STRUCTURAL PROPERTIES OF 2D POSITIVE STATE-SPACE MODELS

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Abstract. When dealing with two-dimensional (2D) discrete state-space models, reachability, controllability and zero-controllability are introduced in two different forms: a local form, which refers to single local states, and a global form, which instead pertains the infinite set of local states lying on a separation set. In this paper, these concepts are investigated in the context of 2D positive systems. Their combinatorial nature suggests a graph theoretic approach to their analysis, as, indeed, to every 2D positive state-space model of dimension n with m inputs one can associate a 2D influence digraph with n vertices and m sources.

For all these properties, necessary and sufficient conditions, which refer to the structure of the digraph, are provided.

Key words. 2D positive systems, reachability, controllability, zero-controllability, influence digraph, finite memory systems, strong connectedness.

1. Introduction

The interest in two-dimensional (2D) systems goes back to the early seventies [1, 7, 18], and was initially motivated by the relevance of these models in seismology applications, X-ray image enhancement, image deblurring, digital picture processing, etc. More recently, some contributions dealing with river pollution modelling [6] and the discretization of PDE's which describe gas absorption and water stream heating [16], naturally introduced a nonnegativity constraint in 2D system equations. Also, two-dimensional models involving only nonnegative variables were successfully adopted for describing the diffusion process of a tracer into a blood vessel [22]. This kind of instances stimulated, in the late nineties, a systematic analysis of "2D positive systems", i.e. 2D state-space models whose input, state and output variables take positive (or at least nonnegative) values, where the results presented in [6, 16, 22] could be naturally framed.

Research efforts in this context were first oriented to extend positive matrix theory to pairs of matrices [10, 11, 12, 21], thus leading to a satisfactory analysis of the free state evolution of 2D positive systems and a complete characterization of their asymptotic stability [20]. More recently, research efforts in 2D positive systems have concentrated on the analysis of their structural properties, and some preliminary results about reachability and controllability have been presented in [13, 14].

When dealing with 2D systems, the concepts of reachability, controllability and zero-controllability are naturally introduced in two different forms: a weak (local) form, which refers to single "local states", and a strong (global) form, which pertains the infinite set of local states lying on some "separation set" [3, 7]. In this paper, the aforementioned concepts are introduced and investigated in the context of 2D positive systems, driven by nonnegative inputs and described by the following state-updating equation [7]:

$$\mathbf{x}(h+1,k+1) = A_1\mathbf{x}(h,k+1) + A_2\mathbf{x}(h+1,k) + B_1u(h,k+1) + B_2u(h+1,k), \quad (1.1)$$

$$\mathbf{y}(h,k) = C\mathbf{x}(h,k), \quad (1.2)$$

3. Local/global zero-controllability

As a first step, we aim at showing that, when dealing with 2D positive systems, local zero-controllability and global zero-controllability are equivalent properties and they both coincide with the finite memory property [3, 10].

A standard (i.e., not necessarily positive) 2D system is said to be **finite memory** if for every initial global state $\mathcal{X}_0$ there exists $N \in \mathbb{Z}_+$ such that the corresponding free state evolution goes to zero within N separation sets, namely $\mathcal{X}_N = 0$.

Finite memory definition for 2D positive systems is obtained by simply introducing the positivity constraint on the initial global state $\mathcal{X}_0$. Several characterizations of finite memory positive systems have been provided in [10].

It is immediately apparent that, when dealing with positive systems, both local and global zero-controllability are properties which just pertain the free state evolution, as nonnegative inputs could not make the task of obtaining a zero local or global state easier! Basing on this simple remark, which holds true also for 1D positive systems, the proof of the following proposition becomes almost straightforward.

PROPOSITION 3.1. *Given a 2D positive system (1.1)-(1.2), of dimension n, the following facts are equivalent:*

i) the system is locally zero-controllable;

ii) the system is finite memory;

iii) the system is globally zero-controllable.

