{ij} = 0$ and vice versa.

A nonnegative $n \times n$ matrix $M$, $n \geq 2$, is called *reducible* if there exists a permutation matrix $P$ such that

$$P^T M P = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \tag{1}$$

where $M_{11}$ and $M_{22}$ are square submatrices. Otherwise $M$ is *irreducible*.
The zero patterns of the powers $M^\nu$ of an irreducible matrix $M$ can exhibit different behaviours for large values of $\nu$. If there exists an integer $N$ such that $M^\nu \gg 0$ for all $\nu \geq N$, then $M$ is called *primitive*; otherwise there exist positive integers $h$ and $t_{ij}$, $i, j = 1, 2, ..., n$, such that for all $\nu \geq t_{ij}$, $[M^\nu]_{ij}$, the $(i, j)$-th entry of $M^\nu$, is positive if and only if $\nu = t_{ij} + \ell h$. The two cases admit a spectral characterization. Actually, a primitive matrix $M$ has a simple real positive eigenvalue $r$, whose module is strictly greater than the module of any other eigenvalue of $M$. On the other hand, if $M$ is not primitive, its spectrum includes $h$ simple eigenvalues of maximal module, i.e. $r, re^{j\frac{2\pi}{h}}, ..., re^{j\frac{2\pi}{h}(h-1)}$, $r > 0$. The integer $h$ is called the *imprimitivity index* of $M$.

Finite memory, separability and property L

2D homogeneous positive systems considered in this paper are described by the equation:

$$x(h + 1, k + 1) = A\ x(h, k + 1) + B\ x(h + 1, k), \tag{2}$$

where the doubly indexed local state sequence $x(\cdot, \cdot)$ takes values in the positive cone $\mathbf{R}^n_+ := \{x \in \mathbf{R}^n : x_i \geq 0,\ i = 1, 2, ..., n\}$, $A$ and $B$ are nonnegative $n \times n$ matrices, and the initial conditions are assigned by specifying the nonnegative values of the local states on the *separation set* $\mathcal{C}_0 := \{(i, -i) : i \in \mathbf{Z}\}$.

There are essentially two reasons why the investigation of homogeneous positive systems is more difficult in 2D than in 1D case. First of all, the dynamics of a 2D system (2) is determined by the matrix pair $(A, B)$ and, as well known, the algebraic tools we use for studying a pair of linear transformations are not as simple and effective as those available for the analysis of a single linear transformation. In particular, as we shall see, a natural 2D extension of the Perron-Frobenius theorem is not immediately apparent. On the other hand, the free evolution is strongly influenced by the choice of the nonnegative initial local states and, most of all, by the support of the states sequence on $\mathcal{C}_0$.

For a given matrix pair $(A, B)$ we define the characteristic polynomial

$$\Delta_{A,B}(z_1, z_2) := \det(I - Az_1 - Bz_2), \tag{3}$$

and denote by $\mathcal{V}(\Delta_{A,B})$ the corresponding variety, i.e. the set of all (complex) solutions of the equation $\Delta_{A,B}(z_1, z_2) = 0$.

When dealing with a generic (i.e. nonnecessarily positive) 2D system, some natural assumptions on the structure of the characteristic polynomial allow to single out important classes of systems, whose dynamical behaviour exhibits very peculiar, distinguishing features. The classes of systems we will consider in this section are the following ones:

**finite memory systems** whose state evolution goes to zero in a finite number of steps;

**separable systems** whose characteristic polynomial factorizes into

$$\Delta_{A,B}(z_1, z_2) = r(z_1)s(z_2),$$

for suitable polynomials $r(z_1) \in \mathbf{R}[z_1]$ and $s(z_2) \in \mathbf{R}[z_2]$;
**systems with property L**, described by two matrices $A$ and $B$ whose eigenvalues can be ordered into two $n$-tuples

$$\Lambda(A) = (\lambda_1, \lambda_2, ..., \lambda_n) \quad \text{and} \quad \Lambda(B) = (\mu_1, \mu_2, ..., \mu_n) \tag{4}$$

such that, for all $\alpha, \beta$ in $\mathbf{C}$, the spectrum $\Lambda(\alpha A + \beta B)$ is given by

$$\Lambda(\alpha A + \beta B) = (\alpha\lambda_1 + \beta\mu_1, \alpha\lambda_2 + \beta\mu_2, ..., \alpha\lambda_n + \beta\mu_n). \tag{3.12}$$

The nonnegativity hypothesis introduces further constraints on the structure of the above systems, we will explore in some detail. To this purpose, we introduce the Hurwitz products of a matrix pair $(A, B)$ which are inductively defined as

$$A^i \sqcup\!\sqcup^0 B = A^i, \qquad A^0 \sqcup\!\sqcup^j B = B^j \tag{5}$$

and, when $i$ and $j$ are both greater than zero,

$$A^i \sqcup\!\sqcup^j B = A(A^{i-1} \sqcup\!\sqcup^j B) + B(A^i \sqcup\!\sqcup^{j-1} B). \tag{6}$$

Note that the sequence of local states $x(\cdot, \cdot)$ one obtains by assuming zero initial conditions on $\mathcal{C}_0$, except at $(0, 0)$, is represented by the power series

$$\begin{aligned} X(z_1, z_2) &= \sum_{h,k \geq 0} x(h, k) z_1^h z_2^k = (I - Az_1 - Bz_2)^{-1} x(0, 0) \\ &= \sum_{h,k \geq 0} A^h \sqcup\!\sqcup^k B \; x(0, 0) z_1^h z_2^k. \end{aligned}$$

As a consequence, the local state at $(h, k)$, $x(h, k) = A^h \sqcup\!\sqcup^k B \; x(0, 0)$, has to be interpreted as the sum of the elementary contributions along all paths connecting $(0, 0)$ to $(h, k)$ in the two-dimensional grid [Fornasini Marchesini, 1993].

**Proposition 2.1** [Finite memory] [**?**] For a pair of $n \times n$ nonnegative matrices $(A, B)$, the followings are equivalent

*i*) the associated 2D system (2) is finite memory;

*ii*) there is only a finite number of nonzero Hurwitz products

*iii*) $\Delta_{A,B}(z_1, z_2) = 1$;

*iv*) $A + B$ is nilpotent;

*v*) there exists a permutation matrix $P$ such that $P^T AP$ and $P^T BP$ are both upper triangular matrices with zero diagonal.

As a consequence of *v*), if we perform a permutation on the basis of the local state space, so as to reduce matrices $A$ and $B$ into upper triangular form, it is easy to realize that, for any initial global state on $\mathcal{C}_0$, the last $t$ components of the local states on the separation set $\mathcal{C}_t = \{(h, k) : h + k = t\}$ are identically zero.

Separability property can be characterized in terms of $\mathbf{S}(A, B)$, the multiplicative semigroup generated by $A$ and $B$.

**Proposition 2.2** [Separability] [**?**] For a pair of $n \times n$ matrices $A > 0$ and $B > 0$, the following facts are equivalent:

*i*) the associated 2D system (2) is separable, i.e. $\Delta_{A,B}(z_1, z_2) = r(z_1)s(z_2)$;

*ii*) every matrix product in $\mathbf{S}(A,B)$ has zero trace, provided that both $A$ and $B$ appear at least once;

*iii*) $w(A,B)$ is nilpotent, for all $w \in \Xi^*$ such that $|w|_i > 0,\ i = 1,2$.

Proof $i) \Leftrightarrow ii)$ To prove this equivalence we refer to a characterization of separability, presented in [Fornasini Marchesini Valcher, 1993], which states that $(A,B)$ is separable if and only if

$$\mathrm{tr}(A^i ш^j B) = 0, \qquad \forall\ (i,j), \quad i > 0,\ j > 0. \tag{3.8}$$

As tr $(A^i ш^j B) = \sum_{|w|_1=i, |w|_2=j} \mathrm{tr}\ w(A,B)$, and all the words $w(A,B)$ are nonnegative, (3.8) implies *ii*). The converse is always true.

$ii) \Leftrightarrow iii)$ By assumption *ii*), for each $w \in \Xi^*$, with $|w|_1 > 0$ and $|w|_2 > 0$, we have

$$\mathrm{tr}(w(A,B))^k = 0, \qquad k = 1, 2, ...,$$

which implies the nilpotency of $w(A,B)$.
Conversely, the nilpotency of $w(A,B)$ trivially implies that tr $w(A,B) = 0$ ■

Separable nonnegative pairs can be reduced to two different canonical forms. One is obtained by resorting to permutation matrices, i.e. to a reordering of the basis of the local state space, while the other is based on a (complex) similarity transformation, namely a more general change of basis. To construct the canonical forms we need the following Lemma:

**Lemma 3.3** Let $A > 0$ and $B > 0$ constitute a separable pair; then $A + B$ is a reducible matrix.

Proof Consider any $w = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_m} \in \Xi^*$, with $|w|_1 > 0$ and $|w|_2 > 0$. Because of the characterization *ii*) of separability given in Proposition 3.2, each diagonal element of $w(A,B)$ is zero, and therefore for any sequence of integers $\ell_1, \ell_2, ..., \ell_m \in \{1, 2, ..., n\}$

$$[\psi(\xi_{i_1})]_{\ell_1\ell_2}[\psi(\xi_{i_2})]_{\ell_2\ell_3}...[\psi(\xi_{i_m})]_{\ell_m\ell_1} = 0. \tag{3.9}$$

