A polynomial matrix approach to the behavioral analysis of nD systems

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Abstract

The paper analysises the internal properties (controllability, observability, detectability and extendability) of finite support multidimensional behaviors and the relations with their polynomial matrix descriptions.

1 Introduction

Behavior theory is the study of the trajectories a dynamical system produces according to its evolution laws. It originated in the analysis of 1D systems, and was developed by J.C.Willems in the last two decades [?, ?].

The aim of this paper is to discuss some aspects of the behavior theory of finite support multidimensional signals. Particular attention has been paid to the support structure of multidimensional signals, and to elementary operations (restriction, extension and concatenation) which have a concrete meaning from the signal processing standpoint. These provide a link between the parity checks description of nD behaviors and the concepts of observability and extendability. We investigate, then, the way behavior trajectories are generated and their supports are related to the corresponding inputs.

The use of nD Laurent polynomial matrices is pervasive all over the paper; no attempt has been made, however, to analyse their algebraic properties. For these we refer the reader to [?, ?]. Similarly, infinite support behaviors are marginally touched on. The analysis of their connections with finite nD behaviors falls within the scope of duality theory, and has been carried on in [?].

2 Main definitions and properties

Let $\mathbb{F}$ be a field and denote by $\mathbf{z}$ the n-tuple $(z_1, z_2, ..., z_n)$, so that $\mathbb{F}[\mathbf{z}]$ and $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ are shorthand notations for the polynomial and the Laurent polynomial (*L-polynomial*) rings in the indeterminates $z_1, ..., z_n$, respectively.

For any sequence $\mathbf{w} = \{\mathbf{w}(\mathbf{h})\}_{\mathbf{h}\in\mathbb{Z}^n}$, taking values in $\mathbb{F}^p$, the *support* of $\mathbf{w}$ is the set of points where $\mathbf{w}$ is nonzero, i.e. $\text{supp}(\mathbf{w}) := \{\mathbf{h} = (h_1, h_2, ..., h_n) \in \mathbb{Z}^n : \mathbf{w}(\mathbf{h}) \neq 0\}$. Also, $\mathbf{w}$ can be represented via a formal power series

$$\sum_{h_i\in\mathbb{Z}} \mathbf{w}(h_1, h_2, ..., h_n)\, z_1^{h_1} z_2^{h_2} \cdots z_n^{h_n} = \sum_{\mathbf{h}\in\mathbb{Z}^n} \mathbf{w}(\mathbf{h})\ \mathbf{z}^{\mathbf{h}},$$

where $\mathbf{h}$ stands for the n-tuple $(h_1, h_2, ..., h_n)$ and $\mathbf{z}^{\mathbf{h}}$ for the term $z_1^{h_1} z_2^{h_2} ... z_n^{h_n}$.
On the other hand, power series can be viewed as representing vectors with entries in $\mathcal{F}_\infty := \mathbb{F}^{\mathbb{Z}^n}$, thus setting a bijective map between nD sequences taking values in $\mathbb{F}^p$ and formal power series with coefficients in $\mathbb{F}^p$. This allows us to identify nD sequences with the associated power series, in particular, finite support nD signals, with L-polynomial vectors, and to denote both of them with the same symbol $\mathbf{w}$. Sometimes, mostly when a power series $\mathbf{w}$ is obtained as a Cauchy product, it will be useful to denote the coefficient of $\mathbf{z}^{\mathbf{h}}$ in $\mathbf{w}$ as $(\mathbf{w}, \mathbf{z}^{\mathbf{h}})$.

Linear operators on the sequence space are represented by appropriate matrices with elements in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$, whose primeness features find a counterpart in terms of properties of the associated operators.

Definition An L-polynomial matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$, $p \geq m$, is
• *unimodular* if $p = m$ and $\det G$ is a unit in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$;
• *right factor prime* (rFP) if in every factorization $G = \bar{G}T$, with $\bar{G} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ and $T \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m\times m}$, T is a unimodular matrix;
• *right minor prime* (rMP) if its maximal order minors have no common factors;
• *right variety prime* (rVP) if the ideal I_G, generated by its maximal order minors, includes (nonzero) L-polynomials in $\mathbb{F}[z_i, z_i^{-1}]$ for every $i = 1, 2, ..., n$;
• *right zero prime* (rZP) if the ideal I_G is the ring $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ itself.

An nD (finite) *behavior* $\mathcal{B}$ *with* p *components* is a set of finite support signals (trajectories) taking values in $\mathbb{F}^p$ and endowed with the properties of *linearity* and *shift-invariance*, namely $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{B} \Rightarrow \alpha\mathbf{w}_1 + \beta\mathbf{w}_2 \in \mathcal{B}$ and $\mathbf{z}^{\mathbf{h}}\mathbf{w}_1$, for all $\alpha, \beta \in \mathbb{F}$ and $\mathbf{h} \in \mathbb{Z}^n$.

As every nD behavior $\mathcal{B}$ can be viewed as an $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$-submodule of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, which is a Noetherian module [?], $\mathcal{B}$ is finitely generated, i.e. there exists a finite set of column vectors $\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_m$ in $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ s.t.

$$\mathcal{B} \equiv \Big\{ \sum_{i=1}^{m} \mathbf{g}_i u_i : u_i \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}] \Big\} =: \mathrm{Im}G. \tag{2.1}$$

The L-polynomial matrix $G := \mathrm{row}\{\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_m\}$ is called *generator matrix* of $\mathcal{B}$. Generator matrices of the same behavior $\mathcal{B}$ have the same rank r over the field of rational functions $\mathbb{F}(\mathbf{z})$. Being an invariant w.r.t. all generator matrices of $\mathcal{B}$, r is called the *rank of* $\mathcal{B}$.

