A polynomial matrix approach to the structural properties of 2D positive systems

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Abstract

In this paper, (local/global) reachability and observability [2] are introduced in the context of two-dimensional (2D) positive systems. While local reachability and observability are naturally characterized by resorting to state space techniques [5], their global versions are better investigated via a polynomial approach. Necessary and sufficient conditions for the existence of these properties are provided and, in particular, polynomial canonical forms for globally reachable/observable positive systems with scalar inputs/scalar outputs are provided.

1 Introduction

"2D positive systems", i.e. two-dimensional state-space models whose input, state and output variables take only nonnegative values, have been introduced in the nineties. The aim was that of providing a unifying theoretical framework to a family of interesting problems involving, in their mathematical descriptions, 2D system equations under a nonnegativity constraint. Research efforts were first oriented to extend positive matrix theory to pairs of matrices [3, 4], thus leading to the analysis of the free state evolution and the asymptotic stability of 2D positive systems. More recently, reachability and controllability have been addressed and some results have been presented in [5, 6, 7], by assuming a traditional state-space approach.

The 2D positive systems we consider in the paper are described by the following state-updating equation [2]:

$$\begin{aligned}\mathbf{x}(h+1,k+1) &= A_1\mathbf{x}(h,k+1)+A_2\mathbf{x}(h+1,k) \quad (1)\\ &+ B_1\mathbf{u}(h,k+1)+B_2\mathbf{u}(h+1,k)\end{aligned}$$

$$\mathbf{y}(h,k) = C\mathbf{x}(h,k), \qquad (2)$$

where the n-dimensional **local states** $\mathbf{x}(\cdot,\cdot)$, the m-dimensional inputs $\mathbf{u}(\cdot,\cdot)$ and the p-dimensional outputs $\mathbf{y}(\cdot,\cdot)$ take nonnegative values, A_1, A_2, B_1, B_2 and C are nonnegative matrices of suitable sizes. The initial conditions are assigned by specifying the (nonnegative) values of the state vectors on the **separation set** $\mathcal{S}_0 := \{(h,-h) : h \in \mathbb{Z}\}$, namely by assigning all local states of the initial **global state** $\mathcal{X}_0 := \{\mathbf{x}(h,k) : (h,k) \in \mathcal{S}_0\}$.

Any global state $\mathcal{X}_k := \{\mathbf{x}(h,k-h) : h \in \mathbb{Z}\}$, consisting of all local states lying on $\mathcal{S}_k := \{(h,k-h) : h \in \mathbb{Z}\}$, can be represented either by means of a 2D power series $X_k(z_1,z_2) = \sum_{h\in\mathbb{Z}} \mathbf{x}(h,k-h)z_1^h z_2^{k-h}$ or (if $\mathcal{S}_k$ is known) by means of a 1D power series $X_k(\xi) = \sum_{h\in\mathbb{Z}} \mathbf{x}(h,k-h)\xi^h$, and the state updating along the separation sets can be described, in turn, in 1D or 2D polynomial terms.

As we will see, while local reachability and observability refer to single local states and hence lead to a "point by point" analysis in the discrete grid [5], their global counterparts refer to the infinite set of local states lying on a "separation set", and hence are naturally investigated by resorting to the polynomial descriptions now provided for the global states.

Before proceeding, we introduce some basic definitions and concepts. The Hurwitz products of two $n\times n$ matrices A_1 and A_2 are inductively defined [2] as

$$\begin{aligned} A_1{}^i\sqcup{}^j A_2 &= 0, \quad \text{if either } i \text{ or } j \text{ is negative},\\ A_1{}^i\sqcup{}^0 A_2 &= A_1^i, \quad \text{if } i \geq 0,\\ A_1{}^0\sqcup{}^j A_2 &= A_2^j, \quad \text{if } j \geq 0,\end{aligned}$$

and, if $i,j > 0$,

$$A_1{}^i\sqcup{}^j A_2 = A_1(A_1{}^{i-1}\sqcup{}^j A_2) + A_2(A_1{}^i\sqcup{}^{j-1} A_2).$$

Given a nonnegative vector $\mathbf{v} \in \mathbb{R}_+^n$, we define its

nonzero pattern as the set $p(\mathbf{v}) := \{i \in \{1, 2, \ldots, n\} : v_i \neq 0\}$. Notice that $p(\mathbf{v}_1+\mathbf{v}_2) = p(\mathbf{v}_1) \cup p(\mathbf{v}_2)$. Similar definitions and remarks extend to nonnegative $n \times n$ matrices or to (possibly infinite) families of nonnegative vectors $\{\mathbf{v}_j\}_{j\in J}, \mathbf{v}_j \in \mathbb{R}^n_+$. In the first case, the nonzero pattern is a subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$, in the second case a subset of $\{1, \ldots, n\} \times J$.