As $A$ and $B$ are nonzero, there exist entries $[A]_{ij} > 0$ and $[B]_{hk} > 0$. If $A + B$ were irreducible, there would be integers $p$ and $q$ such that $[(A+B)^p]_{jh} > 0$ and $[(A+B)^q]_{ki} > 0$. Consequently

$$[\psi(\xi_{t_1})]_{j\ell_1}[\psi(\xi_{t_2})]_{\ell_1\ell_2}...[\psi(\xi_{t_p})]_{\ell_{p-1}h} > 0$$

and

$$[\psi(\xi_{s_1})]_{kr_1}[\psi(\xi_{s_2})]_{r_1r_2}...[\psi(\xi_{s_q})]_{r_{q-1}i} > 0$$

for appropriate choices of $\xi_{t_\nu}$ and $\xi_{s_\mu}$ and of the indexes $\ell_\nu$ and $r_\mu$. Therefore

$$[A]_{ij}\ [\psi(\xi_{t_1})]_{j\ell_1}[\psi(\xi_{t_2})]_{\ell_1\ell_2}...[\psi(\xi_{t_p})]_{\ell_{p-1}h}\ [B]_{hk}\ [\psi(\xi_{s_1})]_{kr_1}[\psi(\xi_{s_2})]_{r_1r_2}...[\psi(\xi_{s_q})]_{r_{q-1}i} > 0,$$

which contradicts (3.9) ■

**Proposition 3.4** Let $(A,B)$ be an $n \times n$ nonnegative matrix pair, then the followings are equivalent:

*i*) $\Delta_{A,B}(z_1,z_2) = r(z_1)s(z_2)$;

*ii*) there exists a permutation matrix $P$ such that $P^TAP$ and $P^TBP$ are conformably partitioned into block triangular matrices

$$P^TAP = \begin{bmatrix} A_{11} & * & * & * \\ & A_{22} & * & * \\ & & \ddots & * \\ & & & A_{tt} \end{bmatrix} \quad P^TBP = \begin{bmatrix} B_{11} & * & * & * \\ & B_{22} & * & * \\ & & \ddots & * \\ & & & B_{tt} \end{bmatrix}, \tag{3.10}$$

where $A_{ii} \neq 0$ implies $B_{ii} = 0$. It entails no loss of generality assuming that the nonzero diagonal blocks in $P^T AP$ and $P^T BP$ are irreducible;

*iii*) there exists a nonsingular matrix $T$ such that $\hat{A} = T^{-1}AT$ and $\hat{B} = T^{-1}BT$ are upper triangular matrices and $\hat{a}_{ii} \neq 0$ implies $\hat{b}_{ii} = 0$.

Proof If one of the matrices is zero, the proposition is trivially true, so we will confine ourselves to the case of $A$ and $B$ both nonzero.

*i*) $\Rightarrow$ *ii*) By the previous Lemma, there exists a permutation matrix $P_1$ s.t.

$$P_1^T(A+B)P_1 = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}$$

and, consequently,

$$P_1^T AP_1 + P_1^T BP_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

where $A_{ii}, B_{ii}$ and $C_{ii}$, $i = 1, 2$, are square submatrices. As the nonnegative matrix pairs $(A_{ii}, B_{ii})$ are separable, we can apply the same procedure as before to both of them. By iterating this method we end up with a pair of matrices with the structure (3.10).

*ii*) $\Rightarrow$ *iii*) Let $P$ be a permutation matrix that reduces $A$ and $B$ as in (3.10). Recalling that every square matrix is similar to an upper triangular matrix, consider the matrix $Q = \text{diag}\{Q_{11}, Q_{22}, ..., Q_{tt}\}$, where $Q_{ii}$ are nonsingular square matrices such that $Q_{ii}^{-1}(A_{ii} + B_{ii})Q_{ii}$ is upper triangular. Then $T = PQ$ is the nonsingular matrix we are looking for.

*iii*) $\Rightarrow$ *i*) Obvious ■

Property L:
the matrix pair $(A, B)$, and corresponds to the possibility of factorizing $\Delta_{A,B}(z_1, z_2)$ into the product of linear factors, as follows

$$\Delta_{A,B}(z_1, z_2) = \prod_{i=1}^{n}(1 - \lambda_i z_1 - \mu_i z_2). \tag{7}$$

Under appropriate assumptions, nonnegativity of $A$ and $B$ allows for some precise statements concerning the coupling of their maximal eigenvalues.

**Proposition 3.5** [Property L] Let $(A, B)$ be a nonnegative $n \times n$ matrix pair, endowed with property L w.r.t. the orderings (4), and assume $A + B$ irreducible.
Then there exists a unique index $i$ such that

$$\lambda_i, \mu_i \in \mathbf{R}_+, \qquad \lambda_i \geq |\lambda_j|, \qquad \mu_i \geq |\mu_j|, \qquad j = 1, 2, ..., n,$$

and, for each $\alpha, \beta > 0$, $\alpha\lambda_i + \beta\mu_i$ is the maximal positive eigenvalue of the irreducible matrix $\alpha A + \beta B$.

Proof Denoting by $\nu_1(\alpha), \nu_2(\alpha), ..., \nu_n(\alpha)$ the eigenvalues of $\alpha A + (1-\alpha)B$, property L implies that

$$\nu_j(\alpha) = \alpha\lambda_j + (1-\alpha)\mu_j, \qquad j = 1, 2, ..., n. \tag{3.13}$$

Moreover, for all $\alpha \in (0, 1)$, the matrix $\alpha A + (1-\alpha)B$, having the same zero-pattern as $A + B$, is irreducible and hence has a simple maximal eigenvalue $\nu_{\max}(\alpha)$. We aim to prove that there exists an integer $i$ such that for all $\alpha$, $\nu_{\max}(\alpha) = \alpha\lambda_i + (1-\alpha)\mu_i$, where $\lambda_i$ and $\mu_i$ are real positive eigenvalues of $A$ and $B$, respectively.

Note first that the characteristic polynomial

$$\Delta_{A,B}(z_1, z_2) = \prod_{i=1}^{n}(1 - \lambda_i z_1 - \mu_i z_2). \tag{3.14}$$

belongs to $\mathbf{R}[z_1, z_2]$. So, if one factor $1 - \lambda_i z_1 - \mu_i z_2$ has not real coefficients, also $1 - \bar{\lambda}_i z_1 - \bar{\mu}_i z_2$ appears in (3.14). That amounts to say that, when a nonreal pair $(\lambda_j, \mu_j)$ appears in (4), also the conjugate pair $(\bar{\lambda}_j, \bar{\mu}_j)$ does, and hence both $\nu_j(\alpha) = \alpha\lambda_j + (1-\alpha)\mu_j$ and $\nu_k(\alpha) = \alpha\bar{\lambda}_j + (1-\alpha)\bar{\mu}_j$ belong to $\Lambda(\alpha A + (1-\alpha)B)$. Moreover, $\nu_j(\alpha)$ is real if and only if $\nu_k(\alpha)$ is, and they take the same value. As $\nu_{\max}(\alpha)$, $0 < \alpha < 1$, has to be simple, it cannot coincide with any eigenvalue $\nu_j(\alpha)$ associated with a nonreal pair $(\lambda_j, \mu_j)$.
Therefore, an integer $j(\alpha)$ exists, possibly depending on $\alpha$, such that $(\lambda_j, \mu_j)$ is a real pair and

$$\nu_{\max}(\alpha) = \nu_{j(\alpha)}(\alpha). \tag{3.15}$$

Because of the linear structure of (3.13), we can determine finitely many points, $\alpha_1, \alpha_2, ..., \alpha_r$, $0 < \alpha_1 < \alpha_2 < ... < \alpha_r < 1$, with the property that the index $j(\alpha)$ in (3.15) remains constant on each interval $(\alpha_\mu, \alpha_{\mu+1})$, $\mu = 1, 2, ..., r-1$, and takes different values on different intervals. If $r$ were greater than zero, $\nu_{\max}(\alpha_\mu)$, $\mu = 1, 2, ..., r$, would be a multiple eigenvalue of the irreducible matrix $\alpha A + (1-\alpha)B$, a contradiction. So $r$ has to be zero and $j(\alpha)$ takes in $(0, 1)$ a unique value $i$.
Next, we show that $\lambda_i$ and $\mu_i$ are maximal eigenvalues of $A$ and $B$. Suppose, for instance, that $A$ possesses a positive eigenvalue $\lambda_h > \lambda_i$. As the eigenvalues of $\alpha A + (1-\alpha)B$ are continuous functions of $\alpha$, $|\nu_h(\alpha)|$ would be greater than $|\nu_i(\alpha)|$ for all values of $\alpha$ in a suitable neighbourhood of 1, a contradiction.
Finally, letting $\bar{\alpha} = \alpha/(\alpha+\beta)$ and $1-\bar{\alpha} = \beta/(\alpha+\beta)$, we have that $\bar{\alpha}\lambda_i + (1-\bar{\alpha})\mu_i$ is the maximal positive eigenvalue of $\bar{\alpha}A + (1-\bar{\alpha})B = \frac{1}{\alpha+\beta}(\alpha A + \beta B)$ and, consequently, $\alpha\lambda_i + (1-\alpha)\mu_i$ is the maximal positive eigenvalue of $\alpha A + \beta B$ ■

**Example 1** The pair

$$A = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} \qquad B = \begin{bmatrix} 2/5 & 3/10 \\ 2/5 & 4/5 \end{bmatrix}$$

is endowed with property L w.r.t. the orderings

$$\Lambda(A) = (1,\ -1/2), \qquad \Lambda(B) = (1,\ 1/5).$$

For each $\alpha > 0$ and $\beta > 0$, the maximal eigenvalue $\alpha + \beta$ of $\alpha A + \beta B$ is obtained as a linear combination of the maximal eigenvalues of $A$ and $B$. Note that $A + B$ is strictly positive, and hence irreducible. When we drop the irreducibility assumption, as, for instance, with the pair

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

the maximal eigenvalues of $A$ and $B$ are not necessarily coupled w.r.t. the orderings of the spectra, and hence do not appear in the same linear factor of the characteristic polynomial $\Delta_{A,B}(z_1, z_2)$.

Cyclic structure of 2D-digraphs

A *2D-digraph* $\mathcal{D}^{(2)}$ is a triple $(V, \mathcal{A}, \mathcal{B})$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of *vertices*, and $\mathcal{A}$ and $\mathcal{B}$ are subsets of $V \times V$ whose elements are called $\mathcal{A}$-*arcs* and $\mathcal{B}$-*arcs*, respectively. There is an $\mathcal{A}$-arc from $v_i$ to $v_j$ if $(v_i, v_j)$ is in $\mathcal{A}$, and a $\mathcal{B}$-arc if $(v_i, v_j)$ is in $\mathcal{B}$.
When assigning a *path* $p$ in $\mathcal{D}^{(2)}$ one has to specify, for each pair of consecutive vertices, which kind of arc they are connected by, thus giving $p$ a representation like $(v_{i_0}, v_{i_1})_{\mathcal{A}}$, $(v_{i_1}, v_{i_2})_{\mathcal{B}}$,

$\dots, (v_{i_{k-1}}, v_{i_k})_{\mathcal{B}}$. Sometimes, however, when we are interested only in the vertices $p$ passes through, we drop the subscripts. Also, when the emphasis is only in the initial and final vertices, we adopt the shorthand notation $v_{i_0} \underset{p}{\longrightarrow} v_{i_k}$.