One of the pillars of Willems behavior theory is the notion of (external) controllability, that in a 1D context means that for controllable behaviors the past has no lasting implications about the far future [?]. In the multidimensional case the notions of "past" and "future" are quite elusive. What seems more reasonable, instead, is to investigate to what extent the values a trajectory $\mathbf{w}$ assumes on a subset $\mathcal{S}_1 \subset \mathbb{Z}^n$ influence the values on the subset $\mathcal{S}_2$, disjoint from $\mathcal{S}_1$, and to check if there exists a lower bound on the distance

$$d(\mathcal{S}_1, \mathcal{S}_2) := \min\Big\{ \sum_{i=1}^{n} |h_i - k_i|, \mathbf{h} \in \mathcal{S}_1, \mathbf{k} \in \mathcal{S}_2 \Big\}, \tag{2.2}$$

which guarantees that $\mathbf{w}|\mathcal{S}_2$, the *restriction to* $\mathcal{S}_2$ *of the sequence* $\mathbf{w}$, is independent of $\mathbf{w}|\mathcal{S}_1$. This point of view led to the following definition [?].

($\mathbf{C}_1$) [Controllability] A finite behavior $\mathcal{B}$ is *controllable* if there exists an integer $\delta > 0$ s.t., for any pair of nonempty subsets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathbb{Z}^n$, with $d(\mathcal{S}_1, \mathcal{S}_2) \geq \delta$, and any pair of trajectories $\mathbf{w}_1$ and $\mathbf{w}_2 \in \mathcal{B}$, there exists $\mathbf{v} \in \mathcal{B}$ s.t. $\mathbf{v}|\mathcal{S}_1 = \mathbf{w}_1|\mathcal{S}_1$ and $\mathbf{v}|\mathcal{S}_2 = \mathbf{w}_2|\mathcal{S}_2$.

While definition (C_1) requires to paste together different signals into a new one, the following definition refers to the possibility of finding a legal extension for every portion $\mathbf{w}|\mathcal{S}$ of a behavior trajectory $\mathbf{w}$, by adjusting the sample values in a small area surrounding $\mathcal{S}$. Precisely, once introduced the *ε-extension*, $\varepsilon \geq 0$, of the set $\mathcal{S}$,
$\mathcal{S}^\varepsilon := \{(h_1, h_2, ..., h_n) \in \mathbb{Z}^n : d((h_1, h_2, ..., h_n), \mathcal{S}) \leq \varepsilon\}$,
one can give the following definition.

($\mathbf{C}_2$) [Zero-controllability] A finite behavior $\mathcal{B}$ is *zero-controllable* if there exists an integer $\varepsilon > 0$ s.t., for any nonempty set $\mathcal{S}$ of $\mathbb{Z}^n$ and any $\mathbf{w} \in \mathcal{B}$, there exists $\mathbf{v} \in \mathcal{B}$ satisfying $\mathbf{v}|\mathcal{S} = \mathbf{w}|\mathcal{S}$ and $\mathrm{supp}(\mathbf{v}) \subseteq \mathcal{S}^\varepsilon$.

Properties (C_1) and (C_2) make sense both for finite and infinite support behaviors, and the equivalence (C_1) $\Leftrightarrow$ (C_2) holds for both of them. However, while these conditions are always met [?] by a finite behavior $\mathcal{B}$, and follow from the noetherian module structure of $\mathcal{B}$, for an infinite behavior controllability constitutes an additional constraint w.r.t. linearity and shift invariance [?, ?].

Observability [?] formalizes the possibility of pasting into a legal sequence any pair of trajectories that take the same values on a sufficiently large subset of $\mathbb{Z}^n$. This amounts to say that, however chosen a sequence $\mathbf{w} \in \mathcal{B}$ and a subset $\mathcal{S} \subset \mathbb{Z}^n$, the possible extensions of $\mathbf{w}|\mathcal{S}$ only depend on the values of $\mathbf{w}$ on a boundary region of $\mathcal{S}$.

($\mathbf{O}_1$) [Observability] A finite behavior $\mathcal{B}$ is *observable* if there exists an integer $\delta > 0$ s.t., for any pair of nonempty subsets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathbb{Z}^n$, with $d(\mathcal{S}_1, \mathcal{S}_2) \geq \delta$, and any pair of trajectories $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathcal{B}$, satisfying $\mathbf{w}_1|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathbf{w}_2|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2)$, the trajectory

$$\mathbf{v}(\mathbf{h}) = \begin{cases} \mathbf{w}_1(\mathbf{h}) & \mathbf{h} \in \mathcal{S}_1 \\ \mathbf{w}_1(\mathbf{h}) = \mathbf{w}_2(\mathbf{h}) & \mathbf{h} \in \mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) \\ \mathbf{w}_2(\mathbf{h}) & \mathbf{h} \in \mathcal{S}_2 \end{cases} \tag{2.3}$$

is an element of $\mathcal{B}$.

Observability can be equivalently restated as follows: if the support of a sequence $\mathbf{w} \in \mathcal{B}$ can be partitioned into a pair of disjoint subsets, which are far enough apart, the restrictions of $\mathbf{w}$ to each subset are legal trajectories.

($\mathbf{O}_2$) [Zero-observability] A finite behavior $\mathcal{B}$ is *zero-observable* if there is an integer $\varepsilon > 0$ s.t. for any $\mathbf{w} \in \mathcal{B}$ satisfying $\mathbf{w}|(\mathcal{S}^\varepsilon \setminus \mathcal{S}) = \mathbf{0}$, $\mathcal{S} \subseteq \mathbb{Z}^n$ not empty, the sequence $\mathbf{v}$ which coincides with $\mathbf{w}$ on $\mathcal{S}$ and is zero elsewhere belongs to $\mathcal{B}$.

Proposition 2.1 *Observability and zero observability are equivalent.*

Proof (O_1) $\Rightarrow$ (O_2) Assume that $\mathcal{B}$ fulfills (O_1). Given $\mathcal{S} \subset \mathbb{Z}^n$ and $\mathbf{w} \in \mathcal{B}$ s.t. $\mathbf{w}|(\mathcal{S}^\delta \setminus \mathcal{S}) = \mathbf{0}$, take in ($\mathrm{O}_1$) $\mathbf{w}_1 = \mathbf{w}$, $\mathbf{w}_2 = \mathbf{0}$, $\mathcal{S}_1 = \mathcal{S}$, $\mathcal{S}_2 = \mathcal{C}\mathcal{S}^\delta$. The trajectory $\mathbf{v} \in \mathcal{B}$ satisfying (2.3), coincides with $\mathbf{w}$ on $\mathcal{S}$ and is zero elsewhere. So, (O_2) holds with $\varepsilon = \delta$.