We denote by $\mathbf{1}_n$ the n-dimensional real vector with all entries equal to 1.

A polynomial vector $\mathbf{v}(z_1, z_2) \in \mathbb{R}_+[z_1, z_2]$ is said to be an ***i*-th p-monomial vector** if $\mathbf{v}(1, 1)$ has the same nonzero pattern as the i-th canonical vector $\mathbf{e}_i$, i.e., $p(\mathbf{v}(1, 1)) = \{i\}$, and its nonzero entry is a monomial $cz_1^h z_2^k$ in $\mathbb{R}_+[z_1, z_2]$. A **p-monomial matrix** is a nonsingular (square) matrix whose columns are p-monomial vectors. P-monomial vectors and p-monomial matrices in $\mathbb{R}_+[\xi]$ are defined in an analogous way. Standard monomial vectors and monomial matrices in $\mathbb{R}_+$ can be seen as special cases of their general polynomial versions.

2 Local and global reachability

Definition 2.1 *A 2D state-space model (1)-(2) is*

- **(positively) locally reachable** *[2] if, upon assuming $\mathcal{X}_0 = 0$, for every $\mathbf{x}^* \in \mathbb{R}^n_+$ there exist $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, with $h + k > 0$, and a nonnegative input sequence $\mathbf{u}(\cdot, \cdot)$ s.t. $\mathbf{x}(h, k) = \mathbf{x}^*$. When so, we will say that $\mathbf{x}^*$ is (locally) reachable in $h + k$ steps;*
- **(positively) globally reachable** *[2] if, upon assuming $\mathcal{X}_0 = 0$, for every global state $\mathcal{X}^*$ with entries in $\mathbb{R}^n_+$, there exist $k \in \mathbb{Z}_+$ and a nonnegative input sequence $\mathbf{u}(\cdot, \cdot)$ s.t. the global state $\mathcal{X}_k = \{\mathbf{x}(h, k - h) : h \in \mathbb{Z}\}$ coincides with $\mathcal{X}^*$. When so, we will say that $\mathcal{X}^*$ is (globally) reachable in k steps.*

In the following, the specification "positively" will be omitted when no ambiguities arise. Clearly, global reachability ensures local reachability.

Introduce the **reachability matrix in k steps** [2], $k \in \mathbb{N}$, i.e.

$$\mathcal{R}_k := \left[({A_1}^{i-1} \sqcup\!\sqcup^{j} A_2)B_1 + ({A_1}^{i} \sqcup\!\sqcup^{j-1} A_2)B_2\right]_{i,j\geq 0,\ 0<i+j\leq k}.$$

The reachable set in k steps, i.e. the set of local states that can be reached in any assigned position of the separation set $\mathcal{S}_k$, starting from $\mathcal{X}_0 = 0$ and by applying nonnegative input sequences, obviously coincides with the set of all nonnegative combinations of the columns of $\mathcal{R}_k$, namely with $\mathrm{Cone}(\mathcal{R}_k)$. Consequently, a system is locally reachable if and only if there exists $N \in \mathbb{N}$ s.t. $\mathrm{Cone}(\mathcal{R}_N) = \mathbb{R}^n_+$.

Positive local reachability is trivially equivalent to the possibility of reaching (starting from zero initial conditions) every vector of the canonical basis in $\mathbb{R}^n$ by means of nonnegative inputs, which amounts to saying that there exists $N \in \mathbb{N}$ s.t. the reachability matrix in N steps, $\mathcal{R}_N$, includes an $n \times n$ monomial submatrix.

Proposition 2.2 *[5] Given a 2D system (1)-(2) the following facts are equivalent:*

i) the system is locally reachable;

ii) there exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n$, and n indices $j = j(i) \in \{1, 2, \ldots, m\}$ s.t.

$$p\left(({A_1}^{h_i-1} \sqcup\!\sqcup^{k_i} A_2)B_1\mathbf{e}_j + ({A_1}^{h_i} \sqcup\!\sqcup^{k_i-1} A_2)B_2\mathbf{e}_j\right) = \{i\}; \tag{3}$$

iii) there exists $N \in \mathbb{N}$ such that the reachability matrix in N steps $\mathcal{R}_N$ has an $n \times n$ monomial submatrix.