Proof. i) $\Rightarrow$ ii) Suppose that the system is locally zero-controllable and choose as $\mathcal{X}_0$ the positive global state whose local states $\mathbf{x}(i,-i), i \in \mathbb{Z}$, are all equal to the vector $\mathbf{1}_n$. For every $(h,k) \in \mathbb{Z} \times \mathbb{Z}$, with $h+k > 0$, we have $\mathbf{x}(h,k) = (A_1+A_2)^{h+k}\mathbf{1}_n$. Since there exists (h,k) such that $\mathbf{x}(h,k) = 0$, we have also $(A_1+A_2)^{h+k} = 0$, which ensures [10] the finite memory property of the 2D system described by the positive matrix pair (A_1, A_2).
ii) $\Rightarrow$ iii) For every nonnegative $\mathcal{X}_0$, just leave the system evolve freely.
iii) $\Rightarrow$ ii) Obvious. □

At this point, it is clear that a 2D positive system is locally (globally) controllable if and only if it is both finite memory and locally (globally) reachable. Since finite memory property is very easy to check, our interest will focus on local and global reachability properties. Characterizations of such properties will immediately lead to characterizations of local and global controllability.

4. Local reachability

When dealing with standard 2D systems, local reachability is easily tested by evaluating the column span of the **reachability matrix in k steps** [7], i.e.

$$\begin{aligned}\mathcal{R}_k &= [\,B_1 \quad B_2 \quad A_1B_1 \quad A_1B_2 + A_2B_1 \quad A_2B_2 \quad A_1^2B_1 \quad (A_1{}^1\sqcup{}^1A_2)B_1 + A_1^2B_2 \quad \dots \quad A_2^{k-1}B_2\,] \\ &= \left[(A_1{}^{i-1}\sqcup{}^jA_2)B_1 + (A_1{}^i\sqcup{}^{j-1}A_2)B_2\right]_{i,j\geq 0,\ 0<i+j\leq k}\end{aligned}$$

as k varies over the set $\mathbb{N}$ of positive integers. Indeed, reachable states in k steps, i.e. local states that can be reached in any assigned position of the separation set $\mathcal{C}_k$, starting from $\mathcal{X}_0 = 0$, *i) the system is globally reachable;*

ii) there exists a permutation matrix P such that

$$(5.5) \qquad P^T(A_1+A_2)P = \begin{bmatrix} \star & + & & & 0 \\ \star & 0 & + & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & + \\ \star & 0 & \dots & 0 & 0 \end{bmatrix} \qquad P^T(B_1+B_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ + \end{bmatrix},$$

where $\star$ and $+$ represent a nonnegative and a positive entry, respectively, and

$$(5.6) \qquad P^T(A_1 * A_2)P = \left[\begin{array}{c|c} \star & \\ \star & \\ \vdots & 0_{n\times(n-1)} \\ \star & \end{array}\right] \qquad P^T(B_1 * B_2) = 0.$$

Proof. For the sake of simplicity, as positive (either 1D or 2D global) reachability does not depend on the values of the nonzero entries of all matrices involved, within the proof all nonzero entries will be assumed unitary.

i) $\Rightarrow$ ii) If the system is globally stable then, by Lemma 5.2, the pair (A_1+A_2, B_1+B_2) is (positively) reachable. This ensures [5] that there exists a permutation matrix P such that (5.5) holds. Moreover, conditions (5.3)-(5.4) imply that only one among P^TB_1 and P^TB_2 is nonzero, which gives the second identity in (5.6). Suppose, without loss of generality, $P^TB_1 = \mathbf{e}_n$ and $P^TB_2 = 0$. From

$$P^T[(A_1{}^1\sqcup\!\sqcup^0 A_2)B_1 + (A_1{}^0\sqcup\!\sqcup^1 A_2)B_1] = P^T(A_1+A_2)B_1 = P^T(A_1+A_2)PP^T(B_1+B_2) = \mathbf{e}_{n-1}$$

and conditions (5.3)-(5.4), it follows that only one among $P^T[(A_1{}^1\sqcup\!\sqcup^0 A_2)B_1]$ and $P^T[(A_1{}^0\sqcup\!\sqcup^1 A_2)B_1]$ coincides with $\mathbf{e}_{n-1}$, while the other is zero. But this means that only one among $(P^TA_1P)\mathbf{e}_n$ and $(P^TA_2P)\mathbf{e}_n$ is $\mathbf{e}_{n-1}$, and hence an $(n-1)$th monomial vector, while the other is zero. By proceeding in this way, we show that only one among $(P^TA_1P)\mathbf{e}_i$ and $(P^TA_2P)\mathbf{e}_i$ is an $(i-1)$th monomial vector, $i = 2, \dots, n-1, n$, while the other is zero. This proves the first identity in (5.6).

ii) $\Rightarrow$ i) Conditions (5.5) and (5.6) easily imply that there exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \dots, n$, such that (5.3) and (5.4) hold. So, the 2D system is globally reachable. □

REMARKS As a consequence of the previous proposition, all pairs (h_i, k_i), that make (5.3) and (5.4) satisfied, sum up to n distinct integers $h_i + k_i$ and none of them exceeds n. This means that the set of all such $h_i + k_i, i = 1, 2, \dots, n$, coincides with the set $\{1, 2, \dots, n\}$ and hence, in particular, the global reachability index for 2D (globally reachable) systems with scalar inputs coincides with n. This situation is quite different from the one arising when local reachability is concerned, since the local reachability index can far exceed the system dimension.