(O_2) $\Rightarrow$ (O_1) Assume that $\mathcal{B}$ fulfills condition (O_2). Given $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{Z}^n$, with $d(\mathcal{S}_1, \mathcal{S}_2) > \varepsilon$, and $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathcal{B}$ satisfying $\mathbf{w}_1|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathbf{w}_2|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2)$, the sequence $\mathbf{w}_1 - \mathbf{w}_2 \in \mathcal{B}$ satisfies $(\mathbf{w}_1 - \mathbf{w}_2)|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathbf{0}$. As a consequence, the signal $\mathbf{w}$, which coincides with $\mathbf{w}_1 - \mathbf{w}_2$ in $\mathcal{S}_1$ and is zero elsewhere, is in $\mathcal{B}$, and $\mathbf{v} := \mathbf{w} + \mathbf{w}_2 \in \mathcal{B}$ fulfills (2.3). So, (O_1) holds for $\delta = \varepsilon$. ∎

3 Trajectories recognition

Underlying the definition of controllability is the idea of driving a portion of trajectory into another one, provided that there is room enough for adjustements. Observability, instead, is related with the "dual" issue of recognizing whether a given sequence $\mathbf{v} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ is an element of $\mathcal{B}$. This problem, that tipically arises in fault detection and convolutional encoding contexts, can be managed by resorting to a linear filter (residual generator or syndrome former) that produces an identically zero output signal when the input is an admissible trajectory of $\mathcal{B}$. This requires to find a set of sequences (parity checks) endowed with the property that their convolution with the elements of $\mathcal{B}$ is zero. So, for a given behavior $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, a (finite) *parity check* is a column vector $\mathbf{s} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ that satisfies $\mathbf{s}^T\mathbf{w} = \mathbf{0}$, for all $\mathbf{w} \in \mathcal{B}$. The set $\mathcal{B}^\perp$ of all finite parity checks of $\mathcal{B}$ is the *orthogonal behavior*, and as a submodule of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, it is generated by the columns of some matrix $H \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times q}$, that is

$$\mathcal{B}^\perp = \{\mathbf{s} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p : \mathbf{s} = H\mathbf{x}, \mathbf{x} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^q\}. \quad (3.1)$$

Condition $\mathbf{s}^T\mathbf{w} = \mathbf{0}$, $\forall\ \mathbf{s} \in \mathcal{B}^\perp$, however, needs not imply $\mathbf{w} \in \mathcal{B}$. In general

$$\mathcal{B}^{\perp\perp} := \{\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p : \mathbf{s}^T\mathbf{w} = \mathbf{0}, \forall\ \mathbf{s} \in \mathcal{B}^\perp\} \quad (3.2)$$

properly includes $\mathcal{B}$, and is the set of all L-polynomial vectors obtained by combining the columns of G over the field of rational functions $\mathbb{F}(\mathbf{z})$. It is clear that $\mathcal{B}$ can be identified via a finite set of parity checks if and only if $\mathcal{B} = \mathcal{B}^{\perp\perp}$ or, equivalently,

$$\mathcal{B} = \ker H^T := \{\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p : H^T\mathbf{w} = \mathbf{0}\}. \quad (3.3)$$

In this setting, observability finds a somewhat more substantial interpretation: if $\mathcal{B} = \ker H^T$, the restriction of a trajectory to a set $\mathcal{S}$ still provides a legal signal whenever the distance between $\mathcal{S}$ and the remaining support of the trajectory exceeds the range of action of the parity check matrix H. Proposition 3.1 below shows that kernel representations correspond to observable behaviors, and makes it clear that observability induces further constraints on $\mathcal{B}$, in addition to linearity and shift invariance.

Proposition 3.1 *A behavior $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ is observable iff there is an integer $h > 0$ and an L-polynomial matrix $H^T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{h\times p}$ s.t. $\mathcal{B} = \ker H^T$.*

The proof depends on a the following technical lemma

Lemma 3.2 [?] *Let $m(\mathbf{z})$ be in $\mathbb{F}[\mathbf{z}]$. For any integer $\rho > 0$ there is $p(\mathbf{z}) \in \mathbb{F}[\mathbf{z}]$ s.t.*

$$m(\mathbf{z})p(\mathbf{z}) \in \mathbb{F}[\mathbf{z}^\rho] := \mathbb{F}[z_1^\rho, ..., z_n^\rho]. \blacksquare \quad (3.4)$$

Proof of Proposition 3.1 Assume that $\mathcal{B} = \mathrm{Im}G$, $G \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times m}$, is an observable behavior, and let $\mathcal{B}^\perp = \mathrm{Im}\ H$, $H \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times q}$, denote the orthogonal behavior introduced in (3.1). We will show that $\mathcal{B} \equiv \ker H^T$. Since $H^TG = \mathbf{0}$, it is clear that $\ker H^T \supseteq \mathcal{B}$. To prove the converse, express $\mathbf{w} \in \ker H^T$ as $\mathbf{w} = G\mathbf{n}/d(\mathbf{z})$, $d \in \mathbb{F}[\mathbf{z}]$, $\mathbf{n} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{m\times 1}$. By Lemma 3.2, for every integer $\rho > 0$ there is a suitable polynomial $p(\mathbf{z})$ s.t. $p(\mathbf{z})d(\mathbf{z}) \in \mathbb{F}[z_1^\rho, \dots, z_n^\rho]$. If property ($O_2$) holds w.r.t. $\varepsilon > 0$, and $r > 0$ is an integer s.t. $\mathrm{supp}(\mathbf{w}) \subseteq B(\mathbf{0}, r)$, we choose $\rho > 2r + \varepsilon$. So, the behavior sequence $p(\mathbf{z})d(\mathbf{z})\mathbf{w} = G\mathbf{n}p(\mathbf{z})$ can be written as $\sum_{i_1,i_2,\dots,i_n} c_{i_1,i_2,\dots,i_n} z_1^{\rho i_1} z_2^{\rho i_2} \cdots z_n^{\rho i_n}\ \mathbf{w}$, and thus is the sum of disjoint shifted copies of $\mathbf{w}$, and the distance between two arbitrary copies exceeds ε. So, by (O_2), each copy of $\mathbf{w}$, and hence $\mathbf{w}$ itself, is in $\mathcal{B}$.

