On the positive reachability of 2D positive systems

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Abstract.

Local reachability of two-dimensional (2D) positive systems, by means of positive scalar inputs, is addressed by means of a graph theoretic approach. Some results concerned with equivalent conditions for local reachability as well as upper and lower bounds on the reachability indices are provided.

1 Introduction

Recent years have seen a growing interest in two-dimensional (2D) systems that are subject to a positivity constraint on their dynamical variables [2, 3, 4, 5]. There are actually several different motivations for this interest, coming from various domains of science and technology. Positive 2D systems arise, for instance, when discretizing pollution and self-purification processes along a river stream, or when providing a discrete model for the traffic flow in a motorway. More generally, the positivity assumption is a natural one when describing distributed processes whose variables represent quantities that are intrinsically nonnegative, like pressures, concentrations, population levels, etc.

In this paper we address the positive local reachability property for 2D positive systems with scalar inputs. To this end, we assume a combinatorial point of view. 2D influence graphs (namely direct graphs which exhibit two types of arcs and two types of input flows [3, 4]) are the appropriate tools for formalizing and solving the problem. The results presented here are preliminary and the general solution of the problem seems nontrivial.

2D positive systems considered in this paper are described by the following state-updating equation [1]:

$$\mathbf{x}(h+1,k+1) = A_1\mathbf{x}(h,k+1)+A_2\mathbf{x}(h+1,k)+B_1u(h,k+1)+B_2u(h+1,k), \quad (1)$$

where the local states $\mathbf{x}(\cdot,\cdot)$ and the scalar input $u(\cdot,\cdot)$ take nonnegative values, A_1 and A_2 are nonnegative $n \times n$ matrices, B_1 and B_2 are nonnegative n-dimensional column vectors, and the initial conditions are assigned by specifying

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the nonnegative values of the state vectors on the *separation set* $\mathcal{C}_0 := \{(h,k) : h,k \in \mathbb{Z}, h+k=0\}$, namely by assigning all **local states** of the initial **global state** $\mathcal{X}_0 := \{\mathbf{x}(h,k) : (h,k) \in \mathcal{C}_0\}$.

Hurwitz products of two $n \times n$ matrices A_1 and A_2 are inductively defined as

$$
\begin{aligned}
&A_1{}^i\sqcup\!\sqcup^j A_2 = 0, \qquad \text{when either } \ i \text{ or } \ j \ \text{ is negative},\\
&A_1{}^i\sqcup\!\sqcup^0 A_2 = A_1^i, \ \text{ for } \ i \geq 0, \qquad A_1{}^0\sqcup\!\sqcup^j A_2 = A_2^j, \ \text{ for } \ j \geq 0,\\
&A_1{}^i\sqcup\!\sqcup^j A_2 = A_1(A_1{}^{i-1}\sqcup\!\sqcup^j A_2) + A_2(A_1{}^i\sqcup\!\sqcup^{j-1} A_2), \ \text{ for } \ i,j > 0.
\end{aligned}
$$

A **2D influence graph** $\mathcal{D}^{(2)}$ is a sextuple $(s, V, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2)$, where s is the *source*, $V = \{v_1, v_2, \ldots, v_n\}$ is the set of *vertices*, $\mathcal{A}_1$ and $\mathcal{A}_2$ are subsets of $V \times V$ whose elements are called $\mathcal{A}_1$-*arcs* and $\mathcal{A}_2$-*arcs*, respectively, meanwhile $\mathcal{B}_1$ and $\mathcal{B}_2$ are subsets of $s \times V$ whose elements are called $\mathcal{B}_1$-*arcs* and $\mathcal{B}_2$-*arcs*, respectively. To every 2D positive system (1), of size n, with scalar inputs we associate a 2D influence graph $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ of source s, with n vertices, $v_1, v_2, \ldots, v_n$. There is an $\mathcal{A}_1$-arc (an $\mathcal{A}_2$-arc) from v_j to v_i iff the (i,j)th entry of A_1 (of A_2) is nonzero. There is a $\mathcal{B}_1$-arc (a $\mathcal{B}_2$-arc) from s to v_i iff the ith entry of B_1 (of B_2) is nonzero.