Even more, for systems with scalar inputs, Proposition 5.1 can be restated in terms of the reachability matrices. Indeed, the reachability matrix (in n steps), $\mathcal{R} := \mathcal{R}_n$, can always be block-partitioned as

$$\mathcal{R} = [\, R_1 \mid R_2 \mid \dots \mid R_n \,],$$

where R_ℓ represents the block matrix including all columns $(A_1{}^{i-1}\sqcup\!\sqcup^j A_2)B_1 + (A_1{}^i\sqcup\!\sqcup^{j-1} A_2)B_2$ with $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, and $i + j = \ell$. By referring to this expression of $\mathcal{R}$, equations (5.1) and (5.2) (and hence global reachability) hold if and only if, for every $i = 1, 2, \ldots, n$, there exists $\ell_i \in \{1, 2, \ldots, n\}$ such that R_{ℓ_i} consists of all zero columns except for one which is an ith monomial vector.

Once we have shown that, in order to have global reachability, conditions (5.1) and (5.2) must be satisfied for suitable pairs (h_i, k_i) with $h_i + k_i \leq n_1$, these two conditions lead the way to a polynomial matrix characterization s of globally reachable positive systems with scalar inputs.

PROPOSITION 5.4. *For an n-dimensional 2D positive system (1.1)-(1.2) with scalar inputs the following facts are equivalent:*

i) the system is globally reachable;

ii) the polynomial reachability matrix

$$\mathcal{R}(\xi) := [\, B_1 + B_2\xi \quad (A_1 + A_2\xi)(B_1 + B_2\xi) \quad \ldots \quad (A_1 + A_2\xi)^{n-1}(B_1 + B_2\xi) \,] \in \mathbb{R}_+[\xi]^{n\times n}$$

can be expressed as

$$\mathcal{R}(\xi) = M \begin{bmatrix} \xi^{\nu_1} & & & \\ & \xi^{\nu_2} & & \\ & & \ddots & \\ & & & \xi^{\nu_n} \end{bmatrix}, \qquad M \text{ a monomial matrix, } \nu_i \geq 0;$$

iii) the polynomial reachability matrix $\mathcal{R}(\xi)$ is nonsingular and its inverse $\mathcal{R}(\xi)^{-1}$ belongs to $\mathbb{R}_+[\xi]^{n\times n}$.

Proof. i) $\Leftrightarrow$ ii) easily follows from (5.1) and (5.2), keeping in mind that the n pairs (h_i, k_i) must satisfy the two constraints: $h_i + k_i \neq h_j + k_j$ for $i \neq j$ and $0 \leq h_i + k_i \leq n - 1$.

ii) $\Leftrightarrow$ iii) Suppose that both $\mathcal{R}(\xi)$ and its inverse $\mathcal{R}(\xi)^{-1}$ belong to $\mathbb{R}_+[\xi]^{n\times n}$. So, at each point $\bar{\xi} \in \mathbb{R}_+$ $\mathcal{R}(\bar{\xi})$ and $\mathcal{R}(\bar{\xi})^{-1}$ are nonnegative matrices satisfying

$$I_n = \mathcal{R}(\bar{\xi})\mathcal{R}(\bar{\xi})^{-1}.$$

Since the only nonnegative matrices endowed with a nonnegative inverse are monomial matrices [17], this implies that $\mathcal{R}(\bar{\xi})$ is a monomial matrix for every $\bar{\xi} \in \mathbb{R}_+$. This is possible (if and) only if $\mathcal{R}(\xi) \in \mathbb{R}_+[\xi]^{n\times n}$ satisfies ii).

The converse is obvious. □

When dealing with systems with several inputs, Lemma 5.2 leads to a characterization of global reachability (controllability) similar to the one given in Proposition 5.3 (in Corollary ??, respectively). This requires, however, to consider the canonical forms available for reachable 1D positive systems with several inputs [4, 19]. As such forms are rather complicate, except when the 1D system matrix ($A_1 + A_2$, in this case) is devoid of zero columns, we restrict ourselves to this special case. The proof of the following proposition can be easily obtained by resorting to the canonical form given in [19] and the same reasonings adopted within the proof of Proposition 5.3.

