Feedback stabilization, regulation and optimal control of Boolean control networks

Ettore Fornasini and Maria Elena Valcher

Abstract— **In this paper various control problems for Boolean control networks (BCNs) are investigated. By resorting to some recent results regarding the infinite-horizon optimal control, we first provide an alternative proof of the fact that the stabilization of a BCN to a given reachable equilibrium point can always be performed by means of a static state-feedback. Secondly, upon deriving necessary and sufficient conditions for the solvability of the output regulation problem, we show that, when such conditions are satisfied, also this problem can be solved by means of a static state-feedback. In both cases, a feedback gain matrix is explicitly derived by making use of the results obtained for the optimal control problem. Finally, some preliminary results about the stabilization problem by means of a static, either time-invariant or time-varying, output feedback are also presented.**

I. Introduction

Research interests in Boolean networks (BNs) and Boolean control networks (BCNs) have a very long tradition. The renewed interest witnessed in recent times, however, must be mainly credited to two reasons: on the one hand, BNs and BCNs (as well as probabilistic BNs) have proved to be effective modeling tools for a number of rapidly evolving research topics, like genetic regulation networks [10], and consensus problems [9], [16]. On the other hand, the algebraic framework developed by D. Cheng and co-authors [1], [3], [4] has allowed to cast both BNs and BCNs into the framework of linear state-space models (operating on canonical vectors), thus benefitting of a large number of powerful algebraic tools, in addition to more traditional graph-based techniques. By resorting to this approach, stability and stabilizability of an equilibrium point or a limit cycle [2], [6], controllability [13], observability, reconstructibility and state estimation [5], have been thoroughly investigated. Also, the optimal control of BCNs has been addressed in a few contributions. In [17] (see also Chapter 15 in [4]) the problem of finding the input sequence that maximizes, on the infinite-horizon, an average payoff that weights both the state and the input at every time $t \in \mathbb{Z}_+$, has been investigated. Meanwhile in [11] and [12] the optimum control problem on a finite horizon, by assuming that the payoff function only depends on the state of the BCN at the end of the control interval, has been explored.

In a pair of recent papers [7], [8], both the finite-horizon and the infinite-horizon optimal control problems for BCNs have been investigated, by assuming a cost function that depends on both the state and the input values at every time instant. In the former case, the optimal solution is expressed as a time-varying static state feedback law. In the latter, the solution is obtained as the limit of the solution over the finite horizon $[0, T]$, and it is therefore a time-invariant static state feedback law. In this paper we first exploit the results obtained for the infinite-horizon optimal control problem to provide an alternative proof of the fact that the stabilization of a BCN to a given reachable equilibrium point can always be performed by means of a static state-feedback. Then we derive necessary and sufficient conditions for the solvability of the output regulation problem, and we show that, when such conditions are satisfied, also this problem can be solved by means of a static state-feedback. In both cases, a feedback gain matrix is explicitly derived by making use of the results obtained for the optimal control problem. Finally, some preliminary results about the stabilization problem by means of a static, either time-invariant or time-varying, output feedback are presented.

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Notation. $\mathbb{Z}_+$ denotes the set of nonnegative integers. Given $k, n \in \mathbb{Z}_+,\ k \le n$, the symbol $[k, n]$ denotes the set of integers $\{k, k+1, \dots, n\}$. We consider Boolean vectors and matrices, taking values in $\mathcal{B} := \{0, 1\}$, with the usual operations (sum $\vee$, product $\wedge$ and negation $\neg$).

δ^i_k is the ith canonical vector of size k, $\mathcal{L}_k$ the set of all k-dimensional canonical vectors, and $\mathcal{L}_{k\times n} \subset \mathcal{B}^{k\times n}$ the set of all $k \times n$ matrices whose columns are canonical vectors. $L \in \mathcal{L}_{k\times n}$ can be represented as a row vector whose entries are canonical vectors in $\mathcal{L}_k$, namely $L = [\, \delta^{i_1}_k \quad \delta^{i_2}_k \quad \dots \quad \delta^{i_n}_k \,]$, for suitable indices $i_1, i_2, \dots, i_n \in [1, k]$. $\mathbf{1}_k$ is the k-dimensional vector with all entries equal to 1. The (ℓ, j)th entry of a matrix M is denoted by $[M]_{\ell j}$, its ith column by $\mathrm{col}_i(M)$, the ℓth entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_\ell$.

Given a matrix $L \in \mathcal{B}^{k\times k}$ (in particular, $L \in \mathcal{L}_{k\times k}$), we associate with it a *digraph* $\mathcal{D}(L)$, with vertices $1, \dots, k$. There is an arc (j, ℓ) from j to ℓ if and only if the (ℓ, j)th entry of L is unitary. A sequence $j_1 \to j_2 \to \dots \to j_r \to j_{r+1}$ in $\mathcal{D}(L)$ is a *path* of length r from j_1 to j_{r+1} provided that $(j_1, j_2), \dots, (j_r, j_{r+1})$ are arcs of $\mathcal{D}(L)$.

There is a bijection between Boolean variables $X \in \mathcal{B}$ and vectors $\mathbf{x} \in \mathcal{L}_2$, defined by the relationship

$$\mathbf{x} = \begin{bmatrix} X \\ \neg X \end{bmatrix}. \qquad (1)$$

The *(left) semi-tensor product* $\ltimes$ between matrices (and, in particular, vectors) is defined as follows [4], [13]: given $L_1 \in \mathbb{R}^{r_1\times c_1}$ and $L_2 \in \mathbb{R}^{r_2\times c_2}$ (in particular, $L_1 \in \mathcal{L}_{r_1\times c_1}$ and

$L_2 \in \mathcal{L}_{r_2 \times c_2}$), we set

$$L_1 \ltimes L_2 := (L_1 \otimes I_{T/c_1})(L_2 \otimes I_{T/r_2}), \quad T := \text{l.c.m.}\{c_1, r_2\},$$

where l.c.m. denotes the least common multiple. The semi-tensor product is an extension of the standard matrix product, by this meaning that if $c_1 = r_2$, then $L_1 \ltimes L_2 = L_1 L_2$. Note that if $\mathbf{x}_1 \in \mathcal{L}_{r_1}$ and $\mathbf{x}_2 \in \mathcal{L}_{r_2}$, then $\mathbf{x}_1 \ltimes \mathbf{x}_2 \in \mathcal{L}_{r_1 r_2}$.
For the properties of the semi-tensor product we refer to [4]. (1) extends to a bijection between $\mathcal{B}^n$ and $\mathcal{L}_{2^n}$, as follows: given $X = [\, X_1 \quad X_2 \quad \ldots \quad X_n \,]^\top \in \mathcal{B}^n$, set

$$\mathbf{x} := \begin{bmatrix} X_1 \\ \neg X_1 \end{bmatrix} \ltimes \begin{bmatrix} X_2 \\ \neg X_2 \end{bmatrix} \ltimes \ldots \ltimes \begin{bmatrix} X_n \\ \neg X_n \end{bmatrix}.$$