A *path* p in $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is a sequence of adjacent arcs and, in particular, an s-*path* is a path which originates from the source s. A path (in particular, an s-path) p is specified by assigning its vertices and the type of arcs they are connected by. If we denote by $|p|_1$ the number of $\mathcal{A}_1$-arcs and $\mathcal{B}_1$-arcs, and by $|p|_2$ the number of $\mathcal{A}_2$-arcs and $\mathcal{B}_2$-arcs occurring in p, then $[|p|_1 \ \ |p|_2]$ is the *composition* of p and $|p| = |p|_1 + |p|_2$ its *length*. A path whose extreme vertices coincide is a *cycle*. In particular, if each vertex appears exactly once as the first vertex of an arc, the cycle is a *circuit*. A 2D influence graph is *strongly connected* is for any two vertices v_i and v_j there is a path (of arbitrary composition) connecting v_i to v_j. $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is strongly connected iff $A_1 + A_2$ is an irreducible matrix.

Two matrices M and N, of the same size, are said to have the same *nonzero pattern* if $m_{ij} \neq 0$ implies $n_{ij} \neq 0$ and viceversa. A vector $\mathbf{v}$ is said to be an i**th monomial vector** if it can be expressed as $\alpha_i \mathbf{e}_i$, where $\mathbf{e}_i$ denotes the ith canonical vector and α_i is some positive real coefficient. A **monomial matrix** is a nonsingular (square) matrix whose columns are monomial vectors.

2 Reachability and positive reachability definitions

A 2D state-space model (1) is **positively locally reachable** [1] if, upon assuming $\mathcal{X}_0 = 0$, for every $\mathbf{x}^* \in \mathbb{R}^n_+$ there exists $(h,k) \in \mathbb{Z} \times \mathbb{Z}$ with $h+k > 0$ and a nonnegative input sequence $\mathbf{u}(\cdot,\cdot)$ such that $\mathbf{x}(h,k) = \mathbf{x}^*$. When so, we will say that $\mathbf{x}^*$ is reached in $h+k$ steps.

As for standard (i.e., not necessarily positive) 2D systems, positive local reachability analysis can be reduced to the analysis of the **reachability matrix**

in k steps [1]

$$\begin{aligned}\mathcal{R}_k &= [\,B_1 \quad B_2 \quad A_1B_1 \quad A_1B_2 + A_2B_1 \quad A_2B_2 \quad A_1^2B_1 \quad \dots \quad A_2^{k-1}B_2\,] \\ &= \left[(A_1{}^{i-1} \sqcup\!\sqcup^{j} A_2)B_1 + (A_1{}^{i} \sqcup\!\sqcup^{j-1} A_2)B_2\right]_{i,j\geq 0,\ 0<i+j\leq k}\end{aligned}$$

as k varies over the set of positive integers. In fact, the set X_k^+ of all local states that can be reached in k steps, by means of nonnegative inputs and starting from initial zero conditions, obviously coincides with the set of all nonnegative combinations of the columns of $\mathcal{R}_k$, namely $X_k^+ = \text{Cone}\mathcal{R}_k$.

As for 1D positive systems, the chain of reachability cones does not necessarily reach stationarity and, indeed, certain positive states can be reached only asymptotically. Moreover, positive local reachability is trivially equivalent to the possibility of reaching (starting from zero initial conditions) every vector of the canonical basis in $\mathbb{R}^n$ by means of nonnegative inputs, which in turn amounts to saying that there exists some $k \in \mathbb{N}$ such that the reachability matrix in k steps, $\mathcal{R}_k$, includes an $n \times n$ monomial submatrix. This is, of course, a structural property of the system, by this meaning that it only depends on the nonzero patterns of the system matrices and not on the specific values of their nonzero elements. However, differently from the 1D positive case, the **reachability index** I_R of a (locally reachable) 2D positive system, namely the minimum index k such that $X_k^+ = \text{Cone}\mathcal{R}_k = \mathbb{R}_+^n$, is not bounded by n.