Let us now address global reachability. When dealing with a polynomial description of the forced state evolution, the global state on the k-th separation set $X_k(z_1, z_2) = \sum_{h\in\mathbb{Z}} \mathbf{x}(h, k - h)z_1^h z_2^{k-h}$ can be expressed in terms of the input sequences on the separation sets $\mathcal{S}_t, 0 \leq t \leq k - 1$, as follows

$$X_k(z_1, z_2) = \mathcal{R}_k(z_1, z_2) \begin{bmatrix} U_{k-1}(z_1, z_2) \\ U_{k-2}(z_1, z_2) \\ \vdots \\ U_0(z_1, z_2) \end{bmatrix}, \tag{4}$$

where

$$\mathcal{R}_k(z_1, z_2) = [(B_1z_1 + B_2z_2)\ (A_1z_1 + A_2z_2)(B_1z_1 + B_2z_2)\ \ldots\ (A_1z_1 + A_2z_2)^{k-1}(B_1z_1 + B_2z_2)] \tag{5}$$

and $U_t(z_1, z_2) = \sum_{h\in\mathbb{Z}} \mathbf{u}(h, t - h)z_1^h z_2^{t-h}, t = 0, 1, \ldots, k-1$. Starting from this 2D polynomial description, we obtain a characterization of global reachability.

Proposition 2.3 *The 2D system (1)-(2) is globally reachable if and only if there exists some index $N \in \mathbb{N}$ such that the 2D polynomial matrix $\mathcal{R}_N(z_1, z_2)$ given in (5) includes an $n \times n$ p-monomial submatrix, i.e.*

$$M \cdot \mathrm{diag}\{z_1^{\mu_1} z_2^{\nu_1}, z_1^{\mu_2} z_2^{\nu_2}, \ldots, z_1^{\mu_n} z_2^{\nu_n}\}, \tag{6}$$

for some monomial matrix M and some $\mu_i, \nu_i \geq 0$.

Proof Clearly, it suffices to ensure the reachability of the "elementary global states" consisting of all zero (local) states except for one of them, which coincides with the monomial vector $\mathbf{e}_i$, $i \in \{1, 2, \ldots, n\}$. Consequently, the 2D system (1)-(2) is globally reachable if and only if, for every $i \in \{1, 2, \ldots, n\}$, there are $k_i \in \mathbb{N}$ and $h_i \in \mathbb{Z}$ such that $X_{k_i}(z_1, z_2) = \mathbf{e}_i z_1^{h_i} z_2^{k_i-h_i}$ is a global state reachable in k_i steps. This means that

$$\mathbf{e}_i z_1^{h_i} z_2^{k_i-h_i} = \mathcal{R}_{k_i}(z_1, z_2) \begin{bmatrix} U^{(i)}_{k_i-1}(z_1, z_2) \\ U^{(i)}_{k_i-2}(z_1, z_2) \\ \vdots \\ U^{(i)}_0(z_1, z_2) \end{bmatrix},$$

for some $U_t^{(i)}(z_1,z_2), t = 0,1,\ldots,k_i-1$. By the nonnegativity assumption, there is no loss of generality assuming that each $U_t^{(i)}(z_1,z_2)$ has finite support, namely it is a Laurent polynomial. On the other hand, the nonnegativity of the coefficients of all polynomial matrices and vectors involved ensures that the above condition holds true if and only if there exists at least one column of $\mathcal{R}_{k_i}(z_1,z_2)$ taking the structure $\mathbf{e}_i c_i z_1^{\mu_i} z_2^{\nu_i}, c_i \in \mathbb{R}_+$. So, the proposition statement holds for $N = \max\{k_1,k_2,\ldots,k_n\}$. ■

Remark The characterization of Proposition 2.3 may be restated in terms of polynomial reachability matrices in the single variable ξ. Indeed, the 2D system (1)-(2) is globally reachable if and only if there exists $N \in \mathbb{N}$ such that $\mathcal{R}_N(\xi) = [(B_1+B_2\xi)\ (A_1+A_2\xi)(B_1+B_2\xi)\ \ldots\ (A_1+A_2\xi)^{N-1}(B_1+B_2\xi)]$ includes an $n \times n$ p-monomial submatrix.

As a corollary of the previous result, we get

Corollary 2.4 *If the 2D system (1)-(2) is globally reachable then* $[\,A_1+A_2\xi \quad B_1+B_2\xi\,]$ *includes an* $n \times n$ *p-monomial matrix.*

Proof The nonnegativity of the coefficients, together with the fact that the product of two polynomials is a monomial if and only if they are both monomials, allows to saying that

$$(A_1+A_2\xi)^{k_i-1}(B_1+B_2\xi)\mathbf{e}_j = \mathbf{e}_i \cdot (c_i \cdot \xi^{\mu_i}) \qquad (7)$$

$c_i \in \mathbb{R}_+, \mu_i \in \mathbb{Z}_+$, implies either that the j-th column of $B_1+B_2\xi$ is an i-th p-monomial vector (if $k_i = 1$) or that some column of $A_1+A_2\xi$ is an i-th p-monomial vector (if $k_i > 1$). Since (7) must be verified for every $i \in \{1,2,\ldots,n\}$, the result immediately follows. ■

The following lemmas lead the way to further characterizations of global reachability.