II. Infinite-horizon optimal control of BCNs

A *Boolean control network* (BCN) is described by the following equations

$$\begin{aligned} X(t+1) &= f(X(t), U(t)), \\ Y(t) &= h(X(t)), \qquad t \in \mathbb{Z}_+, \end{aligned} \tag{2}$$

where $X(t)$, $U(t)$ and $Y(t)$ denote the n-dimensional state variable, the m-dimensional input and the p-dimensional output at time t, taking values in $\mathcal{B}^n, \mathcal{B}^m$ and $\mathcal{B}^p$, respectively. f, h are (logic) functions, i.e. $f : \mathcal{B}^n \times \mathcal{B}^m \to \mathcal{B}^n$ and $h : \mathcal{B}^n \to \mathcal{B}^p$. By resorting to the semi-tensor product $\ltimes$, state, input and output Boolean variables can be represented as canonical vectors in $\mathcal{L}_N,\ N = 2^n,\ \mathcal{L}_M,\ M = 2^m$, and $\mathcal{L}_P,\ P = 2^p$, respectively, and the BCN (2) satisfies [4] the following *algebraic description*:

$$\begin{aligned} \mathbf{x}(t+1) &= L \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \\ \mathbf{y}(t) &= H\mathbf{x}(t) \end{aligned} \tag{3}$$

where $\mathbf{x}(t) \in \mathcal{L}_N, \mathbf{u}(t) \in \mathcal{L}_M$ and $\mathbf{y}(t) \in \mathcal{L}_P$. $L \in \mathcal{L}_{N \times NM}$ and $H \in \mathcal{L}_{P \times N}$ are matrices whose columns are all canonical vectors of size N and P, respectively. For every choice of the input variable at t, namely for every $\mathbf{u}(t) = \delta_M^j$, $L \ltimes \mathbf{u}(t) =: L_j$ is a matrix in $\mathcal{L}_{N \times N}$. So, we can think of the state equation of the BCN (3) as a Boolean switched system,

$$\mathbf{x}(t+1) = L_{\sigma(t)} \mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{4}$$

where $\sigma(t), t \in \mathbb{Z}_+$, is a switching sequence taking values in $[1, M]$. For every $i \in [1, M]$, we refer to the BN

$$\mathbf{x}(t+1) = L_i \mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{5}$$

as to the ith subsystem of the Boolean switched system (4). L can be expressed as $L = [\, L_1 \quad L_2 \quad \ldots \quad L_M \,]$.

Before proceeding, we need the concept of reachability.

Definition 1: [4] Given a BCN (3), we say that $\mathbf{x}_f = \delta_N^j$ is *reachable* from $\mathbf{x}_0 = \delta_N^h$ if there exists $\tau \in \mathbb{Z}_+$ and an input $\mathbf{u}(t), t \in [0, \tau - 1]$, that leads the state trajectory from $\mathbf{x}(0) = \mathbf{x}_0$ to $\mathbf{x}(\tau) = \mathbf{x}_f$.

A state $\mathbf{x}_f = \delta_N^j$ is reachable from $\mathbf{x}_0 = \delta_N^h$ if and only if [4] there exists $\tau \in \mathbb{Z}_+$ such that the Boolean sum of the matrices $L_i, i \in [1, M]$, namely

$$L_{tot} := \bigvee_{i=1}^{M} L_i,$$

satisfies $[L_{tot}^\tau]_{jh} = 1$. In the sequel, we will denote the set of states reachable from $\mathbf{x}_0$ as $\mathcal{R}(\mathbf{x}_0)$.

In a recent contribution [8] we have addressed the following infinite-horizon optimal control problem:

Given the BCN (3), with initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{L}_N$, determine an input sequence that minimizes the cost function:

$$J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \sum_{t=0}^{+\infty} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t), \tag{6}$$

where $\mathbf{c}^\top := [\, \mathbf{c}_1^\top \quad \mathbf{c}_2^\top \quad \ldots \quad \mathbf{c}_M^\top \,] \in \mathbb{R}^{NM}$ is nonnegative.

We have shown that the optimum index

$$J^*(\mathbf{x}_0) := \min_{\mathbf{u}(\cdot)} J(\mathbf{x}_0, \mathbf{u}(\cdot)),$$

takes a finite value if and only if there exists at least one periodic state-input trajectory $(\mathbf{x}(t), \mathbf{u}(t))_{t \in \mathbb{Z}_+}$, of zero cost, that can be reached from $\mathbf{x}_0$. This amounts to saying that there exist $T > 0$, $\tau \geq 0$ and $\mathbf{u}(t), t \in \mathbb{Z}_+$, such that

$$(\mathbf{x}(t), \mathbf{u}(t)) = (\mathbf{x}(t+T), \mathbf{u}(t+T)),\ \forall\, t \in \mathbb{Z}_+,\ t \geq \tau, \tag{7}$$

$$\mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t) = 0,\ \forall\, t \in \mathbb{Z}_+,\ t \geq \tau. \tag{8}$$

Conditions (7) and (8) can be easily checked, by making use of either the graph associated with the BCN or the matrices $L_i, i \in [1, N]$, and of the vector $\mathbf{c}$ (see [8] for the details). If $\mathcal{H}$ denotes the set of all states that belong to a periodic zero-cost state-input trajectory, $J^*(\delta_N^h) = 0$ for every $h \in \mathcal{H}$. On the other hand, for every state $\delta_N^j, j \notin \mathcal{H}$, it is sufficient to determine the minimum cost state-input trajectory $(\mathbf{x}(t), \mathbf{u}(t))_{t \in \mathbb{Z}_+}$ starting from $\mathbf{x}(0)$ and reaching some state $\delta_N^h, h \in \mathcal{H}$, in a finite number (at most $N - 1$) of steps. $J^*(\delta_N^h)$ is just the cost associated with that minimum cost state-input trajectory.