If we denote by $\alpha(p)$ and $\beta(p)$ the number of $\mathcal{A}$-arcs and $\mathcal{B}$-arcs occurring in $p$, then $[\alpha(p) \ \ \beta(p)]$ is the *composition* of $p$ and $|p| = \alpha(p) + \beta(p)$ its *length*. A path whose extreme vertices coincide, i.e. $v_{i_0} = v_{i_k}$, is called a *cycle*. In particular, if each vertex in a cycle appears exactly once as the first vertex of an arc, the cycle is called a *circuit*.

**Definition** A 2D-digraph $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ is called

i) *strongly connected* if for every pair of vertices $v_i$ and $v_j$ in $V$ there is a path $p$ connecting $v_i$ to $v_j$;

ii) *2D-strongly connected* if for every pair of vertices $v_i$ and $v_j$ in $V$ there are two paths $v_i \underset{p_1}{\longrightarrow} v_j$ and $v_i \underset{p_2}{\longrightarrow} v_j$, connecting $v_i$ to $v_j$, for which

$$\det \begin{bmatrix} \alpha(p_1) & \beta(p_1) \\ \alpha(p_2) & \beta(p_2) \end{bmatrix} \neq 0. \tag{8}$$

The 2D-digraph $\mathcal{D}^{(2)}$ is naturally associated with a 1-*digraph* (i.e. a standard digraph) $\mathcal{D}^{(1)} = (V, \mathcal{E})$, having the same vertices as $\mathcal{D}^{(2)}$ and $\mathcal{E} := \mathcal{A} \cup \mathcal{B}$ as its set of arcs. So, property i) corresponds to the fact that $\mathcal{D}^{(1)}$ is strongly connected (in the ordinary sense), while property ii) requires something more, namely that for every pair of vertices, $v_i$ and $v_j$, the ratio $\beta(p)/\alpha(p)$, (that is considered $\infty$ when $\alpha(p) = 0$), between the number of $\mathcal{B}$-arcs and $\mathcal{A}$-arcs is not invariant as $p$ varies over the set of all paths connecting $v_i$ to $v_j$.

*In this paper all 2D-digraphs will be assumed strongly connected with both sets $\mathcal{A}$ and $\mathcal{B}$ nonempty.*

We associate with the (finite) set $\{\gamma_1, \gamma_2, \dots, \gamma_t\}$ of all circuits in $\mathcal{D}^{(2)}$ (arbitrarily ordered) the *circuit matrix*

$$L(\mathcal{D}^{(2)}) := \begin{bmatrix} \alpha(\gamma_1) & \beta(\gamma_1) \\ \alpha(\gamma_2) & \beta(\gamma_2) \\ \vdots & \vdots \\ \alpha(\gamma_t) & \beta(\gamma_t) \end{bmatrix} \in \mathbb{N}^{t \times 2}, \tag{9}$$

and denote by $M(\mathcal{D}^{(2)})$ the $\mathbb{Z}$-module generated by its rows. Since every cycle $\gamma$ in $\mathcal{D}^{(2)}$ decomposes into a certain number of circuits, it follows that there exist $n_1, n_2, \dots, n_t \in \mathbb{N}$, such that

$$[\alpha(\gamma) \ \ \beta(\gamma)] = [n_1 \ n_2 \ \dots \ n_t] \ \ L(\mathcal{D}^{(2)}), \tag{10}$$

i.e. $[\alpha(\gamma) \ \ \beta(\gamma)]$ is an element of $M(\mathcal{D}^{(2)}) \cap \mathbb{N}^2$. In general, however, $M(\mathcal{D}^{(2)}) \cap \mathbb{N}^2$ properly includes the set of integer pairs representing the compositions of the cycles in $\mathcal{D}^{(2)}$.

As a submodule of $\mathbb{Z}^2$, $M(\mathcal{D}^{(2)})$ admits a basis consisting either of one or of two elements. In the first case $M(\mathcal{D}^{(2)})$ has only two possible bases, namely $\{[\ell \ m]\}$, for some positive integers $\ell$ and $m$, and its opposite $\{-[\ell \ m]\}$, and every circuit $\gamma_j$ in $\mathcal{D}^{(2)}$ consists of $k_j\ell$ $\mathcal{A}$-arcs and $k_j m$ $\mathcal{B}$-arcs, for a suitable $k_j$ in $\mathbb{N}$. On the other hand, when $L(\mathcal{D}^{(2)})$ has rank 2, we can consider its Hermite form over $\mathbb{Z}$

$$\bar{H} := \begin{bmatrix} H \\ \\ 0 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & h_{22} \\ & \\ \multicolumn{2}{c}{0} \end{bmatrix} = \bar{U} L(\mathcal{D}^{(2)}), \tag{11}$$

$\bar{U} \in \mathbb{Z}^{t \times t}$ unimodular, and assume (without loss of generality) that $h_{11}$ and $h_{22}$ are positive integers, and $0 \leq h_{12} < h_{22}$. The rows of $H$ provide a particular basis of $M(\mathcal{D}^{(2)})$, $\{[h_{11} \ h_{12}], [0 \ h_{22}]\}$, and the rows $[w_{11} \ \ w_{12}]$ and $[w_{21} \ \ w_{22}]$ of $W = UH$, as $U$ varies over the group of unimodular matrices in $\mathbb{Z}^{2\times 2}$, give all possible bases of $M(\mathcal{D}^{(2)})$. Since the determinants of all matrices $UH$ have the same modulus, which is the g.c.d. of the second order minors of $L(\mathcal{D}^{(2)})$, all parallelograms $\{\varepsilon[w_{11}\ \ w_{12}] + \delta[w_{21}\ \ w_{22}] : \varepsilon, \delta \in [0, 1)\}$, have the same area, that coincides with the number of integer pairs they include [**?**, **?**].

The cyclic structure of $\mathcal{D}^{(2)}$ and the module $M(\mathcal{D}^{(2)})$ provide enough information to decide whether the 2D-digraph is 2D-strongly connected, as shown in the following proposition.

**Proposition 2.1** *Let $\mathcal{D}^{(2)}$ be a strongly connected 2D-digraph. The following facts are equivalent*

*i) $\mathcal{D}^{(2)}$ is 2D-strongly connected;*

*ii) there are a vertex $v_i$ and two cycles $\gamma$ and $\tilde{\gamma}$, passing through $v_i$, for which*

$$\det \begin{bmatrix} \alpha(\gamma) & \beta(\gamma) \\ \alpha(\tilde{\gamma}) & \beta(\tilde{\gamma}) \end{bmatrix} \neq 0;$$

*iii) there are two circuits $\gamma_i$ and $\gamma_j$ satisfying*

$$\det \begin{bmatrix} \alpha(\gamma_i) & \beta(\gamma_i) \\ \alpha(\gamma_j) & \beta(\gamma_j) \end{bmatrix} \neq 0;$$

*iv) rank $L(\mathcal{D}^{(2)}) = 2$;*

*v) $M(\mathcal{D}^{(2)})$ has a basis consisting of two elements.*

As it is well-known, the lengths of all cycles in a strongly connected 1-digraph $\mathcal{D}^{(1)}$, with imprimitivity index $h$, are multiples of $h$, and there exists a positive integer $T$ such that, for all integers $t \in [T, +\infty) \cap (h)$, there is a cycle in $\mathcal{D}^{(1)}$ of length $t$ [**?**]. A similar statement holds for a 2D-strongly connected digraph $\mathcal{D}^{(2)}$, upon considering for each cycle $\gamma$ in $\mathcal{D}^{(2)}$ not just its length, but its composition $[\alpha(\gamma)\ \beta(\gamma)]$. In this case the module $(h)$ and the half-line $[T, +\infty)$ have to be replaced by $M(\mathcal{D}^{(2)})$ and by a suitable convex cone in $\mathbb{R}^2_+$, respectively.

**Proposition 2.3** *Let $\mathcal{D}^{(2)}$ be a strongly connected 2D-digraph, and let*

$$\mathcal{S} := \left\{ [\alpha(\gamma)\ \ \beta(\gamma)] \in \mathbb{N}^2 : \gamma \text{ a cycle in } \mathcal{D}^{(2)} \right\}$$

*be the set of compositions of all cycles in $\mathcal{D}^{(2)}$.*

*i) If $M(\mathcal{D}^{(2)})$ has rank 1 and is generated by $[\ell\ m] \in \mathbb{N}^2$, there exists $\tau \in \mathbb{N}$ s.t.*

$$\{t\ [\ell\ \ m] : t \in \mathbb{N}, t \geq \tau\} \subseteq \mathcal{S} \subseteq \{t\ [\ell\ \ m] : t \in \mathbb{N}\}. \tag{12}$$

*ii) If $M(\mathcal{D}^{(2)})$ has rank 2 and $\mathcal{K} \subseteq \mathbb{R}^2_+$ denotes the solid (i.e., with nonempty interior) convex cone generated by the rows of $L(\mathcal{D}^{(2)})$, there exists $[u\ \ w] \in M(\mathcal{D}^{(2)}) \cap \mathcal{K}$ such that*

$$M(\mathcal{D}^{(2)}) \cap \Big([u\ \ w] + \mathcal{K}\Big) \subseteq \mathcal{S} \subseteq M(\mathcal{D}^{(2)}) \cap \mathcal{K}. \tag{13}$$

**Example 2.1** In the 2D-digraph $\mathcal{D}^{(2)}$ of Fig. 2.1 $\mathcal{A}$-arcs are represented by thicklines and $\mathcal{B}$-arcs by thinlines (this notation will be adopted in all subsequent pictures). The circuit matrix

$$L(\mathcal{D}^{(2)}) = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 3 & 0 \end{bmatrix}$$

is right prime and hence generates the $\mathbb{Z}$-module $\mathbb{Z}^2$. The set of all vectors $[\alpha(\gamma)\ \ \beta(\gamma)]$ that correspond to some cycle $\gamma$ in $\mathcal{D}^{(2)}$ is represented in Fig. 2.2, and we see that all integer pairs inside $[7\ \ 3] + \mathcal{K}$, $\mathcal{K}$ the cone generated in $\mathbb{R}^2_+$ by $[3\ \ 0]$ and $[1\ \ 1]$, correspond to a cycle in $\mathcal{D}^{(2)}$.