Conversely, let $\mathcal{B} = \ker H^T$, and set $\varepsilon = 2s$, with $s > 0$ an integer s.t. $B(\mathbf{0}, s) \supseteq \mathrm{supp}(H^T)$. If $\mathcal{S}$ is a subset of $\mathbb{Z}^n$ and $\mathbf{w} \in \mathcal{B}$ satisfies $\mathbf{w}|(\mathcal{S}^\varepsilon \setminus \mathcal{S}) = \mathbf{0}$, the sequence $\mathbf{v}$ which coincides with $\mathbf{w}$ on $\mathcal{S}$ and it is zero elsewhere belongs to $\mathcal{B}$. Consequently, $\mathcal{B}$ is zero-observable. ■

As a consequence of Proposition 3.1, observability formalizes the "local nature" of the system laws or, equivalently, the existence of a bound on the size of all windows (in $\mathbb{Z}^n$) we have to look at when deciding whether a signal belongs to $\mathcal{B}$. Letting $\mathcal{B}|\mathcal{S} := \{\mathbf{w}|\mathcal{S} : \mathbf{w} \in \mathcal{B}\}$, the above localization property finds a formal statement as follows:

(O_3) [Local-detectability] A finite behavior $\mathcal{B}$ is *locally-detectable* if there is an integer $\nu > 0$ s.t. every signal $\mathbf{w}$ satisfying $\mathbf{w}|\mathcal{S} \in \mathcal{B}|\mathcal{S}$ for every $\mathcal{S} \subset \mathbb{Z}^n$ with $\mathrm{diam}(\mathcal{S}) \leq \nu$, is in $\mathcal{B}$.

Proposition 3.3 *Local detectability and observability are equivalent.*

Proof Assume that $\mathcal{B}$ satisfies (O_3) for a certain $\nu > 0$. Given $\mathcal{S} \subset \mathbb{Z}^n$ and $\mathbf{w} \in \mathcal{B}$ s.t. $\mathbf{w}|(\mathcal{S}^\nu \setminus \mathcal{S}) = \mathbf{0}$, let $\mathbf{v}$ be the sequence which coincides with $\mathbf{w}$ on $\mathcal{S}^\nu$ and it's zero elsewhere. Consider any window $\mathcal{W}$, with $\mathrm{diam}(\mathcal{W}) \leq \nu$. If $\mathcal{W}$ is included in $\mathcal{S}^\nu$, then $\mathbf{v}|\mathcal{W} = \mathbf{w}|\mathcal{W} \in \mathcal{B}|\mathcal{W}$, otherwise we have $\mathcal{W} \cap \mathcal{S} = \emptyset$, and therefore $\mathbf{v}|\mathcal{W} = \mathbf{0}|\mathcal{W} \in \mathcal{B}|\mathcal{W}$. So, by ($O_3$), $\mathbf{v}$ is a legal trajectory, and (O_2) holds for $\varepsilon = \nu$.

Conversely, assume that $\mathcal{B}$ is observable. By Proposition 3.1, there exists an L-polynomial matrix $H \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times q}$ s.t. $\mathcal{B} = \ker H^T$. Let $\nu > 0$ be an integer s.t. $\mathrm{supp}(H^T) \subseteq B(\mathbf{0}, \nu)$, and suppose that $\mathbf{v}$ is any signal satisfying $\mathbf{v}|\mathcal{S} \in \mathcal{B}|\mathcal{S}$ for every $\mathcal{S} \subset \mathbb{Z}^n$ with $\mathrm{diam}(\mathcal{S}) \leq 2\nu$. If $\bar{\mathcal{S}} := -\mathrm{supp}(H^T)$, the computation of the coefficient of $\mathbf{z}^\mathbf{k}$ in $H^T\mathbf{v}$ involves only samples of $\mathbf{v}$ indexed in $\mathbf{k}+\bar{\mathcal{S}} := \{\mathbf{h} \in \mathbb{Z}^n : \mathbf{h}-\mathbf{k} \in \bar{\mathcal{S}}\} = -\mathrm{supp}(\mathbf{z}^\mathbf{k}H^T)$. On the other hand, since $\mathrm{diam}(\mathbf{k} + \bar{\mathcal{S}}) \leq 2\nu$, there exists $\mathbf{w}_\mathbf{k} \in \mathcal{B}$ which satisfies $\mathbf{v}|(\mathbf{k} + \bar{\mathcal{S}}) = \mathbf{w}_\mathbf{k}|(\mathbf{k} + \bar{\mathcal{S}})$, and this result holds for every $\mathbf{k} \in \mathbb{Z}^n$. So, the coefficient of $\mathbf{z}^\mathbf{k}$ in $H^T\mathbf{v}$ is the same as in $H^T\mathbf{w}_\mathbf{k} \equiv \mathbf{0}$, and hence $\mathbf{v} \in \ker H^T = \mathcal{B}$. ■

A different approach to observability consists in regarding behaviors with p components as elements in the lattice of submodules of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, and analysing whether observable elements enjoy some special ordering properties. Keeping in with the same spirit, we investigate how an observable behavior is affected by certain "extension operations" that merge lattice elements into larger ones. There are essentially two natural ways to perform these extensions: one consists in embedding $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, and therefore each of its submodules, in the rational vector space $\mathbb{F}(\mathbf{z})^p$, the other in considering $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ as a submodule of $\mathcal{F}^p_\infty$, the set of trajectories with p components whose supports possibly extend to the whole space $\mathbb{Z}^n$.

Once a behavior $\mathcal{B}$ with p components is given, in the first case we have to consider the smallest vector subspace of $\mathbb{F}(\mathbf{z})^p$ including $\mathcal{B}$

$$\mathcal{B}_{\rm rat} := \Big\{\sum_{i=1}^{r} \mathbf{w}_i a_i : \mathbf{w}_i \in \mathcal{B},\ a_i \in \mathbb{F}(\mathbf{z}),\ r \in \mathbb{N}\Big\}, \quad (3.5)$$

and restrict our attention to the submodule $\mathcal{B}_{\rm rat} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ of finite support sequences. This in general properly includes $\mathcal{B}$, and hence is a larger element of the lattice. In the other case, we merge $\mathcal{B}$ in

$$\mathcal{B}_{\infty} := \Big\{\sum_{i=1}^{r} \mathbf{w}_i a_i : \mathbf{w}_i \in \mathcal{B},\ a_i \in \mathcal{F}_{\infty},\ r \in \mathbb{N}\Big\}, \quad (3.6)$$

the smallest $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$-submodule of $\mathcal{F}_{\infty}^p$ which includes $\mathcal{B}$ and is closed w.r.t. scalar multiplications by elements of $\mathcal{F}_{\infty}$. Again we confine ourselves to the finite elements $\mathcal{B}_{\infty} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, which include all trajectories of $\mathcal{B}$.