The optimal solution can always be obtained as a static state-feedback. To obtain the feedback law (as well as the optimal cost function), let $\mathbf{m}^*$ be the vector whose jth entry is obtained according to the following **Algorithm**:

- if $j \in \mathcal{H}$ then $[\mathbf{m}^*]_j := 0$;
- if $j \notin \mathcal{H}$, then $[\mathbf{m}^*]_j$ is the solution of the minimization problem: $[\mathbf{m}^*]_j = \min_{i \in [1,M]} [\mathbf{c}_i^\top + (\mathbf{m}^*)^\top L_i]_j$.

Also, if $j \in \mathcal{H}$, let $i^*(j)$ be any index $i \in [1, M]$ such that the pair (δ_N^j, δ_M^i) belongs to a zero cost state-input trajectory. If $j \notin \mathcal{H}$, set $i^*(j) := \arg\min_{i \in [1,M]} [\mathbf{c}_i^\top + (\mathbf{m}^*)^\top L_i]_j$.
Then [8] the optimal cost function is $J^*(\mathbf{x}_0) = (\mathbf{m}^*)^\top \mathbf{x}_0$, while the optimal control input can be implemented by means of the static state-feedback law: $\mathbf{u}(t) = K\mathbf{x}(t)$, where

$$K = [\, \delta_M^{i^*(1)} \quad \delta_M^{i^*(2)} \quad \ldots \quad \delta_M^{i^*(N)} \,].$$

In the following sections we will show that some control problems for BCNs can be solved upon restating them as infinite-horizon optimal control problems.

III. Stabilization to a given state

The first problem we address is that of stabilization of a BCN to some equilibrium point $\mathbf{x}_e$.

Definition 2: [2], [4], [6] A BCN (3) is *stabilizable* to $\mathbf{x}_e \in \mathcal{L}_N$ if for every $\mathbf{x}(0) \in \mathcal{L}_N$ there exist $\mathbf{u}(t), t \in \mathbb{Z}_+$, and $\tau \in \mathbb{Z}_+$ such that $\mathbf{x}(t) = \mathbf{x}_e$ for every $t \geq \tau$.

The problem solution is rather immediate.

Proposition 1: [4], [6], [15] A BCN (3) is stabilizable to $\mathbf{x}_e \in \mathcal{L}_N$ if and only if the following conditions hold

1) $\mathbf{x}_e$ is an equilibrium point of the ith subsystem (5), for some $i \in [1, M]$, i.e. $\mathbf{x}_e = L \ltimes \delta_M^i \ltimes \mathbf{x}_e$;
2) $\mathbf{x}_e$ is reachable from every initial state $\mathbf{x}(0)$, i.e., $\mathbf{x}_e \in \cap_{\mathbf{x}(0)\in\mathcal{L}_N} \mathcal{R}(\mathbf{x}(0))$.

What is more interesting is the fact that if a BCN (3) is stabilizable to $\mathbf{x}_e$, then stabilization is achievable by means of a static state-feedback law [6], [15]. We want to show that the same result can be obtained by casting this problem into the optimal control set-up, and by resorting to the results of the previous section. Assume $\mathbf{x}_e = \delta_N^{j^*}$, and set

$$I(\mathbf{x}_e) := \{i \in [1, M] : \mathbf{x}_e = L \ltimes \delta_M^i \ltimes \mathbf{x}_e\}.$$

Introduce the cost vector $\mathbf{c}^\top := [\, \mathbf{c}_1^\top \quad \mathbf{c}_2^\top \quad \dots \quad \mathbf{c}_M^\top \,]$, with

$$[\mathbf{c}_i]_j = \begin{cases} 0, & \text{if } i \in I(\mathbf{x}_e) \text{ and } j = j^*; \\ 1, & \text{otherwise.} \end{cases} \tag{9}$$

Theorem 1: Given $\mathbf{x}_e = \delta_N^{j^*}$, the BCN (3) is stabilizable to $\mathbf{x}_e$ if and only if $J^*(\mathbf{x}_0) = \min_u J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \min_u \sum_{t=0}^{+\infty} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t)$, with $\mathbf{c}$ given in (9), is finite for each $\mathbf{x}_0 \in \mathcal{L}_N$.

Proof: If the BCN is stabilizable to $\mathbf{x}_e$ then, by Proposition 1 point 2), for every $\mathbf{x}_0$ there exists $\tau \in \mathbb{Z}_+$ and an input sequence $\tilde{\mathbf{u}}(t), t \in [0, \tau-1]$, that drives the BCN state, say $\tilde{\mathbf{x}}(t)$, to $\mathbf{x}_e$ at time τ. On the other hand, by Proposition 1 point 1), the set $I(\mathbf{x}_e)$ is not empty and for every $i \in I(\mathbf{x}_e)$ we have $\mathbf{c}^\top \ltimes \delta_M^i \ltimes \mathbf{x}_e = 0$. We therefore have $J^*(\mathbf{x}_0) \leq \sum_{t=0}^{\tau-1} \mathbf{c}^\top \ltimes \tilde{\mathbf{u}}(t) \ltimes \tilde{\mathbf{x}}(t) < +\infty$.
Conversely, if $J^*(\mathbf{x}_0) < +\infty$ for every $\mathbf{x}_0 \in \mathcal{L}_N$, there exists $\tau \in \mathbb{Z}_+$ such that $\mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t) = 0, \ \forall\ t \geq \tau$. By the way the vector $\mathbf{c}$ has been defined, this implies, in particular, that $\mathbf{x}(\tau) = \mathbf{x}_e$, and the arbitrariety of $\mathbf{x}_0$ ensures that point 2) of Proposition 1 holds. Also, if $\mathbf{u}(\tau) = \delta_M^i$, then $i \in I(\mathbf{x}_e)$ and hence point 1) of Proposition 1 holds, too. Consequently, the BCN (3) is stabilizable to $\mathbf{x}_e$. ∎

This result allows to reduce the solution of the stabilization problem to the solution of an infinite-horizon optimal control problem. In particular, the Algorithm described in the previous section, with $\mathcal{H} = \{j^*\}$, can be used to derive the state-feedback matrix K. Note that $J^*(\mathbf{x}_0)$ will always be equal to the length of the shortest path from $\mathbf{x}_0$ to $\mathbf{x}_e = \delta_N^{j^*}$.

IV. Regulation problem

A classical control theory problem is the regulation of the output trajectory to a given constant value, say $\mathbf{y}_e$. Clearly, this can be seen as a natural extension of the stabilization problem addressed in the previous section. The regulation problem is formalized in the following definition.