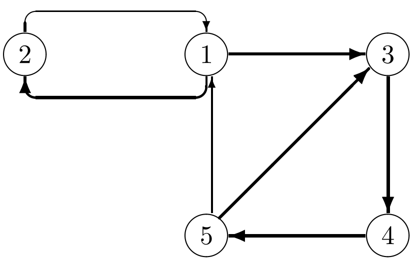


Fig. 2.1 Structure of $\mathcal{D}^{(2)}$

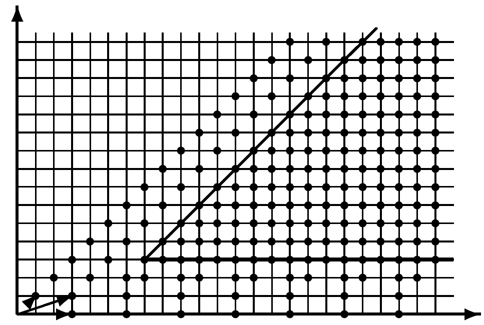

Fig. 2.2 Cycles in $\mathcal{D}^{(2)}$

**Example 2.2** In the 2D-digraph of Fig. 2.3 the circuit matrix

$$L(\mathcal{D}^{(2)}) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

has rank 1 and generates the module $\mathbb{Z} \cdot [1 \;\; 1]$. The compositions $[\alpha(\gamma) \;\; \beta(\gamma)]$ of all cycles are represented in Fig. 2.4.

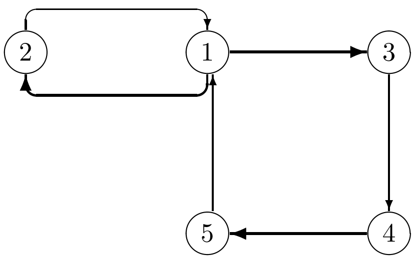


Fig. 2.3 Structure of $\mathcal{D}^{(2)}$

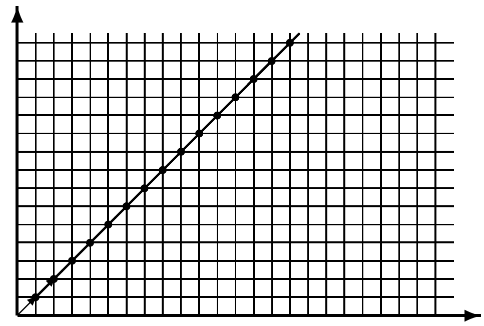

Fig. 2.4 Cycles in $\mathcal{D}^{(2)}$

Paths and imprimitivity classes

As a consequence of (**??**), once a particular 2D-imprimitivity class has been selected as a reference, all classes can be unambiguously indexed by the elements of the quotient module $\mathbb{Z}^2/M(\mathcal{D}^{(2)})$, in the sense that each class is indexed by a coset $[\alpha(p) \;\; \beta(p)] + M(\mathcal{D}^{(2)})$, $p$ being any path that reaches the class, starting from the reference one. We may ask under what conditions the above correspondence, mapping 2D-imprimitivity classes into cosets, is bijective, which amounts to say that for every coset $[h \;\; k] + M(\mathcal{D}^{(2)})$ there is a path $p$ starting from the reference class and having composition $[\alpha(p) \;\; \beta(p)] \equiv [h \;\; k] \bmod M(\mathcal{D}^{(2)})$.
Clearly, when $M(\mathcal{D}^{(2)})$ has rank 1, the quotient module $\mathbb{Z}^2/M(\mathcal{D}^{(2)})$ includes infinitely many elements, and no bijection exists between the (finite) set of 2D-imprimitivity classes and $\mathbb{Z}^2/M(\mathcal{D}^{(2)})$. On the other hand, when the module $M(\mathcal{D}^{(2)})$ has rank 2, this correspondence always exists. The result follows from Proposition 3.3, below, which shows that every integer pair of the cone $\mathcal{K}$, generated by the rows of $L(\mathcal{D}^{(2)})$, represents the composition of some path in $\mathcal{D}^{(2)}$.

**Proposition 3.3** *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a strongly connected 2D-digraph and $\mathcal{K}$ the solid convex cone generated in $\mathbb{R}^2_+$ by the rows of $L(\mathcal{D}^{(2)})$. For every integer pair $[h \;\; k]$ in $\mathcal{K}$ there exist a pair of vertices $v_i$ and $v_j$ and a path $v_i \xrightarrow[p]{} v_j$ such that*

$$[\alpha(p) \;\; \beta(p)] = [h \;\; k]. \tag{14}$$

**Example 3.3** Consider the strongly connected 2D-digraph $\mathcal{D}^{(2)}$ of Fig. 3.1. The integer pairs giving the compositions of the paths in $\mathcal{D}^{(2)}$ are represented in Fig. 3.3, below, where full circles correspond to cycles and empty circles to open paths.

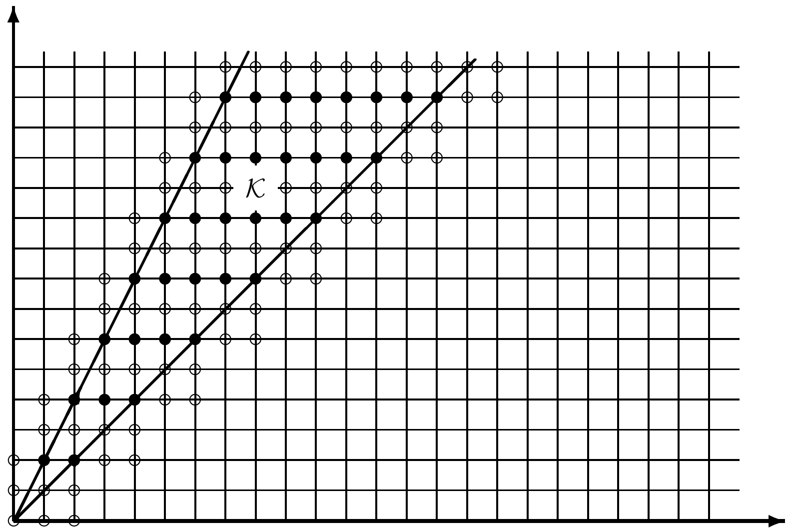


Fig. 3.3 Cycles and paths in $\mathcal{D}^{(2)}$

**Example 3.4** Consider the 2D-digraph of Fig. 3.4. It is immediate to verify that $M(\mathcal{D}^{(2)})$ has rank 1 and basis $\{[2 \;\; 2]\}$. The set of all pairs corresponding to paths/cycles in $\mathcal{D}^{(2)}$ is represented in Fig. 3.5.

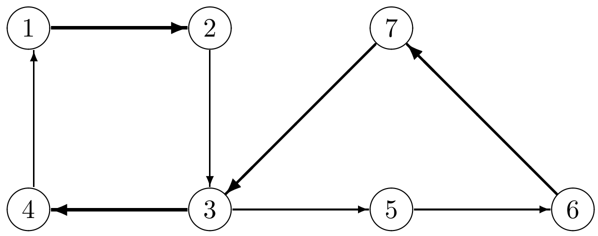


Fig. 3.4 Structure of $\mathcal{D}^{(2)}$

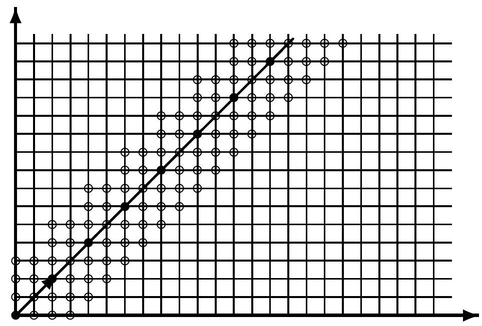

Fig. 3.5 Cycles and paths in $\mathcal{D}^{(2)}$

**Corollary 3.4** *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a strongly connected 2D-digraph. If $M(\mathcal{D}^{(2)})$ has rank 2, for every pair $[h \;\; k] \in \mathbb{N}^2$ there is a pair of vertices $v_i$ and $v_j$ and a path $v_i \xrightarrow[p]{} v_j$ such that*

$$[\alpha(p) \;\; \beta(p)] \equiv [h \;\; k] \qquad \bmod M(\mathcal{D}^{(2)}). \tag{15}$$

Dynamical characterizations of irreducible matrix pairs

Given a nonnegative matrix $F = [f_{ij}] \in \mathbb{R}_+^{n\times n}$, it is possible to associate it with an essentially unique 1-digraph, $\mathcal{D}^{(1)}(F)$, with $n$ vertices, $v_1, v_2, \ldots, v_n$. There is an arc from $v_i$ to $v_j$ if and only if $f_{ji} > 0$.

This correspondence is highly noninjective, yet several properties of the multiplicative semigroup generated by $F$ and of the asymptotic behavior of $F^\nu$, as $\nu$ tends to $+\infty$, only depend on $\mathcal{D}(F)$. More precisely, paths and cycles in $\mathcal{D}(F)$ are strictly related to the nonzero patterns of the powers of $F$, since the $(i,j)$th entry of $F^\nu$ is positive if and only if there exists a path of length $\nu$ from $v_j$ to $v_i$. On the other hand, the structure of a 1-digraph $D$ can obviously be investigated in terms of the algebraic properties of any nonnegative matrix $F$ for which $\mathcal{D}^{(1)}(F) = D$.