PROPOSITION 5.5. *For a 2D positive system (1.1)-(1.2) with m inputs and $A_1 + A_2$ devoid of zero columns, the following facts are equivalent:*

i) the system is globally reachable;

ii) there exist $r \in \mathbb{N}$, with $r \leq m$, and permutation matrices, P and Q, of suitable dimensions, such that

$$[P^T(A_1 + A_2)P \mid P^T(B_1 + B_2)Q] = \left[\begin{array}{cccc|cccc|c} F_{11} & F_{12} & \dots & F_{1r} & \mathbf{e}_{n_1} & 0 & \dots & 0 & \\ F_{21} & F_{22} & \dots & F_{2r} & 0 & \mathbf{e}_{n_2} & \dots & 0 & \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & G_{rem} \\ F_{r1} & F_{r2} & \dots & F_{rr} & 0 & 0 & \dots & \mathbf{e}_{n_r} & \end{array}\right],$$

where

$$F_{ii} = \begin{bmatrix} \star & + & & & 0 \\ \star & 0 & + & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & + \\ \star & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{n_i \times n_i}, \quad F_{ij} = \begin{bmatrix} \star & 0 & & & 0 \\ \star & 0 & & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \\ \star & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{n_i \times n_j}, \text{for } i \neq j,$$

$\star$ and $+$ represent a nonnegative and a positive entry, respectively, and G_{rem} collects the "unnecessary" column of G. Moreover, $P^T(A_1 * A_2)P$ has all zero columns except, possibly, for those corresponding to the first columns of the blocks (namely the columns of indices $1, n_1 + 1, n_1 + n_2 + 1, \dots$) and $P^T(B_1 * B_2) \begin{bmatrix} I_r \\ 0 \end{bmatrix} = 0$.

6. Local and global observability

The concepts of positive global and local reachability have been formally introduced in Definitions 2.1 and 2.2 by referring to arbitrary global or local nonnegative states to be reached (starting from zero global initial conditions) by means of nonnegative inputs. Equivalent definitions can be introduced by referring to the nonzero patterns of the (global/local) states and not to their specific nonnegative values. Indeed, the following equivalent definitions could be clearly employed.

DEFINITION 6.1. *A 2D state-space model (1.1)-(1.2) is*

• **locally reachable** *if, upon assuming $\mathcal{X}_0 = 0$, for every boolean vector $\mathbf{x}_B^* \in \{0,1\}^n$ there exist $(h,k) \in \mathbb{Z} \times \mathbb{Z}$, with $h + k > 0$, and a nonnegative input sequence $\mathbf{u}(\cdot,\cdot)$ such that $\mathbf{x}(h,k)$ has the same nonzero pattern as $\mathbf{x}_B^*$;*

• **globally reachable** *if, upon assuming $\mathcal{X}_0 = 0$, for every sequence of n-dimensional boolean vectors $\{\mathbf{x}_B(h)\}_{h\in\mathbb{Z}}$, there exist $N \in \mathbb{Z}_+$ and a nonnegative input sequence $\mathbf{u}(\cdot,\cdot)$ such that the sequence of local states $\mathbf{x}(h, N-h)$ on the separation set $\mathcal{C}_N := \{(h, N-h) : h \in \mathbb{Z}\}$ satisfies $p(\mathbf{x}(h, N-h)) = p(\mathbf{x}_B(h))$, $\forall\, h \in \mathbb{Z}$, namely the nonzero patterns of the two sequences ordinately coincide.*

Once we think of reachability in terms of nonzero patterns (of local/global states), we may naturally introduce observability definitions based, again, on the nonzero patterns of the free output evolutions. This solution immediately enlightens a duality relation between these properties we will later explore.

DEFINITION 6.2. *A 2D state-space model (1.1)-(1.2) is*

• **locally observable** *if, upon assuming that* $\mathbf{u}(h,k) = 0, \forall\ h,k \in \mathbb{Z}$, *and that the initial global state* $\mathcal{X}_0$ *consists of a single nonzero state* $\mathbf{x}(0,0) = \mathbf{x}^*$, *the knowledge of the nonzero pattern of the free output evolution* $\mathbf{y}_\ell(h,k)$ *in every point* $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ *allows to uniquely determine the nonzero pattern of* $\mathbf{x}^*$;

• **globally observable** *if, upon assuming that* $\mathbf{u}(h,k) = 0, \forall\ h,k \in \mathbb{Z}$, *the knowledge of the nonzero pattern of the free output evolution* $\mathbf{y}_\ell(h,k)$ *in every point* $(h,k) \in \mathbb{Z} \times \mathbb{Z}, h+k \geq 0$, *allows to uniquely determine the nonzero pattern of all local states on the separation set* $\mathcal{C}_0$.