Definition 3: The *regulation problem to the output value* $\mathbf{y}_e \in \mathcal{L}_P$ is solvable for the BCN (3) if for every $\mathbf{x}(0) \in \mathcal{L}_N$ there exist $\mathbf{u}(t), t \in \mathbb{Z}_+$, and $\tau \in \mathbb{Z}_+$ such that $\mathbf{y}(t) = \mathbf{y}_e$ for every $t \geq \tau$.

The problem solution requires some notation. We first introduce the set $\mathcal{X}(\mathbf{y}_e) := \{\delta_N^j : H\delta_N^j = \mathbf{y}_e\}$, which is nothing but the indistinguishability class in 1 step corresponding to the output value $\mathbf{y}_e$ [5]. We also denote by $\mathcal{Z}(\mathbf{y}_e)$ the subset of all states $\tilde{\mathbf{x}}$ of $\mathcal{X}(\mathbf{y}_e)$ for which there exists a state-input trajectory $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{u}}(t)), t \in \mathbb{Z}_+$, satisfying $\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}$ and $\tilde{\mathbf{x}}(t) \in \mathcal{X}(\mathbf{y}_e), \forall\ t \in \mathbb{Z}_+$.

Proposition 2: The regulation problem to the value $\mathbf{y}_e$ is solvable for the BCN (3) if and only if the following conditions hold

1) $\mathcal{X}(\mathbf{y}_e)$ contains a state trajectory, or, equivalently, $\mathcal{Z}(\mathbf{y}_e) \neq \emptyset$;
2) the set $\mathcal{Z}(\mathbf{y}_e)$ is reachable from every initial state $\mathbf{x}(0)$, i.e., $\mathcal{Z}(\mathbf{y}_e) \cap \mathcal{R}(\mathbf{x}(0)) \neq \emptyset$ for every $\mathbf{x}(0) \in \mathcal{L}_N$.

Proof: [Sufficiency] Let $\mathbf{x}(0)$ be any state in $\mathcal{L}_N$. If 1) and 2) hold, there exists $\tilde{\mathbf{x}} \in \mathcal{Z}(\mathbf{y}_e) \cap \mathcal{R}(\mathbf{x}(0))$. Consequently, $\tau \in \mathbb{Z}_+$ and an input $\tilde{\mathbf{u}}(t), t \in [0, \tau-1]$, can be found leading the state trajectory from $\mathbf{x}(0)$ to $\mathbf{x}(\tau) = \tilde{\mathbf{x}}$. As $\tilde{\mathbf{x}} \in \mathcal{Z}(\mathbf{y}_e)$, there exists $\tilde{\mathbf{u}}(t), t \in [\tau, +\infty)$, such that $\mathbf{x}(t) \in \mathcal{Z}(\mathbf{y}_e)$ for every $t \geq \tau$ and hence $\mathbf{y}(t) = \mathbf{y}_e$ for every $t \geq \tau$.
[Necessity] Follows the same lines as the sufficiency part. ∎

Also in this case the problem solution can be expressed as a static state-feedback, and we derive this result again from the solution of the infinite-horizon optimal control problem. Introduce the cost vector $\mathbf{c}^\top := [\, \mathbf{c}_1^\top \quad \mathbf{c}_2^\top \quad \dots \quad \mathbf{c}_M^\top \,]$, with

$$[\mathbf{c}_i]_j = \begin{cases} 0, & \text{if } j \in \mathcal{Z}(\mathbf{y}_e) \text{ and } L \ltimes \delta_M^i \ltimes \delta_N^j \in \mathcal{Z}(\mathbf{y}_e); \\ 1, & \text{otherwise.} \end{cases} \tag{10}$$

Theorem 2: Given the output value $\mathbf{y}_e$, the regulation problem to the value $\mathbf{y}_e$ is solvable for the BCN (3) if and only if the optimal control problem

$$J^*(\mathbf{x}_0) = \min_u J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \min_u \sum_{t=0}^{+\infty} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t),$$

with $\mathbf{c}$ given in (10) has a finite solution for every $\mathbf{x}_0 \in \mathcal{L}_N$.

Proof: If the regulation problem is solvable then, by Proposition 2, for every $\mathbf{x}_0$ there exists $\tau \in \mathbb{Z}_+$ and an input sequence $\tilde{\mathbf{u}}(t), t \in [0, \tau-1]$, that drives the BCN to some state trajectory $\tilde{\mathbf{x}}(t), t \geq \tau$, included in $\mathcal{Z}(\mathbf{y}_e)$ Therefore $J^*(\mathbf{x}_0) \leq \sum_{t=0}^{\tau-1} \mathbf{c}^\top \ltimes \tilde{\mathbf{u}}(t) \ltimes \tilde{\mathbf{x}}(t) < +\infty$.
Conversely, suppose that $J^*(\mathbf{x}_0) < +\infty$ for every $\mathbf{x}_0 \in \mathcal{L}_N$. Then there exists $\tau \in \mathbb{Z}_+$ such that $\mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t) = 0$, $\forall\ t \geq \tau$. By the way the vector $\mathbf{c}$ has been defined and the arbitrariety of $\mathbf{x}_0$, this implies, in particular, that $\mathbf{x}(\tau) \in \mathcal{Z}(\mathbf{y}_e)$, and hence points 1) and 2) of Proposition 2 hold. This ensures that the regulation problem is solvable. ∎

Also in this case, we may apply the Algorithm described in the section II, for $H = \mathcal{Z}(\mathbf{y}_e)$, to derive the state-feedback matrix K. In this case $J^*(\mathbf{x}_0)$ will be equal to the length of the shortest path from $\mathbf{x}_0$ to $\mathcal{Z}(\mathbf{y}_e)$.

V. Output feedback stabilization

As we have seen, the problem of stabilizing a BCN to some state $\mathbf{x}_e$, under the necessary and sufficient conditions given in Proposition 1, can be solved by means of a static state feedback law. A similar result has been derived for the output regulation problem. So, the question spontaneously arises: under what conditions can we solve these problems by resorting to an output feedback?

Definition 4: A BCN (3) is *output feedback stabilizable* to the state $\mathbf{x}_e \in \mathcal{L}_N$ if there exists $K_y \in \mathcal{L}_{M\times P}$ such that the output feedback law $\mathbf{u}(t) = K_y \mathbf{y}(t)$, $t \in \mathbb{Z}_+$, drives every $\mathbf{x}(0) \in \mathcal{L}_N$ to the state $\mathbf{x}_e$ in a finite number of steps, namely $\exists \tau \in \mathbb{Z}_+$ such that $\mathbf{x}(t) = \mathbf{x}_e$ for every $t \geq \tau$.