Lemma 2.5 *If the 2D system (1)-(2) is globally reachable then the 1D positive system described by the pair* (A_1+A_2, B_1+B_2) *is (positively) reachable.*

Proof By the previous Proposition 2.3, the 2D system (1)-(2) is globally reachable if and only if there exists $N \in \mathbb{N}$ such that the polynomial matrix $\mathcal{R}_N(\xi)$ includes an $n \times n$ p-monomial submatrix as in (6). Since this condition holds true for an arbitrary ξ, it must hold true for $\xi = 1$. This means that the reachability matrix in N steps of the pair (A_1+A_2, B_1+B_2) includes the monomial submatrix M and hence the pair (A_1+A_2, B_1+B_2) is positively reachable. ■

Lemma 2.6 *If there exist an integer* $\ell \in \mathbb{Z}_+$ *and a nonzero polynomial* $p(\xi) \in \mathbb{R}_+[\xi]$ *such that*

$$(A_1+A_2\xi)^\ell(B_1+B_2\xi)\mathbf{e}_j = p(\xi)\mathbf{e}_i, \qquad (8)$$

for some $i \in \{1,2,\ldots,n\}$ *and* $j \in \{1,2,\ldots,m\}$*, then there exists* $\bar\ell \in \mathbb{Z}_+, 0 \le \bar\ell \le n-1$*, and a nonzero* $\bar p(\xi) \in \mathbb{R}_+[\xi]$ *such that* $(A_1+A_2\xi)^{\bar\ell}(B_1+B_2\xi)\mathbf{e}_j = \bar p(\xi)\mathbf{e}_i$.

Proof We associate with the polynomial matrix pair $(A_1+A_2\xi, B_1+B_2\xi)$ a directed graph with n vertices and m sources: there is an arc connecting vertex j to vertex i if and only if the (i,j)-th entry of $A_1+A_2\xi$ is nonzero and, similarly, there is an arc connecting source j to vertex i if and only if the (i,j)-th entry of $B_1+B_2\xi$ is nonzero. Each arc is thus weighted by some nonzero polynomial $c+d\xi$, $c,d \in \mathbb{R}_+$.

In graph theoretic terms, condition (8) holds for some nonzero polynomial $p(\xi) \in \mathbb{R}_+[\xi]$ if and only if there exists a "deterministic path" [1] from the j-th source to the i-th vertex in the directed graph associated with the pair $(A_1+A_2\xi, B_1+B_2\xi)$, namely a path starting from the j-th source and reaching, after $\ell+1$ steps, the vertex i and no other vertex. Such a condition holds, in turn, if and only if there exists a deterministic path from the j-th source to the i-th vertex in the directed graph associated with (A_1+A_2, B_1+B_2). But then, we may resort to the result obtained by Coxson and Larson in [1] and say that if such a path exists, then there exists a path from source j to vertex i of length not larger than n. This amounts to saying that $(A_1+A_2)^{\bar\ell}(B_1+B_2)\mathbf{e}_j = c \cdot \mathbf{e}_i$ holds true for some $\bar\ell < n$ and $c > 0$ and hence that $(A_1+A_2\xi)^{\bar\ell}(B_1+B_2\xi)\mathbf{e}_j = \bar p(\xi)\mathbf{e}_i$ holds true for some $\bar\ell < n$ and $\bar p(\xi) \in \mathbb{R}_+[\xi] \setminus \{0\}$. ■

Proposition 2.7 *The 2D system (1)-(2) of size* n *is globally reachable if and only if the polynomial matrix* $\mathcal{R}_n(\xi)$ *includes an* $n \times n$ *p-monomial submatrix.*