Proposition 3.4 *Let $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ be a behavior of rank r. The following statements are equivalent:*
(1) $\mathcal{B}$ is observable;
(2) $\mathcal{B} \equiv \mathcal{B}_{\infty} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$;
(3) $\mathcal{B} \equiv \mathcal{B}_{\rm rat} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$;
(4) $\mathcal{B}$ is maximal in the set of all submodules of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ of rank r;
(5) $s\mathbf{w} \in \mathcal{B} \ \Rightarrow\ \mathbf{w} \in \mathcal{B}$, for every $\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ and every nonzero $s \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$;
(6) $\mathcal{B} = \mathcal{B}^{\perp\perp}$.

Proof (1) $\Rightarrow$ (2) As $\mathcal{B}$ is observable, there exists $H \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times q}$ s.t. $\mathcal{B} = \ker H^T = \{\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p : H^T\mathbf{w} = \mathbf{0}\}$. If $\mathbf{w} \in \mathcal{B}_{\infty} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, then $\mathbf{w} = \sum_i \mathbf{w}_i a_i,\ a_i \in \mathcal{F}_{\infty}, \mathbf{w}_i \in \mathcal{B}$, and therefore $H^T\mathbf{w} = H^T\Big(\sum_i \mathbf{w}_i a_i\Big) = \sum_i (H^T\mathbf{w}_i)a_i = \mathbf{0}$. Thus $\mathbf{w} \in \ker H^T = \mathcal{B}$, which implies $\mathcal{B} \supseteq \mathcal{B}_{\infty} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$. The reverse inclusion is obvious.

(2) $\Rightarrow$ (3) Follows immediately from $\mathcal{B} \subseteq \mathcal{B}_{\rm rat} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p \subseteq \mathcal{B}_{\infty} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$.

(3) $\Rightarrow$ (4) If $\mathcal{B}' \supseteq \mathcal{B}$ and rank$\mathcal{B}'$ = rank$\mathcal{B}$, it is clear that $\mathcal{B}$ and $\mathcal{B}'$ generate the same $\mathbb{F}(\mathbf{z})$-subspace of $\mathbb{F}(\mathbf{z})^p$ and, consequently, $\mathcal{B}_{\rm rat} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p = \mathcal{B}'_{\rm rat} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$. So, the inclusions $\mathcal{B}_{\rm rat} \cap \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p \supseteq \mathcal{B}' \supseteq \mathcal{B}$ and assumption (3) together imply $\mathcal{B}' = \mathcal{B}$, and hence $\mathcal{B}$ is maximal.

(4) $\Rightarrow$ (5) Suppose $s\mathbf{w} \in \mathcal{B}$, $s \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$. The behavior $\mathcal{B}'$ generated by $\mathcal{B}$ and $\mathbf{w}$ has the same rank of $\mathcal{B}$, and hence, by the maximality assumption, coincides with $\mathcal{B}$.

(5) $\Rightarrow$ (6) As $\mathcal{B}$ and $\mathcal{B}^{\perp\perp}$ have the same rank r and $\mathcal{B}^{\perp\perp} \supseteq \mathcal{B}$, both behaviors generate the same $\mathbb{F}(\mathbf{z})$-subspace of $\mathbb{F}(\mathbf{z})^p$. In particular, $\mathbf{w} \in \mathcal{B}^{\perp\perp}$ implies $\mathbf{w} \in (\mathcal{B}^{\perp\perp})_{\rm rat} = \mathcal{B}_{\rm rat}$. So, there exist $p_i, s_i \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$ and $\mathbf{w}_i \in \mathcal{B}$, s.t. $\mathbf{w} = \sum_{i=1}^r \mathbf{w}_i\ p_i/s_i$, which implies $s\mathbf{w} \in \mathcal{B}$, $s = \ell$.c.m.$\{s_i\}$. By assumption (5), $\mathbf{w}$ is in $\mathcal{B}$.

(6) $\Rightarrow$ (1) Since $\mathcal{B}^{\perp}$ is a submodule of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, there exists a suitable L-polynomial matrix H s.t. $\mathcal{B}^{\perp} = {\rm Im}H$. So $\mathcal{B}^{\perp\perp} = \{\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p : \mathbf{v}^T\mathbf{w} = \mathbf{0}, \forall \mathbf{v} \in {\rm Im}H\} = \ker H^T$. By assumption (6), $\mathcal{B}$ coincides with $\ker H^T$, and hence is observable. ■

When no information on the support is given, a positive outcome of the parity checks on a finite window $\mathcal{S}$, does not guarantee that a behavior sequence can be found interpolating the available data on $\mathcal{S}$. A noteworthy exception is represented by the case when $\mathcal{S}$ is surrounded by a sufficiently large boundary region where the signal is zero. If so, extending the data out of $\mathcal{S}$ via the zero sequence leads to a signal which satisfies the parity checks all over $\mathbb{Z}^n$. Clearly, it would be desirable if the extension into a legal trajectory could be accomplished without any assumption on the data values on the boundary region. A discussion of this problem relies on the definition of what we mean by " satisfying the parity checks" on a set $\mathcal{S} \subset \mathbb{Z}^n$.

Definition Let $\mathcal{B} = \ker H^T$ be an observable behavior. A sequence $\mathbf{v} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ *satisfies the parity checks of $\mathcal{B}$ in* $\mathbf{h} \in \mathbb{Z}^n$ if $\Big(H^T\mathbf{v}, \mathbf{z}^{\mathbf{h}}\Big) = 0, \forall\ \mathbf{i} \in \mathbf{h} + {\rm supp}(H^T)$, where $\mathbf{h} + {\rm supp}(H^T) := \{\mathbf{h}+\mathbf{j} : \mathbf{j} \in {\rm supp}(H^T)\}$. Moreover, if $\mathcal{S}$ is any subset of $\mathbb{Z}^n$, $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}$ on $\mathcal{S}$ if it satisfies them in each point of $\mathcal{S}$, i.e. $\Big(H^T\mathbf{v}, \mathbf{z}^{\mathbf{i}}\Big) = 0, \forall\ \mathbf{i} \in \mathcal{S} + {\rm supp}(H^T)$.