In this section we aim to extend the above correspondence to matrix pairs, by associating with every pair $(A, B)$ of $n \times n$ nonnegative matrices a 2D-digraph $\mathcal{D}^{(2)}(A, B)$ with $n$ vertices, $v_1, v_2, \ldots, v_n$. There is an $\mathcal{A}$-arc (a $\mathcal{B}$-arc) from $v_j$ to $v_i$ if and only if the $(i,j)$th entry of $A$ (of $B$) is nonzero [1]. For instance, the pair of positive matrices

$$(A, B) = \left( \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right)$$

corresponds to the 2D-digraph of Fig. 4.1.

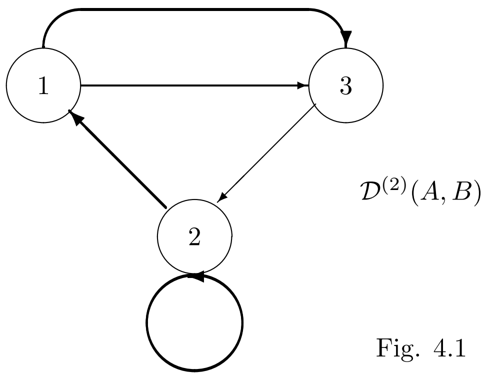


Fig. 4.1

We will show that the combinatorial properties of a pair $(A, B)$ with a 2D-strongly connected digraph can be viewed as natural generalizations of those of an irreducible matrix, i.e. a matrix with a strongly connected digraph. Moreover, the dynamical behavior of the 2D state model described by $(A, B)$ eventually exhibits a two-dimensional periodic pattern, and the "extremal" zeros of its characteristic polynomial are periodically distributed on a torus. This motivates the following definition.

**Definition** A pair $(A, B)$ of $n \times n$ positive matrices is *irreducible* if $\mathcal{D}^{(2)}(A, B)$ is 2D-strongly connected.

It is worth noticing that this amounts to require that $A + B$ is irreducible and $L_{A,B}$ has rank 2. So, in particular, all pairs $(A, B)$ with $A + B$ primitive are irreducible, but the converse is not true.

---

As in the sequel we always refer to the 2D-digraph $\mathcal{D}^{(2)}(A, B)$, associated with a specific matrix pair $(A, B)$, we denote the circuit matrix $L(\mathcal{D}^{(2)}(A, B))$ by $L_{A,B}$ and the corresponding module by $M_{A,B}$.

In positive matrix theory, the irreducibility of a single matrix $F$ has received several equivalent descriptions. Among the others, there are algebraic and system theoretic characterizations, which connect this property to the zero-patterns of the powers of $F$ and to the behavior of the associated state model. More precisely, a positive matrix $F \in \mathbb{R}^{n\times n}$ is irreducible if and only if positive integers $h$ and $T$ can be found, such that for every $t \geq T$

$$\sum_{i=t+1}^{t+h} F^i \gg 0, \tag{16}$$

or equivalently, if for every positive initial condition $\mathbf{x}(0) > 0$ the dynamical model

$$\mathbf{x}(t+1) = F\mathbf{x}(t), \qquad t = 0, 1, \dots \tag{17}$$

produces state vectors satisfying

$$\sum_{i=t+1}^{t+h} \mathbf{x}(i) \gg \mathbf{0},$$

for sufficiently large values of $t$.
Similar results hold true for irreducible matrix pairs, if we refer to the *2D system* [**?**]

$$\mathbf{x}(h+1, k+1) = A\mathbf{x}(h, k+1) + B\mathbf{x}(h+1, k), \qquad h, k \in \mathbb{Z},\ h+k \geq 0, \tag{18}$$

where the doubly indexed *local states* $\mathbf{x}(h,k)$ are elements of $\mathbb{R}^n_+$ and *initial conditions* are given by assigning a sequence $\mathcal{X}_0 := \{\mathbf{x}(\ell, -\ell) : \ell \in \mathbb{Z}\}$ of nonnegative local states on the *separation set* $\mathcal{S}_0 := \{(\ell, -\ell) : \ell \in \mathbb{Z}\}$.
If the initial conditions on $\mathcal{S}_0$ are all zero, except at $(0,0)$, we have

$$\mathbf{x}(h,k) = (A^h ⊔⊔^k B)\ \mathbf{x}(0,0), \qquad \forall\ h,k \in \mathbb{N},$$

where the *Hurwitz products* $A^h ⊔⊔^k B$ of $A$ and $B$ are inductively defined [?] as

$$A^h ⊔⊔^0 B = A^h, \quad h \geq 0, \qquad \text{and} \qquad A^0 ⊔⊔^k B = B^k, \quad k \geq 0, \tag{19}$$

and, when $h$ and $k$ are both positive,

$$A^h ⊔⊔^k B = A(A^{h-1} ⊔⊔^k B) + B(A^h ⊔⊔^{k-1} B). \tag{20}$$

One easily sees that $A^h ⊔⊔^k B$ is the sum of all matrix products that include the factors $A$ and $B$, $h$ and $k$ times, respectively.
For an arbitrary set of initial conditions $\mathcal{X}_0$, each local state in an arbitrary point $(h,k) \in \mathbb{Z}^2$, $h+k \geq 0$, can be obtained by linearity as

$$\mathbf{x}(h,k) = \sum_{\ell} (A^{h-\ell} ⊔⊔^{k+\ell} B)\ \mathbf{x}(\ell, -\ell), \tag{21}$$

where the Hurwitz product $A^{h-\ell} ⊔⊔^{k+\ell} B$ is assumed zero when either $h-\ell$ or $k+\ell$ is negative.

An alternative description of irreducible matrix pairs, that is reminiscent of that in (16), can be obtained by replacing the power matrices with the Hurwitz products, and the half-line $[T, +\infty)$ with a suitable solid convex cone. In fact, it turns out that a positive matrix pair $(A, B)$ is irreducible if and only if there are a finite "window" $\mathcal{F}$ and a solid convex cone such that, independently of how the window has been positioned within the cone, the sum of all Hurwitz products $A^r ⊔⊔^s B$ corresponding to integer pairs in the window, is strictly positive.

If $\mathcal{X}_0$ consists of a single nonzero local state at $(0,0)$, condition (23) can be restated as

$$\sum_{[i \quad j]\in[h \quad k]+\mathcal{F}} \mathbf{x}(i,j) \gg 0, \tag{22}$$

for every pair $[h \quad k] \in \mathbb{N}^2$ s.t. $[h \quad k] + \mathcal{F} \subset \mathcal{K}^*$. When there is an infinite number of nonzero local states on $\mathcal{S}_0$, the state evolution possibly affects the whole half-plane $\{(h,k) \in \mathbb{Z}^2 : h+k \geq 0\}$. We may ask whether there is a separation set $\mathcal{S}_\nu = \{(h,k) \in \mathbb{Z}^2 : h+k = \nu\}$ such that condition (22) is fulfilled by all pairs $[h \quad k]$ beyond $\mathcal{S}_\nu$, i.e. satisfying $h+k \geq \nu$.
This is clearly impossible if no upper bound exists on the distance between consecutive nonzero local states on $\mathcal{S}_0$. If we confine ourselves to *admissible* sets of initial conditions, namely to nonnegative sequences $\mathcal{X}_0$ which satisfy the following constraint: *there is an integer $N > 0$ such that $\sum_{\ell=h}^{h+N} \mathbf{x}(\ell,-\ell) > 0$ for all $h \in \mathbb{Z}$,* irreducibility can be characterized as follows.

**Proposition 4.1** *Let $(A,B)$ be a pair of $n \times n$ positive matrices. The following facts are equivalent:*
*i)$(A,B)$ is irreducible;*
*ii)there are a solid convex cone $\mathcal{K}^*$ and a finite set $\mathcal{F} \subset \mathbb{N}^2$ such that*

$$\sum_{[r \quad s]\in[h \quad k]+\mathcal{F}} A^r \sqcup^s B \gg 0, \qquad \forall\ [h \quad k] \in \mathbb{N}^2 \text{ s.t. } [h \quad k] + \mathcal{F} \subset \mathcal{K}^*; \tag{23}$$

*iii)there is a finite set $\mathcal{F} \subset \mathbb{N}^2$ such that for every admissible set of initial conditions $\mathcal{X}_0$ a positive integer $T$ can be found such that*

$$\sum_{[i \quad j]\in[h \quad k]+\mathcal{F}} \mathbf{x}(i,j) \gg 0, \qquad \forall\ [h \quad k] \in \mathbb{Z}^2 \text{ s.t. } h+k \geq T. \tag{24}$$

The above proposition makes it clear that as in the case of a single positive matrix, where condition (16) involves $h$ consecutive powers of $F$ ($h$ the imprimitivity index), (23) involves the Hurwitz products corresponding to $h^{(2)}$ integer pairs within any shifted version of the window $\mathcal{F}$.

Characteristic polynomials of irreducible matrix pairs

Up to this point, we have considered nonnegative matrix pairs only from the point of view of the corresponding graph. Important tools for analysing the properties of a pair $(A,B)$ are its *characteristic polynomial*, defined as

$$\Delta_{A,B}(z_1,z_2) := \det(I_n - Az_1 - Bz_2) = \sum_{h,k\in\mathbb{N}} d_{hk} z_1^h z_2^k, \qquad d_{00} = 1,$$

and the associated *variety* $\mathcal{V}(\Delta_{A,B})$, namely the set of points $(\lambda,\mu) \in \mathbb{C}^2$ such that $\det(I_n - A\lambda - B\mu) = 0$. Indeed, many features of the pair and of the corresponding 2D system, like internal stability, finite memory, separability, etc. [**?**], can be expressed directly in terms of $\Delta_{A,B}(z_1,z_2)$. The positivity constraint on the matrix entries makes these tools even more powerful, as there exists a strict relation between the cyclic structure of the 2D-digraph $\mathcal{D}^{(2)}(A,B)$ and the *support of* $\Delta_{A,B}(z_1,z_2)$, defined as

$$\text{supp}(\Delta_{A,B}) := \{(h_i,k_i) \in \mathbb{N}^2 : d_{h_i k_i} \neq 0\}.$$

In this section we aim to enlighten certain connections between $\mathrm{supp}(\Delta_{A,B})$ and the circuit matrix $L_{A,B}$, and to show that the *support matrix*

$$S_{A,B} := \begin{bmatrix} h_1 & k_1 \\ h_2 & k_2 \\ \vdots & \vdots \\ h_r & k_r \end{bmatrix} \tag{25}$$

and $L_{A,B}$ provide the same information about the irreducibility of $(A,B)$. This approach is intimately connected with the classical Perron-Frobenius theory for a single positive matrix, and suggests the possibility of obtaining a description of irreducible pairs in terms of the associated characteristic polynomials.