The output feedback stabilization problem is quite challenging to be solved in a computationally tractable way. Clearly, if K_y defines an output feedback law, then $K = K_y H$ defines a state feedback law. So, a possible way could be that of determining whether the set of all state-feedback matrices includes at least one matrix expressed as $K = K_y H$ for some $K_y \in \mathcal{L}_{M\times P}$ (see [14]). However, in general, the search cannot be restricted to the matrices K that implement paths of minimum length from each state to the equilibrium state $\mathbf{x}_e$ [6]. Consequently, the test may need to be performed on a quite large set of state feedback matrices. We illustrate this concept by means of an example.

Example 1: Consider a BCN (3) with $N = 6^1, M = 2$, $P = 2$ and

$$\begin{aligned} L_1 &:= L \ltimes \delta_2^1 = [\,\delta_6^2 \quad \delta_6^6 \quad \delta_6^6 \quad \delta_6^3 \quad \delta_6^6 \quad \delta_6^1\,], \\ L_2 &:= L \ltimes \delta_2^2 = [\,\delta_6^1 \quad \delta_6^3 \quad \delta_6^4 \quad \delta_6^5 \quad \delta_6^4 \quad \delta_6^6\,], \\ H &= [\,\delta_2^1 \quad \delta_2^2 \quad \delta_2^1 \quad \delta_2^1 \quad \delta_2^2 \quad \delta_2^2\,] \end{aligned}$$

The BCN can be represented by the following digraph, obtained by overlapping the digraphs $\mathcal{D}(L_1)$ and $\mathcal{D}(L_2)$.

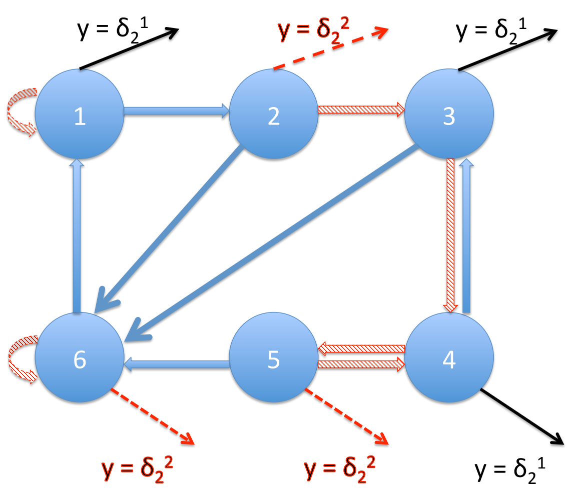

Fig. 1. *Digraph corresponding to the BCN of Example 1.*

Light blue thick arrows represent arcs of $\mathcal{D}(L_1)$, red thick dashed arrows represent arcs of $\mathcal{D}(L_2)$. Black continuous arrows stem from states whose associated output is δ_2^1, while red dashed lines stem from states whose ouput is $y = \delta_2^2$.

[1] $N = 6$ is not a power of 2, but the analysis is not affected by this fact

Assume $\mathbf{x}_e = \delta_6^6$. It is easy to see that $\mathbf{x}_e$ is reachable from every state and $\mathbf{x}_e = L \ltimes \delta_2^2 \ltimes \mathbf{x}_e$.
As both conditions of Proposition 1 are satisfied, the BCN is stabilizable to $\mathbf{x}_e$. If we search for the stabilizing state-feedback matrices that correspond to minimum distance paths from each δ_6^j to $\mathbf{x}_e = \delta_6^6$, we find two possible solutions

$$\begin{aligned} K_1 &= [\,\delta_2^1 \quad \delta_2^1 \quad \delta_2^1 \quad \delta_2^1 \quad \delta_2^1 \quad \delta_2^2\,], \\ K_2 &= [\,\delta_2^1 \quad \delta_2^1 \quad \delta_2^1 \quad \delta_2^2 \quad \delta_2^1 \quad \delta_2^2\,]. \end{aligned}$$

Neither of these matrices can be expressed as $K_i = K_{yi} H$ for some $K_{yi} \in \mathcal{L}_{2\times 2}$, otherwise the last two columns of K_1 or K_2 should be identical. On the other hand, it is easy to see that $K_y = [\,\delta_2^1 \quad \delta_2^2\,] = I_2$ (corresponding to $K_y H = H$, namely $\mathbf{u}(t) = \mathbf{y}(t)$) determines a stabilizing output feedback. ♠

A necessary (but not sufficient) condition for static output-feedback stabilization is given in the following proposition.

Proposition 3: Given a BCN (3), a necessary condition for the existence of a static output feedback stabilizing the BCN to $\mathbf{x}_e$ is that there exists an input value $\bar{\mathbf{u}}$ such that $L \ltimes \bar{\mathbf{u}} \ltimes \mathbf{x}_e = \mathbf{x}_e$, and $\mathbf{x}(t) = \mathbf{x}_e, \forall\, t \in \mathbb{Z}_+$ is the only periodic state trajectory corresponding to the constant input $\mathbf{u}(t) = \bar{\mathbf{u}}, t \in \mathbb{Z}_+$, that is entirely included in $\mathcal{X}(H\mathbf{x}_e)$.

Example 2: Consider a BCN with $N = 4, M = 2, P = 2$

$$\begin{aligned} L_1 &:= L \ltimes \delta_2^1 = [\,\delta_4^1 \quad \delta_4^2 \quad \delta_4^4 \quad \delta_4^1\,], \\ L_2 &:= L \ltimes \delta_2^2 = [\,\delta_4^4 \quad \delta_4^3 \quad \delta_4^3 \quad \delta_4^3\,], \\ H &= [\,\delta_2^1 \quad \delta_2^1 \quad \delta_2^2 \quad \delta_2^2\,] \end{aligned}$$

The BCN can be represented by the following digraph.