As implied by condition $H^T\mathbf{v} = \mathbf{0}$, knowing the data on a finite window $\mathcal{W}$ allows to check the signal only on those subsets $\mathcal{S}$ of $\mathcal{W}$ satisfying the inclusion $\mathcal{S}^{\nu} \subseteq \mathcal{W}$, $\nu > 0$ being an integer selected according to the size of the support of H^T. Once the parity checks have been successfully performed on a sequence $\mathbf{v}$ in a subset $\mathcal{S}$ which fulfills the above inclusion, the question arises whether there exists a behavior signal that fits on $\mathcal{S}$ the available data. In general, even under the observability assumption, the answer is negative. When the hypotheses on $\mathcal{B}$ are properly strengthened, however, an integer $\varepsilon > 0$ can be found, s.t. a positive check on $\mathcal{S}^{\varepsilon}$ guarantees the existence of some $\mathbf{w} \in \mathcal{B}$ which coincides with $\mathbf{v}$ in $\mathcal{S}$. The formal definition of this property is the following.

(E) [Extendability] An observable behavior $\mathcal{B} = \ker H^T$ is *extendable* if there is an integer $\varepsilon > 0$ s.t., for every subset $\mathcal{S} \subset \mathbb{Z}^n$ and every $\mathbf{v} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, which satisfies on $\mathcal{S}^{\varepsilon}$ the parity checks of $\mathcal{B}$, a trajectory $\mathbf{w} \in \mathcal{B}$ can be found s.t. $\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S}$.

Proposition 3.5 below characterizes extendable behaviors as those described by ZP parity check matrices.

Proposition 3.5 *A finite behavior $\mathcal{B}$ is extendable iff $\mathcal{B} = \ker H^T$, for some left zero-prime (ℓZP) matrix H^T.*

Proof Showing that the left zero-primeness of H^T implies property (E) does not depend on the finiteness of the signal supports. Hence the necessity part of the proof mimics that given in [?] for infinite 2D behaviors and will be omitted. The sufficiency part is based on the following lemma.

Lemma [?] *Let H^T be an element of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{q\times p}$. The map $H^T : \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p \to \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^q\ :\ \mathbf{w} \mapsto H^T\mathbf{w}$ is onto iff H^T is ℓZP.*

Suppose that $\mathcal{B}$ satisfies property (E) for some $\varepsilon > 0$. As $\mathcal{B}$ is observable, it can be described as $\mathcal{B} = \ker H^T$, and it is not restrictive assuming H^T ℓFP. To prove that H^T is ℓZP, we use the lemma and show that $H^T \mathbf{x} = \mathbf{a}$ has an L-polynomial solution for all $\mathbf{a}$ in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^q$. As H^T has full row rank over $\mathbb{F}(\mathbf{z})$, the equation has a rational solution $\mathbf{v} = \mathbf{n}/d$, $\mathbf{n} \in \mathbb{F}[\mathbf{z}]^p$, $d \in \mathbb{F}[\mathbf{z}]$, and hence $H^T \mathbf{n} = d\,\mathbf{a}$.

Set $\rho = 2\delta_1 + 4\varepsilon$, where δ_1 and δ_a are the radii of two balls, with center in the origin, including $\mathrm{supp}(H^T)$ and $\mathrm{supp}(\mathbf{a})$, respectively. By Lemma 3.2, there exists $p(\mathbf{z}) \in \mathbb{F}[\mathbf{z}]$ s.t. pd belongs to $\mathbb{F}[z_1^\rho, ..., z_n^\rho]$ and, by the assumption on ρ, equation $H^T p\ \mathbf{n} = pd\,\mathbf{a}$, implies that the support $\mathcal{Q}$ of $pd\mathbf{a}$ is the disjoint union of finitely many shifted copies $\mathcal{Q}_\mathbf{i}$ of $\mathrm{supp}(\mathbf{a})$, whose mutual distance is lower bounded by $2(\varepsilon + \delta_1)$, namely $\mathcal{Q} = \cup_\mathbf{i} \mathrm{supp}(c_i\ \mathbf{z}^{\rho\mathbf{i}}\mathbf{a}) = \cup_\mathbf{i}\ \mathcal{Q}_\mathbf{i}$.

Consequently, the sequence $p\mathbf{n}$ fulfills the parity checks on the set $C(\mathcal{Q}^{\delta_1})$. So, by the extendability assumption, a sequence $\mathbf{w} \in \mathcal{B}$ can be found, coinciding with $\mathbf{v}$ on $C(\mathcal{Q}^{\delta_1+\varepsilon})$. As the support of the finite sequence $\mathbf{y} := \mathbf{w} - p\mathbf{n}$ is included in $\cup_\mathbf{i} \mathcal{Q}_\mathbf{i}^{\delta_1+\varepsilon}$, $\mathbf{y}$ can be rewritten as $\mathbf{y} = \sum_\mathbf{i} \mathbf{y}_\mathbf{i}$, $\mathbf{y}_\mathbf{i}$ being the restriction of $\mathbf{y}$ to $\mathcal{Q}_\mathbf{i}^{\delta_1+\varepsilon}$. Also, by the choice of ρ we made, all supports of $H^T\mathbf{y}_\mathbf{i}$ are disjoint, and hence $H^T\mathbf{y} = pd\mathbf{a}$ implies $H^T\mathbf{y}_\mathbf{i} = c_i\ \mathbf{z}^{\rho\mathbf{i}}\mathbf{a}$, thus proving that the original equation has an L-polynomial solution. ∎

4 Trajectories generation

Once a behavior $\mathcal{B}$ is represented via a finite set of generators $\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_m$, it is natural to look at $G := [\mathbf{g}_1\ \mathbf{g}_2\ ...\ \mathbf{g}_m]$ as a transfer matrix, and hence to consider $\mathcal{B}$ as the image of an input/output map acting on $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m$. When an i/o description is adopted, it is often imperative to associate trajectories of $\mathcal{B}$ and input sequences bijectively. In data transmission the meaning of this requirement is clear, as input signals represent information messages to be retrieved from the received codewords.