**Proposition 5.1** *Let $A$ and $B$ be positive matrices with $A+B$ irreducible and $\rho(A+B)=r$. For any $\theta$ and $\omega \in \mathbb{R}$ the following facts are equivalent:*
*i)$(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$ belongs to $\mathcal{V}(\Delta_{A,B})$;*
*ii)for every cycle $\gamma$ in $\mathcal{D}^{(2)}(A,B)$, including $\alpha(\gamma)$ $\mathcal{A}$-arcs and $\beta(\gamma)$ $\mathcal{B}$-arcs,*

$$\alpha(\gamma)\theta + \beta(\gamma)\omega \equiv 0 \qquad \mod 2\pi; \tag{26}$$

*iii)the characteristic polynomial of the pair $(A,B)$ satisfies*

$$\Delta_{A,B}(z_1,z_2) = \Delta_{A,B}(z_1e^{i\theta}, z_2e^{i\omega}); \tag{27}$$

*iv)for every pair $(h,k) \in \mathrm{supp}(\Delta_{A,B})$*

$$h\theta + k\omega \equiv 0 \qquad \mod 2\pi. \tag{28}$$

We aim to show that the circuit matrix and the support matrix of any pair $(A,B)$ generate the same $\mathbb{Z}$-module, thus extending an analogous result [**?**] connecting the support of $\det(I-zF)$ with the lengths of all circuits in $\mathcal{D}^{(1)}(F)$. The proof depends upon the following technical lemma.

**Proposition 5.3** *Let $(A,B)$ be a pair of $n\times n$ positive matrices, with $A+B$ irreducible. The $\mathbb{Z}$-modules generated by the rows of $L_{A,B}$ and by the rows of $S_{A,B}$ coincide.*

The Perron-Frobenius theorem undoubtely constitutes the most significative result about irreducible matrices, as it clarifies their spectral structure and provides useful information on the asymptotic behavior of the associated state models. Interestingly enough, the varieties of irreducible matrix pairs exhibit features that appear as natural extensions of the properties enlightened by Perron-Frobenius theorem, a result that further corroborates the definition of irreducibility introduced in section 4.

**Proposition 5.4** [2D Perron-Frobenius theorem] *Let $(A,B)$ be an irreducible pair of $n\times n$ positive matrices, with $\rho(A+B)=r$, and let*

$$\bar{H} := \begin{bmatrix} H \\ \\ 0 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & h_{22} \\ & 0 \end{bmatrix} \in \mathbb{N}^{t\times 2}$$

*be the Hermite form of $L_{A,B}$. The variety $\mathcal{V}(\Delta_{A,B})$ intersects the polydisc*

$$\mathcal{P}_{r^{-1}} := \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| \le r^{-1}, |z_2| \le r^{-1}\} \tag{29}$$

*only in a finite number of points of its distinguished boundary* $\mathcal{T}_{r^{-1}} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = r^{-1}, |z_2| = r^{-1}\}$*, namely in the points* $(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$*, one obtains by varying* $(\theta, \omega)$ *in the set*

$$\left\{ \left( \alpha\frac{2\pi}{h_{11}} + \beta\frac{2\pi h_{12}}{h_{11}h_{22}},\ \beta\frac{2\pi}{h_{22}} \right); \alpha, \beta \in \mathbb{N} \right\}. \tag{30}$$

*Moreover,* $(r^{-1}, r^{-1})$ *is a regular point of the variety and there exists a strictly positive vector* $\mathbf{w}$ *such that*

$$\left( I_n - r^{-1}A - r^{-1}B \right) \mathbf{w} = \mathbf{0}. \tag{31}$$

Primitivity and strictly positive asymptotic dynamics

An issue that arises quite naturally when considering the asymptotic behaviour of positive systems is that of guaranteeing that the states eventually become strictly positive vectors. For 1D positive systems

$$x(h+1) = A\ x(h), \qquad\qquad x(0) > 0,$$

the primitivity of the system matrix A [Minc, 1988] is necessary and sufficient for $x(h)$ beeing strictly positive when $h$ is large enough.

For 2D systems described as in (2), we say that the state evolution eventually becomes strictly positive if there exists a positive integer $T$ such that $x(h,k) \gg 0$ for all $(h,k)$, $h+k \geq T$. Clearly it's impossible that every nonzero initial global state $\mathcal{X}_0 = \{x(i,-i) : i \in \mathbf{Z}\}$ produces a strictly positive asymptotic dynamics. Actually, when $\mathcal{X}_0$ includes only a finite number of nonzero states, the support of the free evolution is included in a quarter plane causal cone of $\mathbf{Z} \times \mathbf{Z}$.

As a consequence, we have to take into account not only the properties of the matrix pair $(A, B)$, but also the zero-pattern of the nonnegative initial global state $\mathcal{X}_0$, and we will confine our attention to global states which satisfy the following condition: there exists an integer $M$ such that

$$\sum_{h=1}^{M} x(i+h, -i-h) > 0, \qquad \forall i \in \mathbf{Z}. \tag{4.1}$$

In other words, the maximal distance between two consecutive positive states on the separation set $\mathcal{C}_0$ is upper bounded by $M$.

In the sequel we will provide a set of sufficient conditions on the pair $(A, B)$ guaranteeing a strictly positive asymptotic dynamics for all initial global states satisfying (4.1).

**Proposition 4.1** Suppose that $A > 0$ and $B > 0$ are $n \times n$ positive matrices and there exists $(i,j)$ such that $A^i \sqcup\!\sqcup^j B$ is primitive. Then, for each initial global state satisfying (4.1), there exists a positive integer $T$ such that $x(h,k) \gg 0$ whenever $h+k \geq T$.

**Remark** The existence of a primitive Hurwitz product $A^i \sqcup\!\sqcup^j B$ implies that, for a suitable $p > 0$, $(A^i \sqcup\!\sqcup^j B)^p$, and hence $(A+B)^{(i+j)p}$, are strictly positive. Therefore $A+B$ is primitive. The converse in general is not true, as shown by the following example. The pair of irreducible matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.2}$$

has a primitive sum. However, as $B = A^2$, each Hurwitz product can be expressed as $A^h \sqcup\!\sqcup^k B = \binom{h+k}{h} A^{h+2k}$, and hence is not primitive.

**Corollary 4.2** If $A > 0$ and $B > 0$ are $n \times n$ matrices and there exists a word $w \in \Xi^*$ s.t. $w(A,B)$ is primitive, then for all initial global states satisfying (4.1) the asymptotic behaviour of system (2) is strictly positive.

Primitivity

The analysis of irreducibility just carried on allows to derive in a different way, and to partially extend, some results on primitive matrix pairs presented in a previous contribution. In [**?**] the notion of primitivity for a positive pair $(A, B)$ was introduced as a strict positivity constraint on the asymptotic dynamics of the associated 2D state model.
Introducing irreducibility has required a careful analysis of the equivalent characterizations available for a single irreducible matrix, and has highlighted the graph-theoretic approach as particularly suitable to this purpose. So, irreducible pairs have been first defined in terms of the associated 2D-digraph and later endowed with alternative descriptions.

In this framework primitivity is introduced as a special case of irreducibility, as an irreducible pair is called primitive if its circuit matrix $L_{A,B}$ is right prime, and it is characterized in several alternative ways by resorting to the results derived in the previous sections. Indeed, the proof of the following proposition is an immediate corollary of the previous propositions.

**Proposition 6.1** *Let $(A, B)$ be an irreducible pair of $n \times n$ positive matrices, with $\rho(A+B) = r$. The following facts are equivalent:*

*i) $(A, B)$ is primitive;*

*ii) $S_{A,B}$ is a right prime integer matrix;*

*iii???)$M_{A,B}$ coincides with $\mathbb{Z}^2$;*

*iv) there exists a strictly positive Hurwitz product;*

*v) there is a solid convex cone $\mathcal{K}^*$ in $\mathbb{R}^2_+$ such that for all $(h, k) \in \mathbb{N}^2 \cap \mathcal{K}^*$ the Hurwitz product $A^h \sqcup\!\sqcup^k B$ is strictly positive;*

*vi) for every admissible set of initial conditions there is a positive integer $T$ such that $\mathbf{x}(h, k) \gg 0$ for all $(h, k) \in \mathbb{N}^2$, $h + k \geq T$;*

*vii) the variety $\mathcal{V}(\Delta_{A,B})$ intersects the polydisk $\mathcal{P}_{r^{-1}}$ only in $(r^{-1}, r^{-1})$.* ■

It is worthwhile to remark that the results of the paper easily extend to $k$D-digraphs, i.e. digraphs with $k$ kinds of arcs, and to $k$D systems evolving on $\mathbb{Z}^k$, for any $k \in \mathbb{N}$. We preferred, however, to discuss only the case $k = 2$ and to avoid the notational and graphical burden connected with the general case, since, in our opinion, it tends to obscure the main features of the theory, without providing any conceptual advantage.

Further aspects of the asymptotic dynamics

The problems that will be addressed in this section concern some aspects of the two-dimensional dynamics which entail a finer analysis of the asymptotic behaviour. Indeed our interest here does not merely concentrate on nonzero patterns; it involves also the values of the local state and a qualitative description of the vectors distribution along the separation sets $\mathcal{C}_t = \{(i, j) : i + j = t\}$ as $t$ goes to infinity.
The first problem is that concerning the zeroing of state oscillations on the separation sets $\mathcal{C}_t$, as $t \to +\infty$, when scalar positive systems are considered.