Example 1 Consider the following 2D positive system: $(A_1, A_2, B_1, B_2) = \Big([\,0 \quad \mathbf{e}_3 \quad \mathbf{e}_1 \quad 0 \quad \mathbf{e}_6 \quad \mathbf{e}_7 \quad \mathbf{e}_4\,], [\,\mathbf{e}_2 \quad 0 \quad 0 \quad \mathbf{e}_5 \quad 0 \quad 0 \quad 0\,], [\,\mathbf{e}_1 + \mathbf{e}_4\,], [\,0\,]\Big)$, which corresponds to the 2D graph of Fig. 2.1, where thick lines represent $\mathcal{A}_1$-arcs and $\mathcal{B}_1$-arcs, while thin lines $\mathcal{A}_2$-arcs and $\mathcal{B}_2$-arcs.

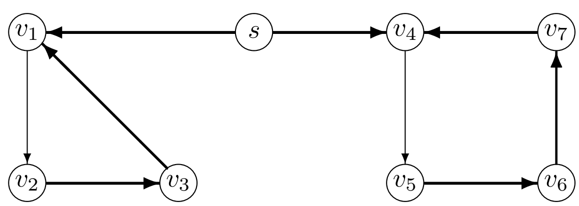

Fig. 2.1 $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ corresponding to Example 1

In this case, the reachability index proves to be 13 while the system dimension is $n = 7$. The above structure can be generalized. If the 2D influence graph of a 2D positive system consists of two loops, including n_1 vertices and $n_1 + 1$ vertices, respectively, connected by two arcs of type 2, while all the remaining are of type 2, just like in Fig. 2.1, I_R turns out to be of the same order as $n_1 \cdot (n_1 + 1)$, namely of the same order as $n^2/4$, since $n = n_1 + (n_1 + 1)$.

Example 1 has proved that for a locally reachable 2D positive system the reachability index may reach the value $n^2/4$. It seems reasonable to conjecture that $n^2/4$ represents an upper bound for the reachability index of every 2D positive system, however, up to now a formal proof of this result is not available. A necessary condition for positive reachability is the following one.

If the positive system (1) is positively locally reachable then the matrix $[\,A_1 \quad A_2 \quad B_1 \quad B_2\,]$ includes an $n \times n$ monomial submatrix.

If the system is locally reachable, there exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n$, such that $(A_1{}^{h_i-1} ⧢^{k_i} A_2)B_1 + (A_1{}^{h_i} ⧢^{k_i-1} A_2)B_2$ is an ith monomial vector. If $h_i + k_i = 1$, the ith monomial vector is a column of B_1 or of B_2, otherwise, if $h_i + k_i > 1$, it is a column of A_1 or of A_2 (possibly both).

As for 1D positive systems, local reachability property admits an interesting and useful characterization in terms of the 2D influence graph associated with the system. Indeed, saying that $(A_1{}^{h_i-1} ⧢^{k_i} A_2)B_1 + (A_1{}^{h_i} ⧢^{k_i-1} A_2)B_2$ is an ith monomial vector just means that every s-path p of composition $[|p|_1 \ |p|_2] = [h_i \ k_i]$ necessarily reaches the vertex v_i alone. If so, we will say that the vertex v_i is **deterministically reached** by all s-paths of composition $[h_i \ k_i]$. As a consequence, the 2D system (1) is positively locally reachable iff for every $i \in \{1, 2, \ldots, n\}$ the vertex v_i is deterministically reached by all s-paths of a given composition $[h_i \ k_i]$. Moreover, I_R coincides with

$$\max_i \min_{h_i, k_i} \{h_i + k_i : \text{ all } s\text{-paths of composition } [h_i \ k_i] \text{ deterministically reach } v_i\}.$$

In the sequel, we will confine our attention to 2D positive systems (1) having one of the two input-to-state matrices which is zero, and assume w.l.o.g. $B_2 = 0$ and, consequently, denote B_1 as B, for the sake of simplicity. These systems are described by the following equation:

$$\mathbf{x}(h+1, k+1) = A_1\mathbf{x}(h, k+1) + A_2\mathbf{x}(h+1, k) + Bu(h, k+1), \qquad (2)$$

where A_1, A_2 are in $\mathbb{R}_+^{n \times n}$ and B is in $\mathbb{R}_+^n$.