Proof By Proposition 2.3, the system is globally reachable if and only if there exists some $N \in \mathbb{N}$ such that the polynomial matrix $\mathcal{R}_N(\xi)$ includes an $n \times n$ p-monomial submatrix. However, by the previous Lemma 2.6, if $(A_1+A_2\xi)^\ell(B_1+B_2\xi)\mathbf{e}_j = c \cdot \xi^{\nu_i}\mathbf{e}_i$, for some nonzero monomial $c \cdot \xi^{\nu_i} \in \mathbb{R}_+[\xi]$, then there exists $\bar\ell \in \mathbb{Z}_+, 0 \le \bar\ell \le n-1$ and a nonzero $\bar p(\xi) \in \mathbb{R}_+[\xi]$ such that $(A_1+A_2\xi)^{\bar\ell}(B_1+B_2\xi)\mathbf{e}_j = \bar p(\xi)\mathbf{e}_i$. This implies, in particular, that

$$(A_1+A_2\xi)^{\ell-\bar\ell}\left[(A_1+A_2\xi)^{\bar\ell}(B_1+B_2\xi)\mathbf{e}_j\right] = (A_1+A_2\xi)^{\ell-\bar\ell}\left[\bar p(\xi)\mathbf{e}_i\right] = c \cdot \xi^{\nu_i}\mathbf{e}_i.$$

Therefore the i-th column of $(A_1+A_2\xi)^{\ell-\bar\ell}$ must be an i-th p-monomial vector, $\bar p(\xi)$ is necessarily a monomial, and $\mathcal{R}_n(\xi)$ has an i-th monomial column $\bar p(\xi)\mathbf{e}_i$. ■

A nice polynomial canonical form can be obtained for globally reachable systems with scalar inputs.

Proposition 2.8 *For a 2D system (1)-(2) of dimension n with scalar inputs the following facts are equivalent:*

i) the system is globally reachable;

ii) there exists a permutation matrix P such that

$$
\begin{aligned}
P^T(A_1+A_2\xi)P &= \begin{bmatrix} \star & a_{12} & & & 0 \\ \star & 0 & a_{23} & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{n-1,n} \\ \star & 0 & \dots & 0 & 0 \end{bmatrix}, \\
P^T(B_1+B_2\xi) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_n \end{bmatrix}
\end{aligned}
\qquad (9)
$$

where $a_{i,i+1}, b_n \in \mathbb{R}_+[\xi]$ are nonzero monomials and $\star$ denotes a polynomial in $\mathbb{R}_+[\xi]$ of degree at most 1;

iii) $\mathcal{R}_n(\xi) \in \mathbb{R}_+[\xi]^{n\times n}$ is a p-monomial matrix;

iv) $\mathcal{R}_n(\xi)$ is nonsingular and $\mathcal{R}_n(\xi)^{-1} \in \mathbb{R}_+[\xi^{-1}]^{n\times n}$.

Proof ii) $\Rightarrow$ iii) $\Rightarrow$ i) are obvious. i) $\Rightarrow$ ii) By Corollary 2.4, global reachability ensures that the $n\times(n+1)$ polynomial matrix $[\,A_1+A_2\xi \quad B_1+B_2\xi\,]$ includes an $n\times n$ p-monomial matrix. Suppose that $B_1+B_2\xi$ is not a p-monomial vector. Since it cannot be zero, then it must either have at least two nonzero entries (case 1) or be a vector of the following type $p(\xi)\mathbf{e}_i$, for some polynomial $p(\xi)$ of lag 1 (case 2). On the other hand, since the block matrix must include an $n\times n$ p-monomial matrix, such a matrix must be $A_1+A_2\xi$. It is easily seen that, under these hypotheses, both in case 1 and in case 2, none of the vectors $(A_1+A_2\xi)^k(B_1+B_2\xi)$ can be p-monomial. So, $B_1+B_2\xi$ is necessarily a p-monomial vector. It entails no loss of generality assuming $B_1+B_2\xi=\mathbf{e}_n$. In fact, we can always reduce ourselves to this case by permuting either the vector components or the matrices B_1 and B_2, possibly both. Clearly, at most one column of $A_1+A_2\xi$ is not p-monomial and the set of the remaining $n-1$ columns of $A_1+A_2\xi$ includes an i-th p-monomial vector for $i=1,2,\dots,n-1$. Suppose that the last column of $A_1+A_2\xi$ is not p-monomial. This implies that, on the one hand, all remaining columns of $A_1+A_2\xi$ are p-monomial, on the other hand $(A_1+A_2\xi)(B_1+B_2\xi)$ is not p-monomial. For these reasons, both in case $(A_1+A_2\xi)(B_1+B_2\xi)$ has at least two nonzero entries (case 1) and in case it is a vector like $p(\xi)\mathbf{e}_i$, for some polynomial $p(\xi)$ of lag 1 (case 2), also the following powers $(A_1+A_2\xi)^i(B_1+B_2\xi), i>1$, are not p-monomial. Suppose now that $(A_1+A_2\xi)(B_1+B_2\xi)$ is a p-monomial vector. Clearly the nonzero entry cannot be in the last (namely n-th) position, otherwise all powers $(A_1+A_2\xi)^i(B_1+B_2\xi)$ would have the same structure, and it entails no loss of generality assuming that the only nonzero entry lies in the $n-1$-th row. We can now repeat the same reasoning we just applied to the last row and claim that if the $n-1$-th column would not be p-monomial then all the other columns in $A_1+A_2\xi$ would not be, and hence all remaining powers $(A_1+A_2\xi)^i(B_1+B_2\xi), i\geq 2$, would not be p-monomial. In this way we have proven that (upon a suitable permutation) we can assume that all columns of $A_1+A_2\xi$, except possibly for the first one, have to be p-monomial vectors and