**Definition 1** : A scalar (nonnecessarily nonnegative) global state $\mathcal{X}_0 = \{x(i, -i) : \ i \in \mathbf{Z}\}$ has (finite) mean value $\mu$ if, given any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that, for all $\nu \geq N(\varepsilon)$ and $h \in \mathbf{Z}$

$$\left| \frac{1}{\nu} \sum_{i=h}^{h+\nu-1} x(i, -i) - \mu \right| < \varepsilon. \tag{5.1}$$

The mean value will be denoted as $\mu = \lim_{\nu \to \infty} \nu^{-1} \sum x(i, -i)$, where the summation is extended to all intervals of length $\nu$, and the convergence is uniform w.r.t. the position of the interval along the separation set $\mathcal{C}_0$.

The following properties are straightforward consequences of Definition 1:

*i)* if $\mathcal{X}_0$ has mean value $\mu$, $\mathcal{X}_0 - \mu = \{x(i, -i) - \mu : i \in \mathbf{Z}\}$ has mean value zero;

*ii*) if $\mathcal{X}_0$ has mean value $\mu$, $\mathcal{X}_t = \{x(i,j), i+j=t\}$ has mean value $(A+B)^t\mu$;

*iii*) if $\mathcal{X}_0$ has mean value $\mu$, then $\mathcal{X}_0$ is bounded, i.e. there exists a positive integer $M$ such that $|x(i,-i)| < M, \quad i \in \mathbf{Z}$;

*iv*) the set of scalar global states constitutes a complete subspace of $\ell_\infty(\mathbf{Z})$, the space of bilateral bounded sequences.

Given a bounded scalar global state $\mathcal{X}_0$ with mean value $\mu$, the *oscillation* and (when $\mu \neq 0$) the *oscillation rate* of $\mathcal{X}_0$ are defined as

$$\mathrm{Osc}(\mathcal{X}_0) := \sup_{i,j\in\mathbf{Z}} |x(i,-i) - x(j,-j)| \tag{5.2}$$

and

$$\mathrm{osc}(\mathcal{X}_0) := \frac{\mathrm{Osc}(\mathcal{X}_0)}{|\mu|}, \tag{5.3}$$

respectively. The following technical lemma shows that a convexity assumption on the pair $(A,B)$ guarantees that the oscillations of the local states on the separation sets $\mathcal{C}_t$ are damped down to zero by the 2D system structure, as $t \to +\infty$.

**Lemma 5.1** Assume that in the scalar 2D system (2) $A$ and $B$ are both positive, and $A+B=1$. Then, for all global states $\mathcal{X}_0$ satisfying the mean value condition (5.1), $\mathrm{Osc}(\mathcal{X}_t) \to 0$ as $t \to \infty$.

PROOF As the amplitude of the oscillations along the separation set is unaffected when a constant value is added to all initial local states, there is no loss of generality in assuming that $\mathcal{X}_0$, and hence $\mathcal{X}_t,\ t=1,2,\ldots$, have zero mean.

Let $\varepsilon$ be an arbitrary real number in $(0,1)$. By the zero mean assumption, there exists an integer $N_1 \geq 0$ such that, for all $\nu \geq N_1$,

$$\bar{x}^{(\nu)}(i,-i) := \frac{1}{2\nu+1}\sum_{j=-\nu}^{\nu} x(i+j,-i-j) \tag{5.4}$$

satisfy, for all $i \in \mathbf{Z}$, $|\bar{x}^{(\nu)}(i,-i)| < \varepsilon/4$.

In the sequel, we shall compare the asymptotic behaviour of (2) induced by the original global state $\mathcal{X}_0 = \{x(i,-i), i \in \mathbf{Z}\}$ with that induced by the global state $\bar{\mathcal{X}}_0^{(\nu)} = \{\bar{x}^{(\nu)}(i,-i), i \in \mathbf{Z}\}$. When the initial conditions are provided by $\bar{\mathcal{X}}_0^{(\nu)}$, we get

$$|\bar{x}^{(\nu)}(t+h,-h)| \leq \frac{\varepsilon}{4}\sum_{i=0}^{t}\binom{t}{i}A^{t-i}B^i = \frac{\varepsilon}{4} \tag{5.5}$$

for all $t \geq 0$ and $h \in \mathbf{Z}$. So, all local states $\bar{x}^{(\nu)}(i,j)$ in the half plane $\{(i,j), i+j \geq 0\}$ have an absolute value less than $\varepsilon/4$ and, consequently, $\mathrm{Osc}\big(\bar{\mathcal{X}}_t^{(\nu)}\big) \leq \varepsilon/2$ for all $t \geq 0$. Moreover,

$$\bar{x}^{(\nu)}(t+h,-h) = \sum_{i=-\nu}^{t+\nu} x(i+h,-i-h)\frac{1}{2\nu+1}\sum_{\lambda=-\nu}^{\nu}\binom{t}{i-\lambda}A^{t-i+\lambda}B^{i-\lambda}, \tag{5.6}$$

where $\binom{t}{i-\lambda}$ is zero if $i-\lambda > t$ or $i-\lambda < 0$.

On the other hand, when the initial conditions are provided by $\mathcal{X}_0$, we get

$$x(t+h,-h) = \sum_{i=0}^{t} x(i+h,-i-h)\binom{t}{i}A^{t-i}B^i. \tag{5.7}$$

So, comparing (5.6) and (5.7), we see that the dynamics induced by $\bar{\mathcal{X}}_0^{(\nu)}$ approximates $x(\cdot,\cdot)$ on $\mathcal{C}_t$ within an error given by

$$\begin{aligned}
&e(t+h,-h) := \bar{x}^{(\nu)}(t+h,-h) - x(t+h,-h)\\
&= \sum_{i\in[-\nu,-1]\cup[t+1,t+\nu]} x(i+h,-i-h)\frac{1}{2\nu+1}\sum_{\lambda=-\nu}^{\nu}\binom{t}{i-\lambda}A^{t-i+\lambda}B^{i-\lambda}\\
&+\sum_{i=0}^{t} x(i+h,-i-h)\Big[\sum_{\lambda=-\nu}^{\nu}\binom{t}{i-\lambda}\frac{1}{2\nu+1}A^{t-i+\lambda}B^{i-\lambda} - \binom{t}{i}A^{t-i}B^i\Big]. \qquad (5.8)
\end{aligned}$$

We consider separately the behaviour of the two addenda in (5.8), as $\nu$ and $t$ go to infinity.

**(i)** Since $\mathcal{X}_0 \in \ell_\infty(\mathbf{Z})$, a positive $M$ exists, such that, for all $i \in \mathbf{Z}$, $|x(i,-i)| < M$. Once $\nu$ has been fixed, there exists a positive integer $N_2$ such that, for all $t \geq N_2$, both $\binom{t}{\nu}A^{t-\nu}B^\nu$ and $\binom{t}{t-\nu}A^\nu B^{t-\nu}$ are less than $\varepsilon/4M(2\nu+1)$, and therefore the modulus of the first addendum in (5.8) is less than $\varepsilon/4$.

**(ii)** We resort to the following statement of the classical Bernoulli theorem [Cramer, 1971]: "Let $\sigma \in (0,1)$, and consider

$$\omega := \sum_{tB(1-\sigma)<i<tB(1+\sigma)} \binom{t}{i}A^{t-i}B^i. \qquad (5.9)$$

Then the ratio $\omega/(1-\omega)$ may be made to exceed any given quantity by choosing $t$ sufficiently large".
So, given $\sigma$, a positive $N_3$ exists, such that $1-\omega < \varepsilon/16M$ for all $t \geq N_3$. Moreover, as the values of $\binom{t}{i}A^{t-i}B^i$ at the boundaries of the interval $(tB(1-\sigma), tB(1+\sigma))$ can be made as small as convenient if $t$ is large enough, we can assume also

$$\binom{t}{i}A^{t-i}B^i < \frac{\varepsilon}{2\nu+1}\frac{1}{8M}$$

when $i - \lfloor tB(1-\sigma)\rfloor = 1,2,\dots,\nu$ or $\lfloor tB(1+\sigma)\rfloor - i = 0,1,\dots,\nu-1$. Consequently, for all $t \geq N_3$, the summation in the second addendum of (5.8), when restricted to the values of $i$ satisfying $|i-tB| > t\sigma$, gives

$$\sum_{|i-tB|>t\sigma} x(i+b,-i-h)\Big[\sum_{\lambda=-\nu}^{\nu}\binom{t}{i-\lambda}\frac{1}{2\nu+1}A^{t-i-\lambda}B^{i-\lambda} - \binom{t}{i}A^{t-i}B^i\Big] < \frac{\varepsilon}{4}. \qquad (5.10)$$

Finally, we look for a suitable bound for the complementary part, namely

$$\sum_{tB(1-\sigma)<i<tb(1+\sigma)} x(i+h,-i-h)\Big[\sum_{\lambda=-\nu}^{\nu}\binom{t}{i-\lambda}\frac{1}{2\nu+1}A^{t-i-\lambda}B^{i-\lambda} - \binom{t}{i}A^{t-i}B^i\Big]. \qquad (5.11)$$

Letting $i = t(B+\delta)$, the term in square brackets can be rewritten as

$$\begin{aligned}
\mathcal{T}_i = &\binom{t}{i}A^{t-i}B^i\Big[-1+\frac{1}{2\nu+1}\Big(1+\frac{(1+\frac{\delta}{B})}{(1+\frac{\delta}{A}+\frac{1}{At})}+\frac{(1-\frac{\delta}{A})}{(1+\frac{\delta}{B}+\frac{1}{Bt})}\\
&+\frac{(1+\frac{\delta}{B})(1+\frac{\delta}{B}+\frac{1}{tB})}{(1+\frac{\delta}{A}+\frac{1}{At})+(1+\frac{\delta}{A}+\frac{2}{At})}+\dots+\frac{(1-\frac{\delta}{A})(1-\frac{\delta}{A}-\frac{1}{At})\dots(1-\frac{\delta}{A}-\frac{\nu+1}{At})}{(1+\frac{\delta}{B}+\frac{1}{Bt})\dots(1+\frac{\delta}{B}+\frac{\nu}{Bt})}\Big)\Big]
\end{aligned}$$

As $t$ goes to infinity, all terms $k/At$ and $k/Bt$ can be neglected. Moreover, for small values of $\sigma$, $|\delta|$ is a fortiori small, and all powers $\delta^3, \delta^4 \dots$ can be neglected w.r.t. $\delta^2$. This gives

$$\mathcal{T}_i \simeq \binom{t}{i}A^{t-i}B^i\big(-1+\frac{1}{2\nu+1}(1+2\nu+\gamma\delta^2)\big) = \binom{t}{i}A^{t-i}B^i\frac{\gamma\delta^2}{2\nu+1} \qquad (5.12)$$

where $\gamma$ is a suitable constant. As the absolute value of (5.11) is not greater than

$$\sum_{tB(1-\sigma)<i<tB(1+\sigma)} M\binom{t}{i}A^{t-i}B^i\frac{|\gamma|\sigma^2}{2\nu+1} \leq \frac{M|\gamma|}{2\nu+1}\sigma^2,$$

it can be made smaller than $\varepsilon/4$ when $\sigma$ is small enough. Therefore, for large values of $t$, we have $|e(t+h,-h)| < \varepsilon/2$ and consequently

$$\mathrm{Osc}(\mathcal{X}_t) \leq \mathrm{Osc}(\bar{\mathcal{X}}_t^{(\nu)}) + 2\ \sup_h|e(t+h,-h)| \leq \frac{3}{2}\varepsilon \ \blacksquare \tag{5.13}$$

The following proposition is now an immediate consequence of Lemma 5.1.