3 2D influence graphs devoid of cycles

In this section we consider 2D positive systems (2) whose 2D influence graph is devoid of cycles. This amounts to saying that the system (1) is *finite memory* or, equivalently [2], by the positivity assumption, that $A_1 + A_2$ is nilpotent.

Given a 2D positive system (1), its 2D influence graph $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is devoid of cycles iff the system is finite memory.

Observe, first, that since the source exhibits no incoming arcs, $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is devoid of cycles iff $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$ is. On the other hand, if γ is a cycle in $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$ and the vertex v_i belongs to γ, then $[(A_1+A_2)^{m\cdot|\gamma|}]_{ii} > 0$ for every positive integer m. So, if (1) is finite memory, namely $A_1 + A_2$ is nilpotent, then $(A_1 + A_2)^k = 0$, $\forall\ k \geq n$. Therefore, no cycle γ can exist in $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$. Conversely, if there is a cycle γ in $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$ then condition $(A_1 + A_2)^k = 0$ for every $k \geq n$ cannot be satisfied.

If a 2D positive system (2), with 2D influence graph $\mathcal{D}^{(2)}(A_1,\ A_2, B, 0)$ devoid of cycles, is positively locally reachable then

i) B is a canonical vector, and

ii) the reachability index I_R satisfies $\min\left\{k \in \mathbb{N} : \sum_{i=1}^{k} i \geq n\right\} \leq I_R \leq n.$

i) Since A_1+A_2 is (positive and) nilpotent, it entails no loss of generality [2] assuming that A_1+A_2 (and hence A_1 and A_2, separately) is in upper triangular form with zero diagonal. So, if A_1 and A_2 have the structure

$$\begin{bmatrix} 0 & + & + \\ & \ddots & + \\ & & 0 \end{bmatrix}$$

and the system is positively locally reachable, then, by Proposition 1, in $[\, A_1 \quad A_2 \quad B \quad 0\,]$ there must appear also the nth canonical vector $\mathbf{e}_n$. This necessarily implies $B = \mathbf{e}_n$.

ii) Since $(A_1 + A_2)^n = 0$, all Hurwitz products $A_1{}^i\sqcup\!\sqcup^j A_2$ are zero whenever $i + j \geq n$. So, $X^+_{n+1} = X^+_n$, and, in general, $X^+_k = X^+_n, \forall\, k \geq n$. If $B = \mathbf{e}_n$, it is easily seen that after one step the only outgoing arc from the source reaches vertex v_n. On the other hand, due to the fact that only two types of arcs are available, paths of length 2 with a common initial arc (from the source to vertex v_n) and distinct compositions may reach deterministically at most two vertices. Again, paths of length 3 with a common initial arc and distinct compositions may deterministically reach at most three vertices, and so on. This means that the minimum number of steps required to deterministically reach each vertex is the smallest positive integer k such that $1 + 2 + \ldots + k \geq n$.

4 2D influence graphs consisting of either one or two disjoint circuits

In this section we consider, first, systems (2) with 2D influence graphs consisting of a single circuit, by this meaning that all vertices $v_1, v_2, \ldots, v_n$ belong to a circuit (and each pair of adjacent vertices is connected by one single arc). This assumption amounts to saying that $A_1 + A_2$ is a permutation matrix, while $A_1 * A_2 = 0$, where $*$ denotes the Hadamard product. So, by resorting to a suitable permutation of the state components we can always obtain

$$A_1 + A_2 = \begin{bmatrix} 0 & + & 0 & & 0 \\ 0 & 0 & + & & 0 \\ & & \ddots & \ddots & \\ & & & \ddots & + \\ + & 0 & & & 0 \end{bmatrix}, \tag{3}$$

where $+$ represents a strictly positive entry and each nonzero entry $+$ appears only in one of the two matrices A_1 and A_2. Notice that vertex v_{i+1} accesses vertex v_i, for $i = 1, 2, \ldots, n-1$, while vertex v_1 accesses v_n.