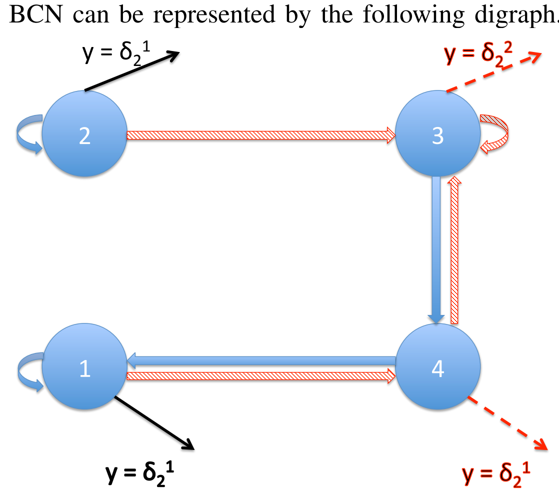

Fig. 2. *Digraph corresponding to the BCN of Example 2.*

Assume $\mathbf{x}_e = \delta_4^1$. It is easy to see that $\mathbf{x}_e$ is reachable from every state and $\mathbf{x}_e = L \ltimes \delta_2^1 \ltimes \mathbf{x}_e$. As both conditions of Proposition 1 are satisfied, the BCN is state-feedback stabilizable to $\mathbf{x}_e$. However, an output feedback stabilizing the BCN to the state δ_4^1 does not exist, since the previous necessary condition is not satisfied. Indeed, the only input value that keeps the system in the equilibrium state is $\mathbf{u} = \delta_2^1$. However, $\delta_4^2 \in \mathcal{X}(H\mathbf{x}_e) = \{\delta_4^i : H\delta_4^i = \delta_2^1\}$ is an equilibrium point of the BCN corresponding to the same input value. ♠

When both conditions of Proposition 1 are satisfied for some specific $\mathbf{x}_e$, we provide an algorithm to explore the existence of an output-feedback law stabilizing the BCN (3) to $\mathbf{x}_e$. To this end, we previously remove output values that never occur (this is the case if there are zero rows in the matrix H) and introduce a suitable permutation of the state and output components. So, we can always assume $\mathbf{x}_e = \delta_N^1$ and $H = \mathrm{diag}\{\mathbf{1}_{n_1}^\top, \mathbf{1}_{n_2}^\top, \dots, \mathbf{1}_{n_P}^\top\}$. In this set-up, Theorem 1 in [14] can be restated, in slightly revised terms, as follows.

Corollary 1: The BCN (3) is output feedback stabilizable to $\mathbf{x}_e = \delta_N^1$ if and only if there exists $K_y \in \mathcal{L}_{M\times P}$ such that

$$(L \ltimes K_y H \Phi_N)^N = [\,\delta_N^1 \quad \delta_N^1 \quad \dots \quad \delta_N^1\,] = \delta_N^1 \mathbf{1}_N^T, \quad (11)$$

where Φ is the so-called *power reducing matrix* [4], i.e. the logical matrix satisfying $\mathbf{x}(t) \ltimes \mathbf{x}(t) = \Phi \mathbf{x}(t), \forall \mathbf{x}(t) \in \mathcal{L}_N$.

If we analyze the structure of the logical matrix $L \ltimes K_y H \Phi_N$, we observe that it is obtained in this way

$$L \ltimes K_y H \Phi_N = [\,\mathrm{blk}_1(L_{i_1}) \quad \mathrm{blk}_2(L_{i_2}) \quad \dots \quad \mathrm{blk}_P(L_{i_P})\,],$$

where $\mathrm{blk}_k(L_{i_k})$ is the $N \times n_k$ matrix obtained by selecting the columns of L_{i_k} with indices in the interval $[(\sum_{\ell=1}^{k-1} n_\ell)+1, (\sum_{\ell=1}^{k-1} n_\ell)+n_k]$, and $K_y = [\,\delta_M^{i_1} \quad \delta_M^{i_2} \quad \dots \quad \delta_M^{i_P}\,]$. Condition (11) is satisfied if and only if

$$L \ltimes K_y H \Phi_N = \begin{bmatrix} 1 & * \\ 0_{N-1} & \tilde{N} \end{bmatrix},$$

for some nilpotent matrix $\tilde{N} \in \mathcal{L}_{(N-1)\times(N-1)}$. In addition, $\tilde{N}$ is nilpotent if and only if all its principal submatrices are nilpotent. This suggests an algorithm to ordinately choose the indices $i_1, i_2, \dots, i_P$ appearing in the matrix K_y.

First, the index i_1 is chosen in such a way that the first column of $\mathrm{blk}_1(L_{i_1})$ is δ_N^1 and the principal submatrix of L_{i_1} obtained by selecting rows and columns of indices $[2, n_1]$ is nilpotent (note that this selection criterion for i_1 is nothing but the necessary condition we have given in Proposition 3). Subsequently, i_2 is chosen in such a way that the principal submatrix of L_{i_2} obtained by selecting rows and columns of indices $[n_1+1, n_1+n_2]$ is nilpotent, and the submatrix of $[\,\mathrm{blk}_1(L_{i_1}) \quad \mathrm{blk}_2(L_{i_2})\,]$ obtained by selecting rows and columns of indices $[2, n_1+n_2]$ is nilpotent. By proceeding in this way, all the indices $i_1, i_2, \dots, i_P$ are chosen. Clearly, if at some stage there is no choice of the index i_k such that the submatrix of $[\,\mathrm{blk}_1(L_{i_1}) \quad \mathrm{blk}_2(L_{i_2}) \quad \dots \quad \mathrm{blk}_k(L_{i_k})\,]$ obtained by selecting rows and columns of indices $[2, n_1+n_2+\dots+n_k]$ is nilpotent, then the previous choices for $i_1, i_2, \dots, i_{k-1}$ must be modified. By proceeding in this way, either a solution is explicitly derived or it is shown that there is no possible solution. The selection criterion allows to restrict the analysis and to not consider all possible P-tuples of indices. So, this represents a sort of branch and bound algorithm.

Remark 1: If we look for all the output feedback matrices that stabilize the BCN to $\mathbf{x}_e = \delta_N^1$, and consider the homogeneous polynomial matrix

$$\mathfrak{L} = \left[\sum_{j_1=1}^{M} \mathrm{blk}_1(L_{i_1}) z_{1,i_1} \middle| \quad \dots \quad \middle| \sum_{j_P=1}^{M} \mathrm{blk}_P(L_{i_P}) z_{P,i_P} \right]$$

in the PM indeterminates $z_{h,i_h}, h \in [1,P], i_h \in [1,M]$, the stabilization to $\mathbf{x}_e = \delta_N^1$ is possible if and only if there exist P indeterminates $z_{1,k_1}, \dots z_{P,k_P}$ such that every entry of the first row of $\mathfrak{L}^N$ includes a monomial (of degree N) in $z_{1,k_1}, \dots z_{P,k_P}$. If so, a stabilizing output feedback matrix is $K_y = [\,\delta_M^{k_1} \quad \delta_M^{k_2} \quad \dots \quad \delta_M^{k_P}\,]$

VI. Time-varying output feedback stabilization

All feedback solutions proposed in the previous sections are time-invariant. There are situations, however, when the output feedback stabilization cannot be achieved by resorting to a time-invariant solution, but it can be by adopting a time varying feedback law $\mathbf{u}(t) = K_y(t)\mathbf{y}(t),\ t \in \mathbb{Z}_+$.