Throughout this section we assume that $\mathcal{B}$ has a full column rank generator matrix G. Under this assumption, G admits (possibly infinitely many) rational left inverses G^{-1}. Each of them, once applied to a trajectory $\mathbf{w} = G\mathbf{u}$, allows to uniquely retrieve the (finite) input sequence $\mathbf{u}$. However, when G^{-1} is applied to a finite support sequence $\mathbf{v} \notin \mathcal{B}$, coming, for instance, from a noisy measurement of $\mathbf{w}$, we may obtain an infinite support sequence, which differs from $\mathbf{u}$ in infinitely many points.

Clearly, when G^{-1} is an L-polynomial matrix this drawback can be avoided. Proposition 4.1 provides equivalent conditions for the existence of an L-polynomial inverse.

Definition Let G be in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ and $\hat{G} = \mathbf{z}^\mathbf{h} G = z_1^{h_1} \cdots z_n^{h_n} G$ in $\mathbb{F}[\mathbf{z}]^{p\times m}$ for some $\mathbf{h} \in \mathbb{N}^n$. If $\mathbb{K}$ denotes the algebraic closure of $\mathbb{F}$, the *L-variety* $\mathcal{V}^L(G)$ *of the maximal order minors of* G is the algebraic set

$$\mathcal{V}^L(G) := \mathcal{V}(\hat{G}) \setminus \big\{(k_1, ..., k_n) \in \mathbb{K}^n,\ \prod_i k_i = 0\big\}, \quad (4.1)$$

where $\mathcal{V}(\hat{G})$ denotes the variety (in $\mathbb{K}$) of the maximal order minors of $\hat{G}$.

The above definition is well-posed, as (4.1) does not depend on the choice of $\hat{G}$.

Proposition 4.1 *Let G be a $p\times m$ L-polynomial matrix. The following statements are equivalent:*
i) G is right zero-prime;
ii) there exists a matrix $P \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{m\times p}$ such that $PG = I_m$;
iii) $\mathcal{V}^L(G)$ is empty.

PROOF *i) ⇒ ii)* Let $m_i(G)$ denotes the i-th maximal order minor of G, $i = 1, 2, ..., \binom{p}{m}$. By the zero-primeness assumption, there exist L-polynomials $h_i \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ s.t. $\sum_i h_i m_i(G) = 1$. If S_i is the $m \times p$ matrix which selects in G the m rows corresponding to $m_i(G)$, from $I_m = \sum_i h_i m_i(G) I_m = \sum_i h_i\big(\mathrm{adj}(S_i G)\big)(S_i G)$, we find that $P := \sum_i h_i\big(\mathrm{adj}(S_i G)\big)S_i$ is a left inverse of G with elements in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$.

ii) ⇒ iii) Let $\mathbf{z}^\mathbf{r} = z_1^{r_1} \cdots z_n^{r_n}$ be a suitable term s.t. $\hat{P} = \mathbf{z}^\mathbf{r} P$ is in $\mathbb{F}[\mathbf{z}]^{m\times p}$. By applying the Binet-Cauchy formula to equation $\hat{P}\hat{G} = z_1^{h_1+r_1} \cdots z_n^{h_n+r_n} I_m$, we get $\sum_i m_i(\hat{P}) m_i(\hat{G}) = z_1^{m(h_1+r_1)} \cdots z_n^{m(h_n+r_n)}$, where $m_i(\hat{P})$ and $m_i(\hat{G})$ are corresponding maximal order minors of $\hat{P}$ and $\hat{G}$, respectively. Then $\mathcal{V}(\hat{G})$ is included in the variety of $z_1^{m(h_1+r_1)} z_2^{m(h_2+r_2)} \cdots z_n^{m(h_n+r_n)}$, which is a subset of $\mathcal{K} := \{(k_1, k_2, ..., k_n) : k_i \in \mathbb{K},\ \prod_i k_i = 0\}$.

iii) ⇒ i) As $\mathcal{K}$ is the variety of $\mathbf{z} = z_1 \cdots z_n$, by assumption *iii)* $\mathcal{V}(\hat{G})$ is included in the variety of $\mathbf{z}$. So, by Hilbert's Nullstellensatz [?], an integer $r > 0$ exists s.t. $z_1^r \cdots z_n^r$ belongs to the ideal generated in $\mathbb{F}[\mathbf{z}]$ by the maximal order minors of $\hat{G}$:

$$z_1^r \cdots z_n^r = \sum_i \bar{h}_i\ m_i(\hat{G}), \qquad \bar{h}_i \in \mathbb{F}[\mathbf{z}]. \qquad (4.2)$$

As each maximal order minor $m_i(\hat{G})$ differs from $m_i(G)$ in a unit of $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$, the zero-primeness of G follows after dividing both members of (4.2) by $z_1^r \cdots z_n^r$. ∎

When the generator matrix G has an L-polynomial inverse, a uniform bound can be found on the support of the input sequences which correspond to the behavior trajectories. Actually, if P is such an inverse, $\mathbf{w} \in \mathcal{B}$ is generated by the input signal $\mathbf{u} = P\mathbf{w}$ whose support cannot exceed "too much" that of $\mathbf{w}$.

(WI) [Wrapping input property] A finite behavior $\mathcal{B}$ has the wrapping input property if there exist a full column rank generator matrix G and a positive integer δ s.t. $\mathbf{w} = G\mathbf{u}$ implies $\mathrm{supp}(\mathbf{u}) \subseteq \big(\mathrm{supp}(\mathbf{w})\big)^\delta$.

Property (WI) does not depend on the particular generator matrix of $\mathcal{B}$, provided that it has full column rank.

Furthermore, when noninjective generator matrices are considered, a particular input can be found, whose support satisfies the above constraint.