**Proposition 5.2** Consider an homogeneous 2D system (2) with $n=1$ (scalar local states) and $A, B > 0$. Assume moreover that the initial global state $\mathcal{X}_0$ has a mean value $\mu > 0$. Then the oscillation rate $\mathrm{osc}(\mathcal{X}_t)$ goes to zero as $t$ goes to infinity.

Proof Lemma 5.1 implies that the oscillation of the global state $\hat{\mathcal{X}}_t$ in the system

$$\hat{x}(h+1,k+1) = \hat{A}\ \hat{x}(h,k+1) + \hat{B}\ \hat{x}(h+1,k),$$

$\hat{A} = A/(A+B)$, $\hat{B} = B/(A+B)$, goes to zero when its initial conditions are given by $\hat{\mathcal{X}}_0 = \mathcal{X}_0$. Since we have $x(i+t,-i) = (A+B)^t\hat{x}(i+t,-i)$ and the mean value of $\mathcal{X}_t$ is $(A+B)^t\mu$,

$$\mathrm{osc}(\mathcal{X}_t) = \frac{\mathrm{Osc}(\mathcal{X}_t)}{\mu(A+B)^t} = \frac{\mathrm{Osc}(\hat{\mathcal{X}}_t)}{\mu} \tag{5.14}$$

goes to zero as $t \to \infty$ $\blacksquare$

If we drop the hypothesis that (2) is a scalar system, the qualitative description of the asymptotic dynamics is by far more interesting, and more difficult. Actually, there is a diversity of questions one may ask, concerning the shape $\mathcal{X}_t$ eventually reaches as $t$ goes to infinity, and answers depend like enough both on the pair $(A, B)$ and on the structure of $\mathcal{X}_0$.

There is, first of all, the question of guaranteeing that the normalized state vector $x(h,k)/||x(h,k)||$ converges towards a unique vector $v$ as $h+k \to \infty$. That is, how can a particular direction in the local state space be recognized as the 2D analogue of a 1D dominant eigenvector? If such a direction exists, a natural issue is to analyse the properties of the scalar sequences $(||x(i+t,-i)||)_{i\in\mathbf{Z}}$ and the possibility of obtaining global states $\mathcal{X}_t$ eventually free from oscillations. Finally, the questions above can be viewed as particular instances of the more general problem of classifying the asymptotic behaviours of the global states and detecting recurrencies that underlie their limiting structure.

The results so far available deal with two rather restrictive classes of positive 2D systems, that is 2D Markov chains [Fornasini 1990] and 2D systems with commutative $A$ and $B$. Further research will lead, it is hoped, to more comprehensive theorems. For sake of brevity, we discuss only some aspects of commutative 2D systems, that partly supplement the treatment of this subject presented in [Fornasini Marchesini 1993].

**Lemma 5.3** Let $A > 0$ and $B > 0$ be $n \times n$ commutative matrices, whose sum $A+B$ is irreducible. Then $A$ and $B$ have a strictly positive common eigenvector $v$

$$Av = r_A v, \qquad Bv = r_B v \tag{5.15}$$

and $r_A, r_B$ are the spectral radii of $A$ and $B$, respectively.

Proof Assume first that $A$ is irreducible, and let $v \gg 0$ be the eigenvector of $A$ corresponding to the eigenvalue $r_A$, that is $Av = r_A v$. The commutativity of $A$ and $B$ and the assumption $B > 0$ imply $A(Bv) = r_A(Bv)$ and $Bv > 0$ respectively. Since an irreducible matrix has exactly one eigenvector [Minc 1988] in $E^n := \{x \in \mathbf{R}^n_+ : \sum_{i=1}^n x_i = 1\}$, and both $v$ and $Bv$ are positive eigenvectors of $A$, we have

$$Bv = \lambda v, \quad \lambda > 0 \tag{5.16}$$

Consequently, $v$ is a strictly positive eigenvector of $B$, corresponding to its maximal eigenvalue $r_B$, and in (5.16) $\lambda = r_B$.
Assume now that $A + B$ is irreducible, and let $A_\varepsilon := A + \varepsilon B, B_\varepsilon := B + \varepsilon A$, where $\varepsilon$ is an arbitrary positive real number. As $A_\varepsilon$ and $B_\varepsilon$ commute and are both irreducible, the first part of the proof gives, for all $\varepsilon > 0$ $A_\varepsilon v^{(\varepsilon)} = r_{A_\varepsilon} v^{(\varepsilon)}, \quad B_\varepsilon v^{(\varepsilon)} = r_{B_\varepsilon} v^{(\varepsilon)}$ where $v^{(\varepsilon)} \gg 0$ is a common eigenvector of $A_\varepsilon$ and $B_\varepsilon$, uniquely determined by the condition $v^{(\varepsilon)} \in E^n$, and $r_{A_\varepsilon}, r_{B_\varepsilon}$ are the spectral radii of $A_\varepsilon$ and $B_\varepsilon$ respectively.
Now eigenvalues and eigenvectors are continuous functions of the entries of the matrices. Hence $A_\varepsilon \to A, \quad B_\varepsilon \to B, \quad r_{A_\varepsilon} \to r_A, \quad r_{B_\varepsilon} \to r_B$ as $\varepsilon \to 0^+$. Moreover, there exists $v \in E^n$ such that $v^{(\varepsilon)} \to v$, and $v$ is a common eigenvector of $A$ and $B$ which fulfills equations (5.15). To conclude the proof, it remains to show that the limiting vector $v$ is strictly positive. Indeed, (5.21) gives $(A+B)v = (r_A + r_B)v$ So, $v$ is a positive eigenvector of the irreducible matrix $A+B$, which implies $v \gg 0$ ■

We are now in a position to provide a stronger version of some results published in [Fornasini Marchesini, 1993], and summarized in the following lemma

**Lemma 5.4** Suppose that in system (2) $A, B$ and the initial global state $\mathcal{X}_0$ satisfy the following assumptions:

(i) $A$ and $B$ are positive commuting matrices
(ii) $A$ and $B$ have a strictly positive common dominant eigenvector v
(iii) There exists $\ell$ and $L$, both positive, such that

$$0 < \ell[1 \quad 1 \quad \dots \quad 1]^T \leq x(i, -i) \leq L[1 \quad 1 \quad \dots \quad 1]^T, \quad \forall i \in \mathbf{Z}. \tag{5.17}$$

Then

$$\lim_{h+k \to +\infty} \frac{x(h,k)}{||x(h,k)||} = \frac{v}{||v||} \ ■$$

**Proposition 5.5** Suppose that in system (2)

a) $A$ and $B$ are primitive commuting matrices
b) there exist an integer $M > 0$ and two positive real numbers $r$ and $R$ such that

$$r \leq [1 \quad 1 \quad \dots \quad 1] \sum_{h=1}^{M} x(i+h, -i-h) \leq R, \quad \forall i \in \mathbf{Z}$$

Then

$$\lim_{h+k \to +\infty} \frac{x(h,k)}{||x(h,k)||} = \frac{v}{||v||}$$

where $v \gg 0$ is a common eigenvector of $A$ and $B$.

Proof As $A+B$ is irreducible, by Lemma 5.3 there exists $v \gg 0$ that satisfies equations (5.15). The primitivity assumption guarantees that $v$ is a dominant eigenvector of both $A$ and $B$. Thus conditions (i) and (ii) of Lemma 5.4 are fulfilled. On the other hand, when $N$ is large enough, all matrices $A^\nu ⊔⊔^{N-\nu}B$, $0 \leq \nu \leq N$, are strictly positive. So, denoting by $s_N$ and $S_N$ their minimum and maximum entries

$$s_N := \min_\nu \min_{h,k}[A^\nu ⊔⊔^{N-\nu}B]_{hk} > 0, \qquad S_N := \max_\nu \max_{h,k}[A^\nu ⊔⊔^{N-\nu}B]_{hk} > 0$$

respectively, and assuming $N \geq M$, we have

$$\begin{aligned} x_j(i+N,-i) &= \sum_{\nu=0}^{N} \mathrm{row}_j(A^\nu ⊔⊔^{N-\nu}B)x(i+N-\nu,-i+\nu-N) \\ &\geq s_N \sum_{\nu=0}^{N} [1 \quad 1 \quad \dots \quad 1] x(i+N-\nu,-i-N+\nu) \geq s_N r \qquad j=1,2,\dots,n \end{aligned}$$

and

$$x_j(i+N,-i) \leq S_N \sum_{\nu=0}^{N} [1 \quad 1 \quad \dots \quad 1] x(i+N-\nu,-i-N+\nu) \leq S_N NR \qquad j=1,2,\dots,n.$$

Therefore, for large values of $N$, $\mathcal{X}_N$ fulfills condition (iii) of Lemma 5.4 , with $\ell = s_N r$ and $L = S_N NR$, and the proof is complete ■

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