Example 3: Consider the BCN of Example 2, and assume, again, $\mathbf{x}_e = \delta_4^1$. A time-invariant output feedback law stabilizing the BCN to $\mathbf{x}_e$ does not exist. However, the time-varying output feedback law $K_y(0) = [\,\delta_2^2 \quad \delta_2^2\,],\ K_y(t) = [\,\delta_2^1 \quad \delta_2^1\,], \forall\ t \geq 1$, stabilizes the BCN to $\mathbf{x}_e$. ♠

The idea behind Example 3 can be generalized, as shown in the following proposition.

Proposition 4: Given a BCN (3) and $\mathbf{x}_e \in \mathcal{L}_N$, suppose that there exists $\bar{\mathbf{u}} = \delta_M^{i^*}$ such that $\mathbf{x}_e = L \ltimes \bar{\mathbf{u}} \ltimes \mathbf{x}_e$ and let $\mathcal{A}_{\bar{\mathbf{u}}}(\mathbf{x}_e)$ denote the domain of attraction of $\mathbf{x}_e$ in the BN $\mathbf{x}(t+1) = L \ltimes \bar{\mathbf{u}} \ltimes \mathbf{x}(t) = L_{i^*}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+$, i.e. the set of all initial states whose associated state trajectory eventually becomes equal to $\mathbf{x}_e$. If there exist $T \in \mathbb{Z}_+$ and an input $\tilde{\mathbf{u}}(0), \tilde{\mathbf{u}}(1), \dots, \tilde{\mathbf{u}}(T-1)$ such that for every $\mathbf{x}(0) \in \mathcal{L}_N$, the state trajectory stemming from $\mathbf{x}(0)$ under the action of the previous input satisfies $\mathbf{x}(T) \in \mathcal{A}_{\bar{\mathbf{u}}}(\mathbf{x}_e)$, then there exists a time-varying output feedback stabilizing the BCN to $\mathbf{x}_e$.

Proof: Assume $\tilde{\mathbf{u}}(t) = \delta_M^{i_t}, t \in [0, T-1]$. Then

$$K_y(t) = \begin{cases} \delta_M^{i_t} \mathbf{1}_P^\top, & t \in [0, T-1]; \\ \delta_M^{i^*} \mathbf{1}_P^\top, & t \geq T; \end{cases}$$

stabilizes the BCN to $\mathbf{x}_e$. ■

Example 4: Consider a BCN with $N = 8, M = 2, P = 2$,

$$\begin{aligned} L_1 &:= L \ltimes \delta_2^1 = [\,\delta_8^2 \quad \delta_8^6 \quad \delta_8^4 \quad \delta_8^5 \quad \delta_8^3 \quad \delta_8^7 \quad \delta_8^3 \quad \delta_8^2\,], \\ L_2 &:= L \ltimes \delta_2^2 = [\,\delta_8^8 \quad \delta_8^2 \quad \delta_8^3 \quad \delta_8^3 \quad \delta_8^4 \quad \delta_8^5 \quad \delta_8^6 \quad \delta_8^7\,], \\ H &= [\,\delta_2^1 \quad \delta_2^2 \quad \delta_2^1 \quad \delta_2^1 \quad \delta_2^2 \quad \delta_2^2 \quad \delta_2^1 \quad \delta_2^2\,] \end{aligned}$$

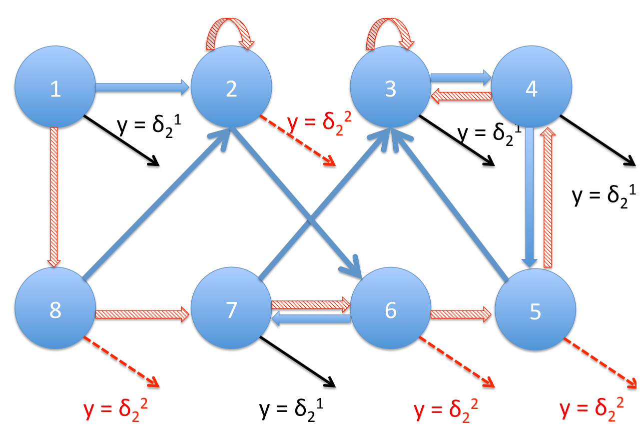

FIG. 3. *Digraph corresponding to the BCN of Example 4.*

Assume $\mathbf{x}_e = \delta_8^3$. Then $\mathbf{x}_e = L \ltimes \delta_2^2 \ltimes \mathbf{x}_e$ and $\mathcal{A}_{\delta_2^2}(\delta_8^3) = \{\delta_8^i; i \in [1,8], i \neq 2\}$. On the other hand, if we apply at $t=0$ and $t=1$ the input values $\tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}(1) = \delta_2^1$, independently of the initial state $\mathbf{x}(0)$, we know that $\mathbf{x}(2) \in \mathcal{A}_{\delta_2^2}(\delta_8^3)$. Therefore the output feedback law

$$K_y(t) = \begin{cases} [\, \delta_2^1 \quad \delta_2^1 \,], & t \in [0,1]; \\ \\ [\, \delta_2^2 \quad \delta_2^2 \,], & t \geq 2; \end{cases}$$

stabilizes the BCN to $\mathbf{x}_e$. ♠

Another set of sufficient conditions for the existence of a time-varying output feedback stabilization is given in the following proposition.

Proposition 5: Given a BCN (3), let $\mathbf{x}_e = \delta_N^1$ be an equilibrium point of the BCN for the set of input values $\mathcal{U}_e \subseteq \mathcal{L}_M$ and assume that $H\mathbf{x}_e = \delta_P^1$. If $\mathbf{x}_e$ is reachable from every initial state $\mathbf{x}(0) = \delta_N^j$ using input sequences $\mathbf{u}^{(j)}(0), \mathbf{u}^{(j)}(1), \dots$ that satisfy the constraint

$$\mathbf{x}^{(j)}(t) \in \mathcal{X}(\delta_P^1) \quad \Rightarrow \quad \mathbf{u}^{(j)}(t) \in \mathcal{U}_e, \tag{12}$$

there exists a time-varying output feedback $K_y(t), t \in \mathbb{Z}_+$ that drives every $\mathbf{x}(0) \in \mathcal{L}_N$ to $\mathbf{x}_e$ in finite time.