Differently from the 1D case, the positive local reachability of such a system $(A_1, A_2, B, 0)$ does not require B to be a monomial vector. When $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ consists of a single circuit, every monomial vector B makes $(A_1, A_2, B, 0)$ positively locally reachable with reachability index $I_R = n$. When B exhibits k nonzero entries and the system is locally reachable, the reachability index may take quite smaller values.

Consider a 2D positive system (2) such that $\mathcal{D}^{(2)}(A_1,\ A_2, B, 0)$ consists of a single circuit and assume w.l.o.g. that $A_1 + A_2$ is expressed as in (3) with $A_1 * A_2 = 0$. If the system is positively locally reachable and B has $k > 1$ nonzero entries, say $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, then

$$I_R \geq \max\{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + i_1\} + 1.$$

Suppose, for the sake of simplicity, that $\max\{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + i_1\} = i_2 - i_1$. By the ordering assumptions introduced on the system vertices and on the labels $i_1, i_2, \ldots, i_k$, it is clear that the minimum $h_1 + k_1$ such that all s-paths of composition $[h_1\ k_1]$ deterministically reach v_{i_1} (keeping in mind that at the first step we get B and hence not a monomial vector) coincides with the length of the s-path that, starting from the source, reaches vertex v_{i_2} at the first step and later enters vertex v_{i_1} without passing through the other vertices v_{i_ℓ} for $\ell \neq 1, 2$. Such an s-path has length $i_2 - i_1 + 1$. Condition $I_R = \max_i \min_{h_i,k_i}\{h_i + k_i :$ all s-paths of composition $[h_i\ k_i]$ deterministically reach $v_i,\} \geq i_2 - i_1 + 1$ completes the proof.
In particular, when $k = 2$ the minimum value of $\max\{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + i_1\} = \max\{(i_2 - i_1), n - (i_2 - i_1)\}$ is just $n/2$ and therefore the minimum value of the reachability index is $\frac{n}{2} + 1$.

For a 2D influence graph consisting of two disjoint circuits we have the following result.

Let $(A_1, A_2, B, 0)$ be a 2D positive system such that $\mathcal{D}^{(2)}(A_1,\ A_2, 0,\ 0)$ consists of two disjoint circuits γ and γ' of length n and n', respectively. If B has only two nonzero entries, one for each cycle, then

$$I_R \leq \text{l.c.m}\{n, n'\} + \max\{n, n'\}.$$

Assume that the vertices in γ are (ordinately) $v_1, v_2, \ldots, v_n$, the vertices in γ' are (ordinately) $v'_1, v'_2, \ldots, v'_{n'}$, and that the two nonzero entries in B correspond to the vertices v_1 and v'_1, as depicted in Figure 4.2.

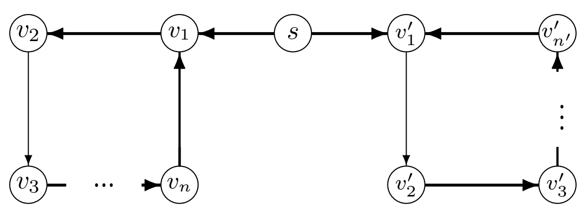

Fig. 4.2 $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ in Proposition 5

Due to the previous assumptions, any vertex $v_j \in \gamma$ ($v'_j \in \gamma'$) is periodically visited after $j, j+n, j+2n, \ldots$ steps ($j, j+n', j+2n', \ldots$ steps, respectively). Moreover, for every $k \in \mathbb{N}$ there exist exactly two s-paths of length k in $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$, and they reach vertices $v_{k \bmod n}$ in γ and $v'_{k \bmod n'}$ in γ', respectively. Such vertices are reached deterministically iff the two s-paths have distinct compositions.