$$
A_1+A_2\xi = \begin{bmatrix} \star & a_{12} & & & 0 \\ \star & 0 & a_{23} & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{n-1,n} \\ \star & 0 & \dots & 0 & 0 \end{bmatrix} \; B_1+B_2\xi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_n \end{bmatrix} \qquad (10)
$$

where $a_{i,i+1}, b_n \in \mathbb{R}_+[\xi]$ are nonzero monomials and the entries denoted by $\star$ are polynomials in $\mathbb{R}_+[\xi]$ (of degree at most 1).

iii) $\Leftrightarrow$ iv) Suppose that $\mathcal{R}_n(\xi)$ belongs to $\mathbb{R}_+[\xi]^{n\times n}$ and its inverse $\mathcal{R}_n(\xi)^{-1}$ to $\mathbb{R}_+[\xi^{-1}]^{n\times n}$. So, at each point $\bar\xi\in\mathbb{R}_+$ $\mathcal{R}_n(\bar\xi)$ and $\mathcal{R}_n(\bar\xi)^{-1}$ are nonnegative matrices satisfying $I_n=\mathcal{R}_n(\bar\xi)\mathcal{R}_n(\bar\xi)^{-1}$. Since the only nonnegative square matrices endowed with nonnegative inverses are monomial, this implies that $\mathcal{R}_n(\bar\xi)$ is monomial for every $\bar\xi\in\mathbb{R}_+$. This is possible (if and) only if $\mathcal{R}_n(\xi)\in\mathbb{R}_+[\xi]^{n\times n}$ is p-monomial and hence satisfies iii). The converse is obvious. ∎

3 Local and global observability

Global and local reachability definitions, given in section 2, could have been equivalently introduced by referring to the nonzero patterns both of the (global/local) states to be reached and of the input sequences, instead of considering their specific nonnegative values. On the other hand, if we aim at introducing observability definitions starting from the free output evolutions of 2D positive systems, and pretend that they provide reasonable dual properties w.r.t. local and global reachability, a nonzero pattern approach is somehow unavoidable.

Definition 3.1 *A 2D state-space model (1)-(2) is*

• **locally observable** *if, upon assuming that the initial global state $\mathcal{X}_0$ consists of a single nonzero local state $\mathbf{x}(0,0)$, the knowledge of the nonzero pattern of the free output evolution $\mathbf{y}_\ell(h,k)$ at each $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ allows to uniquely determine the nonzero pattern of $\mathbf{x}(0,0)$;*

• **globally observable** *if the knowledge of the nonzero pattern of the free output evolution $\mathbf{y}_\ell(h,k)$ at each $(h,k) \in \mathbb{Z} \times \mathbb{Z}, h+k \geq 0$, allows to uniquely determine the nonzero pattern of $\mathcal{X}_0$.*

Global observability trivially implies local observability. In order to explore this latter, we introduce the **observability matrix in k steps**, i.e.

$$\mathcal{O}_k = \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ C({A_1}^1 \sqcup {}^1 A_2) \\ CA_2^2 \\ \vdots \\ CA_2^{k-1} \end{bmatrix} = \left[C({A_1}^i \sqcup {}^j A_2)\right]_{i,j\geq 0,\ 0\leq i+j<k}$$

where k is a positive integer. As a first step, we provide a characterization of local observability.

Proposition 3.2 *Given a 2D system (1)-(2) the following facts are equivalent:*

i) the system is locally observable;

ii) there exist n pairs $(h_i,k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1,2,\dots,n$, and n indices $j = j(i) \in \{1,2,\dots,p\}$ s.t.

$$p\left(\mathbf{e}_j^T C({A_1}^{h_i} \sqcup {}^{k_i} A_2)\right) = \{i\}; \tag{11}$$

iii) there exists $N \in \mathbb{N}$ s.t. the observability matrix in N steps $\mathcal{O}_N$ has an $n \times n$ monomial submatrix.