Proposition 4.2 *Assume that $\mathcal{B}$ has the (WI) property for some full column rank matrix G and some integer $\delta > 0$. Then for every generator matrix $\bar{G} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times q}$ an integer $\bar{\delta} > 0$ can be found s.t. each trajectory $\mathbf{w} \in \mathcal{B}$ can be expressed as $\mathbf{w} = \bar{G}\bar{\mathbf{u}}$ for some input $\bar{\mathbf{u}}$ with* $\mathrm{supp}(\bar{\mathbf{u}}) \subseteq \big(\mathrm{supp}(\mathbf{w})\big)^{\bar{\delta}}$.

Proof Since G and $\bar{G}$ are generator matrices of the same behavior, there exists a full column rank L-polynomial matrix Q, s.t. $G = \bar{G}Q$. Let τ be the radius of a ball, with center in the origin, including $\mathrm{supp}(Q)$, and consider $\mathbf{w} \in \mathcal{B}$. By property (WI), there is $\mathbf{u}$ s.t. $\mathbf{w} = G\mathbf{u}$ and $\mathrm{supp}(\mathbf{u}) \subseteq \big(\mathrm{supp}(\mathbf{w})\big)^{\delta}$. So, $\bar{\mathbf{u}} := Q\mathbf{u}$ satisfies $\mathbf{w} = G\mathbf{u} = \bar{G}Q\mathbf{u} = G\bar{\mathbf{u}}$, and $\mathrm{supp}(\bar{\mathbf{u}}) = \mathrm{supp}(Q\mathbf{u}) \subseteq \big(\mathrm{supp}(\mathbf{u})\big)^{\tau} \subseteq \big(\mathrm{supp}(\mathbf{w})\big)^{\tau+\delta}$. Consequently, the proposition holds for $\bar{\delta} = \tau + \delta$. ■

Interestingly enough, the zero primeness of G is not only sufficient but also necessary for property (WI). So, free behaviors satisfying property (WI) can be identified extendable behaviors.

Proposition 4.3 [?] *A finite behavior has the (WI) property iff it admits a rZP generator matrix.*

The (WI) property introduces severe constraints on the support of the input sequence which produces a trajectory. So, it is not unexpected that it reflects into the strongest primeness property G can be endowed with, namely zero-primeness. Obviously, weaker requirements on the supports of the generating sequences correspond to weaker primeness properties of G. In particular, minor primeness guarantees that the signal producing a behavior sequence $\mathbf{w}$ exhibits a support which slightly exceeds a parallelepipedal box including $\mathrm{supp}(\mathbf{w})$, whereas variety primeness ensures that each projection of $\mathbf{u}$ and $\mathbf{w}$ onto a coordinate hyperplane gives a pair of signals with the (WI) property.
An useful standpoint is to regard a finite support sequence $\mathbf{w} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ as a vector with entries in certain L-polynomial rings that properly include $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$. Actually, $\mathbf{w}$ can be thought of as an element of $\mathbb{F}(z_i^c)[z_i, z_i^{-1}] := \mathbb{F}(z_1, .., z_{i-1}, z_{i+1}, .., z_n)[z_i, z_i^{-1}]$

$$\mathbf{w} = \sum_{h_i\in\mathbb{Z}} \mathbf{w}_{h_i}(z_i^c) z_i^{h_i},$$

or as an element of $\mathbb{F}(z_i)[z_i^c, (z_i^c)^{-1}] := \mathbb{F}(z_i)[z_1, ..., z_{i-1}, z_{i+1}, ..., z_n, z_1^{-1}, .., z_{i-1}^{-1}, z_{i+1}^{-1}, .., z_n^{-1}]$

$$\mathbf{w} = \sum_{\substack{h_1,..,h_{i-1},\\ h_{i+1},..,h_n\in\mathbb{Z}}} \mathbf{w}_{h_1,..,h_{i-1},h_{i+1},..,h_n}(z_i) z_1^{h_1}..z_{i-1}^{h_{i-1}} z_{i+1}^{h_{i+1}}..z_n^{h_n}.$$

Correspondingly, we introduce the following sets

$$\mathrm{supp}_i(\mathbf{w}) := \{\mathbf{h} \in \mathbb{Z}^n : \mathbf{w}_{h_i}(z_i^c) \neq 0\}$$

$$\mathrm{supp}_{i^c}(\mathbf{w}) := \{\mathbf{h} \in \mathbb{Z}^n : \mathbf{w}_{h_1,..,h_{i-1},h_{i+1},..,h_n}(z_i) \neq 0\}.$$

Lemma 4.4 [?] *Let $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ be a full column rank matrix. Then*
i) G is rMP iff G is right (zero) prime in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$ for every $i = 1, 2, ..., n$;
ii) G is rVP iff G is rZP in $\mathbb{F}(z_i)[z_i^c, (z_i^c)^{-1}]$, for every $i = 1, 2, ..., n$. ■

Proposition 4.5 *Let $\mathcal{B}$ be a finite behavior. Then*
i) $\mathcal{B}$ has a rMP generator matrix iff there exist an integer $\delta > 0$ and a full column rank generator matrix G, s.t. $\mathbf{w} \in \mathcal{B}$ implies $\mathbf{w} = G\mathbf{u}$ with

$$\mathrm{supp}(\mathbf{u}) \subseteq \cap_{i=1}^n \big(\mathrm{supp}_i(\mathbf{w})\big)^{\delta}; \tag{4.3}$$

ii) $\mathcal{B}$ has a rVP generator matrix iff there exist an integer $\delta > 0$ and a full column rank generator matrix G, s.t. $\mathbf{w} \in \mathcal{B}$ implies $\mathbf{w} = G\mathbf{u}$ with

$$\mathrm{supp}(\mathbf{u}) \subseteq \cap_{i=1}^n \big(\mathrm{supp}_{i^c}(\mathbf{w})\big)^{\delta}. \tag{4.4}$$

Proof *i)* It is easy to realize that condition (4.3) is equivalent to the set of conditions $\mathrm{supp}_i(\mathbf{u}) \subseteq \big(\mathrm{supp}_i(\mathbf{w})\big)^{\delta}$, $i = 1, 2, ..., n$. These hold true iff G is a rZP matrix in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$, for every $i = 1, 2, ..., n$, namely G is rMP in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$.
ii) The result is shown along the same lines of *i)*, after replacing $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$ with $\mathbb{F}(z_i)[z_i^c, (z_i^c)^{-1}]$. ■

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