Proof: Let $\mathcal{C}_k \subset \mathcal{L}_N$ denote the set of initial states $\mathbf{x}(0) = \delta_N^j$ that can be driven to $\mathbf{x}_e$ at time $t = k$, but not at any time $t < k$, by using input sequences satisfying (12). Clearly $\mathcal{C}_0 = \{\mathbf{x}_e\}$ and there exists $i_{\max} \geq 1$ such that $\mathcal{C}_i \neq \emptyset$ if and only if $i \in [0, i_{\max}]$.
We inductively define the matrices $K_y(t)$, as follows.
[Case $t = 0$] For every $h \in [1,P]$ such that $\mathcal{X}(\delta_P^h) \cap \mathcal{C}_1 \neq \emptyset$, set $\mathrm{col}_h(K_y(0)) = K_y(0)\delta_P^h = \delta_M^{i_{h,0}}$, where the input value $\mathbf{u} = \delta_M^{i_{h,0}}$ maps (at least) one element of $\mathcal{X}(\delta_P^h) \cap \mathcal{C}_1$ in $\mathbf{x}_e$ i.e.

$$\mathbf{x}_e = L \ltimes \delta_M^{i_{h,0}} \ltimes \delta_N^i, \quad \exists \delta_N^i \in \mathcal{X}(\delta_P^h) \cap \mathcal{C}_1,$$

and $\delta_M^{i_{h,0}} \in \mathcal{U}_e$ when $h = 1$. For every $h \in [1,P]$ such that $\mathcal{X}(\delta_P^h) \cap \mathcal{C}_1 = \emptyset$, we let $\tilde{k}$ denote the least positive integer such that $\mathcal{X}(\delta_P^h) \cap \mathcal{C}_{\tilde{k}} \neq \emptyset$. In this case, we set $\mathrm{col}_h(K_y(0)) = K_y(0)\delta_P^h = \delta_M^{i_{h,0}}$, where the input value $\mathbf{u} = \delta_M^{i_{h,0}}$ maps (at least) one element of $\mathcal{X}(\delta_P^h) \cap \mathcal{C}_{\tilde{k}}$ in $\mathcal{C}_{\tilde{k}-1}$, i.e

$$L \ltimes \delta_M^{i_{h,0}} \ltimes \delta_N^i \in \mathcal{C}_{\tilde{k}-1}, \quad \exists \delta_N^i \in \mathcal{X}(\delta_P^j) \cap \mathcal{C}_{\tilde{k}},$$

and, again, $\delta_M^{i_{h,0}} \in \mathcal{U}_e$ when $h = 1$.
[Induction step] Assume that $K_y(0), K_y(1), \dots K_y(t-1)$ have been already selected, and introduce $\mathcal{S}_h^{(t)}$ the (possibly empty) set of states, $\mathbf{x}(t) \in \mathcal{L}_N$, that can be reached at time t, by resorting to the previous output feedback, and such that $H\mathbf{x}(t) = \delta_P^h$. Clearly, $\mathcal{S}_h^{(t)} \subseteq \mathcal{X}(\delta_P^h)$.

- Case $\mathcal{S}_h^{(t)} \neq \emptyset$.

For every $h \in [1,P]$ such that $\mathcal{S}_h^{(t)} \cap \mathcal{C}_1 \neq \emptyset$, we set $\mathrm{col}_h(K_y(t)) = K_y(t)\delta_P^h = \delta_M^{i_{h,t}}$, where the input value $\mathbf{u} = \delta_M^{i_{h,t}}$ maps (at least) one element of $\mathcal{S}_h^{(t)} \cap \mathcal{C}_1$ in $\mathbf{x}_e$, i.e.

$$\mathbf{x}_e = L \ltimes \delta_M^{i_{h,t}} \ltimes \delta_N^i, \ \exists \delta_N^i \in \mathcal{S}_h^{(t)} \cap \mathcal{C}_1,$$

and $\delta_M^{i_{h,t}} \in \mathcal{U}_e$ when $h = 1$.

On the other hand, for every $h \in [1,P]$ such that $\mathcal{S}_h^{(t)} \cap \mathcal{C}_1 = \emptyset$ and $\tilde{k}$ is the least positive integer such that $\mathcal{S}_h^{(t)} \cap \mathcal{C}_{\tilde{k}} \neq \emptyset$, we set $\mathrm{col}_h(K_y(t)) = K_y(t)\delta_P^h = \delta_M^{i_{h,t}}$, where the input value $\mathbf{u} = \delta_M^{i_{h,t}}$ maps (at least) one element of $\mathcal{S}_h^{(t)} \cap \mathcal{C}_{\tilde{k}}$ in $\mathcal{C}_{\tilde{k}-1}$, i.e.

$$L \ltimes \delta_M^{i_{h,t}} \ltimes \delta_N^i \in \mathcal{C}_{\tilde{k}-1}, \ \exists \delta_N^i \in \mathcal{S}_j^{(t)} \cap \mathcal{C}_{\tilde{k}},$$

and $\delta_M^{i_{h,t}} \in \mathcal{U}_e$ when $h = 1$.

- Case $\mathcal{S}_h^{(t)} = \emptyset$.

For every $h \in [1,P]$ such that $\mathcal{S}_h^{(t)} = \emptyset$, $\mathrm{col}_h(K_y(t)) = K_y(t)\delta_P^h$ can be arbitrarily selected.

For a sufficiently large t, we have $\mathcal{S}_1(t) = \{\mathbf{x}_e\}$, while $\mathcal{S}_h(t) = \emptyset$ for every $h \in [2,P]$. Therefore $\mathbf{x}_e$ is a fixed point under the output feedback we have constructed, and the BCN reaches $\mathbf{x}_e$ in a finite number of steps, for every $\mathbf{x}(0)$. ■

Remark 2: In the previous proposition the equilibrium point $\mathbf{x}_e$ is reachable from every initial condition, even if we constrain the input values to belong to $\mathcal{U}_e \subsetneq \mathcal{L}_M$ every time we encounter a state δ_N^j belonging to $\mathcal{X}(H\mathbf{x}_e)$.

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