Proof i) $\Rightarrow$ ii) Suppose, by contradiction, that the system is locally observable but ii) does not hold. This means that there exists $q \in \{1,2,\dots,n\}$ s.t. none of the rows of the observability matrix in k steps, for any $k \in \mathbb{N}$, is a q-th monomial vector. It is easy to verify that the initial states $\mathbf{x}(0,0) = \mathbf{1}_n$ and $\mathbf{x}(0,0) = \mathbf{1}_n - \mathbf{e}_\ell$ have different nonzero patterns but produce free output evolutions endowed with the same nonzero patterns. Thus the system cannot be locally observable.

ii) $\Rightarrow$ i) If ii) holds true, the i-th entry of the local state $\mathbf{x}(0,0)$ is nonzero if and only if $\mathbf{e}_j^T \mathbf{y}_\ell(h_i,k_i) \neq 0$, $i = 1,2,\dots,n$. So, the system is locally observable.

The equivalence of ii) and iii) is obvious. ■

When dealing with 2D systems with scalar outputs, condition ii) above simply becomes: there exist n pairs $(h_i,k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1,2,\dots,n$, s.t. $C({A_1}^{h_i} \sqcup {}^{k_i} A_2)$ is an i-th (row) monomial vector. Notice that, also in this case, all pairs (h_i,k_i) are necessarily distinct, but the case may occur that $h_i + k_i = h_j + k_j$ for $i \neq j$.

In order to address global observability by means of polynomial techniques, we express the free output evolution on each separation set $\mathcal{S}_t$ by means of a power series, $Y_t(z_1,z_2) = \sum_{h\in\mathbb{Z}} \mathbf{y}(h,t-h) z_1^h z_2^{t-h}$, and relate it to the global initial conditions $\mathcal{X}_0$ as follows

$$\begin{aligned} Y_t(z_1,z_2) &= \textstyle\sum_{h\in\mathbb{Z}} C\mathbf{x}(h,t-h) z_1^h z_2^{t-h} \\ &= \textstyle\sum_{h\in\mathbb{Z}} C \sum_{\ell=0}^t ({A_1}^\ell \sqcup {}^{t-\ell} A_2)\mathbf{x}(h-\ell,\ell-h) z_1^h z_2^{t-h} \\ &= C \textstyle\sum_{\ell=0}^t ({A_1}^\ell \sqcup {}^{t-\ell} A_2) z_1^\ell z_2^{t-\ell} X_0(z_1,z_2) \\ &= C(A_1 z_1 + A_2 z_2)^t X_0(z_1,z_2). \end{aligned}$$

Consequently

$$\begin{bmatrix} Y_0(z_1,z_2) \\ \vdots \\ Y_{k-1}(z_1,z_2) \end{bmatrix} = \mathcal{O}_k(z_1,z_2) X_0(z_1,z_2) \tag{12}$$

where

$$\mathcal{O}_k(z_1,z_2) := \begin{bmatrix} C \\ C(A_1z_1 + A_2z_2) \\ \vdots \\ C(A_1z_1 + A_2z_2)^{k-1} \end{bmatrix}. \tag{13}$$

Starting from this 2D polynomial description, we can obtain a characterization of global observability.

Proposition 3.3 *The 2D system (1)-(2) is globally observable if and only if there exists some nonnegative index N such that the 2D polynomial matrix $\mathcal{O}_N(z_1,z_2)$ given in (13) includes an $n \times n$ p-monomial submatrix.*

Proof Of course, if there exists some index $N \in \mathbb{N}$ such that the observability matrix $\mathcal{O}_N(z_1,z_2)$ includes an $n\times n$ p-monomial submatrix, then there exists a suitable selection of separation sets $\mathcal{S}_{k_1}, \mathcal{S}_{k_2}, \dots, \mathcal{S}_{k_n}, k_i \in \mathbb{N}$ and a corresponding suitable choice of output components $j_1, j_2, \dots, j_n \in \{1,2,\dots,p\}$ such that

$$\begin{bmatrix} \mathbf{e}_{j_1}^T Y_{k_1}(z_1,z_2) \\ \mathbf{e}_{j_2}^T Y_{k_2}(z_1,z_2) \\ \vdots \\ \mathbf{e}_{j_n}^T Y_{k_n}(z_1,z_2) \end{bmatrix} = M \cdot \mathrm{diag}\{z_1^{\mu_1} z_2^{\nu_1}, \dots, z_1^{\mu_n} z_2^{\nu_n}\} X_0(z_1,z_2).$$

Since we have already seen that a p-monomial matrix in $\mathbb{R}_+[\xi]^{n\times n}$ exhibits an inverse (in $\mathbb{R}_+[\xi^{-1}]^{n\times n}$) having the same structure, it follows that

$$\mathrm{diag}\{z_1^{-\mu_1} z_2^{-\nu_1}, \dots, z_1^{-\mu_n} z_2^{-\nu_n}\} M^{-1} \cdot \begin{bmatrix} \mathbf{e}_{j_1}^T Y_{k_1}(z_1,z_2) \\ \mathbf{e}_{j_2}^T Y_{k_2}(z_1,z_2) \\ \vdots \\ \mathbf{e}_{j_n}^T Y_{k_n}(z_1,z_2) \end{bmatrix}$$

$$= X_0(z_1, z_2).$$

This allows an entry by entry identification of all local components of the initial global state, and hence the identification of the nonzero pattern of $\mathcal{X}_0$.