It is easy to see that global observability trivially implies local observability as, indeed, among all possible initial global states one may consider those consisting of all zero local states except in $(0,0)$, and for that type of global states the free output evolution just pertains the first orthant.

In order to explore local and global observability properties, we first introduce the **observability matrices in** k **steps**, i.e.

$$\mathcal{O}_k = \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ C(A_1{}^1\sqcup{}^1 A_2) \\ CA_2^2 \\ \vdots \\ CA_2^{k-1} \end{bmatrix} = \left[C(A_1{}^i\sqcup{}^j A_2)\right]_{i,j\geq 0,\ 0\leq i+j<k}$$

as k varies over the set $\mathbb{N}$ of positive integers.

PROPOSITION 6.3. *Given a 2D system (1.1)-(1.2) the following facts are equivalent:*

i) the system is locally observable;

ii) there exist n *pairs* $(h_i,k_i) \in \mathbb{Z}_+\times\mathbb{Z}_+, i=1,2,\dots,n$, *and* n *indices* $j=j(i) \in \{1,2,\dots,m\}$ *such that* $\mathbf{e}_j^T C(A_1{}^{h_i}\sqcup{}^{k_i} A_2)$ *is an* i*th monomial vector;*

iii) there exists $k \in \mathbb{N}$ *such that the observability matrix in* k *steps has a monomial submatrix.*

Proof. i) $\Rightarrow$ ii) Suppose, by contradiction, that the system is locally observable but ii) does not hold. This means that there exists $\ell \in \{1,2,\dots,n\}$ such that none of the rows of the observability matrix in k steps is an ℓth monomial vector, for every $k \in \mathbb{N}$. It is easy to verify that the initial states $\mathbf{x}(0,0) = \sum_{i=1}^n \mathbf{e}^i$ and $\mathbf{x}(0,0) = \sum_{\substack{i=1 \\ i\neq\ell}}^n \mathbf{e}_i$ have different nonzero patterns but produce free output evolutions endowed with the same nonzero patterns. As a consequence, the system cannot be locally observable.

ii) $\Rightarrow$ i) Of course, if ii) holds true, the ith entry of the local state $\mathbf{x}(0,0)$ is nonzero if and only if $\mathbf{e}_j^T \mathbf{y}_\ell(h_i,k_i) \neq 0$, $i=1,2,\dots,n$. Consequently, the system is locally observable.

The equivalence of ii) and iii) is obvious. □

Notice that when dealing with 2D systems with scalar output, the aforementioned condition ii) simply becomes: there exist n pairs $(h_i,k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i=1,2,\dots,n$, such that

$C(A_1{}^{h_i-1}\sqcup^{k_i}A_2)$ is an ith (row) monomial vector. Notice, also in this case, that all pairs (h_i, k_i) are necessarily distinct, but the case may occur that $h_i + k_i = h_j + k_j$ for $i \neq j$.

We aim, now, at addressing global observability for the special case of 2D system with scalar outputs. We preliminarily provide the following lemma.

LEMMA 6.4. *If the 2D system (1.1)-(1.2) with scalar output is globally observable then the 1D positive system described by the pair $(A_1 + A_2, C)$ is (positively) observable and hence [15] there exists a permutation matrix P such that*

$$(6.1)\qquad P^T(A_1 + A_2)P = \begin{bmatrix} 0 & + & & & 0 \\ 0 & 0 & + & & 0 \\ 0 & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & + \\ \star & \star & \dots & \star & \star \end{bmatrix} \qquad CP = [+ \quad 0 \quad 0 \quad \dots \quad 0],$$

where $\star$ and $+$ represent a nonnegative and a positive entry, respectively,

Proof. Global observability must be preserved, in particular, when initial global states $\mathcal{X}_0$ consist of all equal vectors, namely

$$\mathcal{X}_0 = \{\mathbf{x}(i, -i) = \bar{\mathbf{x}} \geq 0, \forall\ h \in \mathbb{Z}\}.$$

But in this situation the 2D system behaves as the 1D positive systems $(A_1 + A_2, C)$, since on every separation set $\mathcal{C}_k$ all free output vectors $\mathbf{y}_\ell(i, k - i)$ coincide with $C(A_1 + A_2)^k\bar{\mathbf{x}}$. Consequently, global observability of the 2D system requires the observability of the 1D positive system $(A_1 + A_2, C)$. □

By exploiting the previous lemma, we may now provide the following characterization.