Set $N := \text{l.c.m.}\{n, n'\}$ and suppose that none of the paths of length $j, j+n, \ldots, j+N$ deterministically reaches v_j. Since after $j+N$ steps we reach, at the same time and with the same composition, v_j and v'_j just like after j steps, the subsequent evolution will periodically repeat the same nonzero pattern, thus preventing the possibility of deterministically reaching v_j. As this reasoning applies to all vertices of γ and γ' (in particular to v_n and $v'_{n'}$), the given bound immediately follows.

5 Strongly connected 2D influence graphs including only two circuits

In this section we aim at addressing 2D positive systems with a strongly connected 2D influence graph that includes only two circuits, γ_1 and γ_2. Even though these assumptions are undoubtedly restrictive, an extension to the general case of 2D positive systems with a strongly connected 2D influence graph seems reasonable. On the other hand, when $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ is not strongly connected, which amounts to saying that the matrix $A_1 + A_2$ is not irreducible, possible bounds on the reachability index of the system $(A_1, A_2, B, 0)$, based on the reachability indices of the irreducible subsystems of $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$, can be obtained.

We first derive a lemma, whose proof is omitted for the sake of brevity.

If $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is a strongly connected 2D influence graph with n vertices and that it includes only two circuits, say γ_1 and γ_2, then

i) every vertex belongs either to γ_1 or to γ_2;

ii) there exists at least one vertex which belongs both to γ_1 and to γ_2;

iii) each path p, with $|p| \geq |\gamma_1|$, includes at least one vertex $v_2 \in \gamma_2$ and, conversely, each path p, with $|p| \geq |\gamma_2|$, includes at least one vertex $v_1 \in \gamma_1$;

iv) $N := |\gamma_1| + |\gamma_2| \geq n+1$.

Let $(A_1, A_2, B, 0)$ be a 2D positive system such that $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ is strongly connected and includes only two circuits γ_1 and γ_2. If $(A_1, A_2, B, 0)$ is positively locally reachable, then $I_R \leq N := |\gamma_1| + |\gamma_2|$.

Suppose, by contradiction, that there exists some vertex r which is deterministically reached by all s-paths of composition $[h+1\ k]$, with $h+1+k \geq N+1$, and cannot be reached deterministically in a smaller number of steps. This amounts to saying that $(A_1{}^h\sqcup\!\sqcup^k A_2)B$ is an r-monomial vector and for every i, j, with $i + j < h + k$, $(A_1{}^i\sqcup\!\sqcup^j A_2)B$ is not an r-monomial vector.

All paths of composition $[h\ k]$ from the vertices corresponding to the nonzero entries of B to the vertex r have length greater than or equal to $N \geq n + 1$. This implies that each of these paths contains at least one circuit. We can assume w.l.o.g. that there exists one such path p the vertex r of which includes the circuit γ_1. It is easily seen that there exists a path p', from at least one vertex corresponding to the nonzero entries of B to the vertex r of composition $[h-\alpha_1(\gamma_1)\ \ k-\alpha_2(\gamma_1)]$. Since $(A_1{}^{h-\alpha_1(\gamma_1)}\sqcup\!\sqcup^{k-\alpha_2(\gamma_1)} A_2)B$ is not an r-monomial vector, it means that there exists also a path q', of composition $[h-\alpha_1(\gamma_1)\ \ k-\alpha_2(\gamma_1)]$, from at least one vertex corresponding to the nonzero entries of B to some other vertex s. Since $|q'| = (h+k)-|\gamma_1| \geq N-|\gamma_1| = |\gamma_2|$, this implies that at least one vertex of q' belongs to γ_1. But then, by suitably adding a circuit γ_1, we can obtain from q' a new path q, from some vertex corresponding to the nonzero entries of B to the vertex s, of composition $[h\ \ k]$. This implies that in $(A_1{}^h\sqcup\!\sqcup^k A_2)B$ both the rth and the sth entries are nonzero, thus contradicting the original assumption. Therefore $h + k < N$.

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