Conversely, suppose by contradiction that the system is globally observable but there exists some index $\ell \in \{1, 2, \dots, n\}$ such that none of the rows of the observability matrix in k steps $\mathcal{O}_k(z_1, z_2)$, for any $k \in \mathbb{N}$, is an ℓ-th p-monomial vector. Two cases may occur: either every row having a nonzero ℓ-th entry has also other nonzero entries, or all rows whose only nonzero entry is the ℓ-th one, exhibit a polynomial of strictly positive lag in the ℓ-th position. If so, we denote by $L > 0$ the smallest such lag. In the first case, it is easy to see that the initial global states

$$\begin{aligned} X_0(z_1, z_2) &= \sum_{h\in\mathbb{Z}} \mathbf{1}_n z_1^h z_2^{-h} \\ \bar{X}_0(z_1, z_2) &= \sum_{h\in\mathbb{Z}} (\mathbf{1}_n - \mathbf{e}_\ell)\, z_1^h z_2^{-h} \end{aligned}$$

have different nonzero patterns but produce free output evolutions endowed with the same nonzero patterns, thus contradicting global observability. Similarly, in the second case, the initial global states

$$\begin{aligned} X_0(z_1, z_2) &= \sum_{h\in\mathbb{Z}} \mathbf{1}_n z_1^h z_2^{-h} \\ \bar{X}_0(z_1, z_2) &= \sum_{h\in\mathbb{Z}} (\mathbf{1}_n - \mathbf{e}_\ell)\, z_1^h z_2^{-h} + \sum_{h\in\mathbb{Z}} \mathbf{e}_\ell z_1^{h(L+1)} z_2^{-h(L+1)} \end{aligned}$$

have different nonzero patterns but produce free output evolutions endowed with the same nonzero patterns, contradicting again global observability. ■

Remark The 2D system (1)-(2) is globally observable if and only if there exists $N \in \mathbb{N}$ such that

$$\mathcal{O}_N(\xi) = \begin{bmatrix} C \\ C(A_1 + A_2\xi) \\ \vdots \\ C(A_1 + A_2\xi)^{N-1} \end{bmatrix}$$

includes an $n \times n$ p-monomial submatrix

Corollary 3.4 *If the 2D system (1)-(2) is globally observable then* $\begin{bmatrix} A_1 + A_2\xi \\ C \end{bmatrix}$ *includes an $n \times n$ p-monomial matrix.*

Starting from Proposition 3.3, it is straightforward to apply the same type of reasonings adopted in section 2 for global reachability, thus obtaining

Proposition 3.5 *The 2D system (1)-(2) of size n is globally observable if and only if the polynomial matrix $\mathcal{O}_n(\xi)$ includes an $n \times n$ p-monomial submatrix.*

A polynomial canonical form can be obtained for globally observable systems with scalar outputs, by resorting to the results derived in this section and to the reasonings adopted within the proof of Proposition 2.8.

Proposition 3.6 *For a 2D system (1)-(2) of size n with scalar outputs the following facts are equivalent:*

i) the system is globally observable;

ii) there exists a permutation matrix P such that

$$\begin{aligned} P^T(A_1 + A_2\xi)P &= \begin{bmatrix} \star & \star & \star & \dots & \star \\ a_{21} & 0 & 0 & & 0 \\ 0 & a_{32} & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \dots & a_{n,n-1} & 0 \end{bmatrix} \\ CP &= \begin{bmatrix} 0 & 0 & 0 & \dots & c_n \end{bmatrix} \end{aligned}$$

where $a_{i,i-1} \in \mathbb{R}_+[\xi]$ are nonzero monomials, $c_n \in \mathbb{R}_+, c_n > 0$, and $\star$ denotes a polynomial in $\mathbb{R}_+[\xi]$ of degree at most 1;

iii) $\mathcal{O}_n(\xi) \in \mathbb{R}_+[\xi]^{n\times n}$ is a p-monomial matrix;

iv) $\mathcal{O}_n(\xi)$ is nonsingular and $\mathcal{O}_n(\xi)^{-1} \in \mathbb{R}_+[\xi^{-1}]^{n\times n}$.

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