PROPOSITION 6.5. *A 2D system (1.1)-(1.2) with scalar output is globally observable if and only if there exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \dots, n$, such that*

$$(6.2)\qquad C(A_1{}^{h_i}\sqcup^{k_i}A_2) \quad \textit{is an } i\textit{th monomial vector},$$

$$(6.3)\qquad C(A_1{}^{h}\sqcup^{k}A_2) = 0,\ \forall\ (h, k) \neq (h_i, k_i) \textit{ with } h + k = h_i + k_i.$$

Proof. Suppose that the system is globally observable and hence, by Lemma 6.4, the positive pair $(A_1 + A_2, C)$ is observable. Also, it entails no loss of generality assuming that the state components have been suitably permuted in order to obtain the previous canonical form for the 1D positive system $(A_1 + A_2, C)$. Of course, this ensures that n distinct pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \dots, n$, with $0 \leq h_i + k_i \leq n - 1$ and $h_i + k_i \neq h_j + k_j$ for $i \neq j$ can be found such that (6.2) holds. Showing that (??) holds is equivalent to proving that

$$A_1 * A_2 = \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \star & \star & \dots & \star & \star \end{bmatrix}.$$

So, we aim at showing that if there exists $i \in \{1, 2, \dots, n-1\}$ such that $[A_1]_{i,i+1} * [A_2]_{i,i+1} \neq 0$, then two initial global states can be found with different nonzero patterns but producing free output evolutions endowed with the same nonzero patterns. Set

$$\mathcal{X}_0 = \{\mathbf{x}(h, -h) = \mathbf{e}_{i+1}, \forall\, h \in \mathbb{Z}\} \qquad \text{and} \qquad \bar{\mathcal{X}}_0 = \{\mathbf{x}(h, -h) = \mathbf{e}_{i+1}, \forall\, h \in \mathbb{Z}\text{even}, \mathbf{x}(h, -h) = 0, \forall\, h \in \mathbb{Z}\text{odd}\}.$$

This is obvious if either $[A_1]_{n,i+1} = [A_2]_{n,i+1} = 0$ or $[A_1]_{n,i+1} * [A_2]_{n,i+1} \neq 0$, since the output is the same on the first separattion set and starting from the separation set $\mathcal{C}_1$ the two global states produce global states endowed with the same nonzero patterns.

If, for instance $[A_1]_{n,i+1} = 0$ and $[A_2]_{n,i+1} = 0$ then......

************* The converse is obvious. □

FIN QUI ********* Of course, by the same reasonings adopted in section 5, the canonical form derived for observable 1D scalar outputs positive systems [15] immediately lead to a canonical form for 2D globally observable systems with scalar outputs. A similar result could be obtained for the multi-output case. Moreover, as for global reachability, it is immediately seen that global observability can be checked within n steps. This ensures, in particular, that in order to determine the nonzero pattern of $\mathcal{X}_0$ we simply need to know the nonzero pattern of the free output evolution on the first n separation sets $\mathcal{C}_k, k = 0, 1, \dots, n-1$.

To conclude, we present the following result, whose proof follows the same lines of the proof of Proposition 5.4.

PROPOSITION 6.6. *For an n-dimensional 2D positive system (1.1)-(1.2) with scalar outputs the following facts are equivalent:*

i) the system is globally observable;

ii) the polynomial observability matrix

$$\mathcal{O}(\xi) := \begin{bmatrix} C \\ C(A_1 + A_2\xi) \\ \vdots \\ C(A_1 + A_2\xi)^{n-1} \end{bmatrix} \in \mathbb{R}_+[\xi]^{n \times n}$$

can be expressed as

$$\mathcal{O}(\xi) = M \begin{bmatrix} \xi^{\nu_1} & & & \\ & \xi^{\nu_2} & & \\ & & \ddots & \\ & & & \xi^{\nu_n} \end{bmatrix}, \qquad M \text{ a monomial matrix;}$$

iii) the polynomial observability matrix $\mathcal{R}(\xi)$ is nonsingular and its inverse $\mathcal{R}(\xi)^{-1}$ belongs to $\mathbb{R}_+[\xi]^{n \times n}$.

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