On the Structure of Positive Behaviors

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Abstract
Positive systems in the behavioral approach are introduced as sets of nonnegative trajectories that satisfy a closure condition with respect to linear combinations with nonnegative coefficients. Completeness and finite memory properties are discussed and compared with the analogous properties of linear shift invariant behaviors.

1 Introduction

One of the primary objectives in system analysis is to obtain a mathematical model of the system structure and a formal representation of its interactions with the environment. In classical control theory, for instance, a black box model is adopted, in which system interactions are modeled via input and output variables, and the system structure essentially reduces to a map that associates with each input function the corresponding output. State space models, on the other hand, allow to incorporate internal variables as some kind of running collection of initial conditions, and to provide a complete description of the system at any time t, at least in so far as determining its future evolution. Indeed, knowledge of these conditions at time t together with a specification of future inputs is all is necessary to specify future outputs.

Recently, a powerful theory of dynamical system, that encompasses (but does not reduce to) input/output and state models, has been introduced by J.C.Willems [2],[3]. According to this theory, a *system* is identified with the family of trajectories it produces. In particular, an input-output model, with input vector $\mathbf{u} \in \mathbb{R}^m$ and output vector $\mathbf{y} \in \mathbb{R}^p$, is the set of all time functions $\begin{bmatrix} \mathbf{u}(\cdot) \\ \mathbf{y}(\cdot) \end{bmatrix}$ that occur at the $m+p$ system "terminals". If state variables are also considered, and $\mathbf{x} \in \mathbb{R}^n$ denotes the state vector, system trajectories are all time functions $\begin{bmatrix} \mathbf{u}(\cdot) \\ \mathbf{x}(\cdot) \\ \mathbf{y}(\cdot) \end{bmatrix}$ whose components fulfill the state equations of the model. In its general setting, however, Willems's approach leaves completely aside any a priori assumption on the (driving, measured, controlled...) variables occurring in system trajectories, and the underlying causality structures are brought to light by a detailed investigation of the laws that constrain the time trajectories. The importance of this point of view shouldn't be underestimated, as in many cases (economical and biological systems, multidimensional models, etc.) system variables interact in an extremely intricated way, and their preliminary classification into inputs and outputs as well as the determination of a state structure, could be misleading or artificial.

The abstract framework of Willem's behavior theory is quite simple. Given a time set T (typically $\mathbb{R}$ or $\mathbb{Z}$), and an alphabet W, consisting of all q-tuples of symbols that can occur at the q terminals of the system, a system Σ (in behavioral form) is any triple

$$\Sigma = (T, W, \mathcal{B}) \tag{1}$$

where $\mathcal{B} \subseteq W^T$ denotes the family of all allowable time trajectories, and is called the *behavior* of Σ. As a matter of fact, at the heart of behavior representations (1) is the problem of efficiently assigning the "system laws" a time function has to satisfy in order to be a trajectory in $\mathcal{B}$. These are usually expressed in terms of equations, and the behavior is exactly their solution set.

If we confine ourselves to the discrete time case $T = \mathbb{Z}$, and assume that $\mathcal{B}$ is invariant w.r.to time shifts, i.e. $\mathbf{w} \in \mathcal{B}$ implies that, for all $k \in \mathbb{Z}$, the trajectory $\sigma^k \mathbf{w}$, defined by

$$(\sigma^k \mathbf{w})(t) = \mathbf{w}(t + k), \ \forall t \tag{2}$$

is in $\mathcal{B}$ too, a very neat *linear behavior theory* is already available. However the linearity assumptions alone, i.e.

i) $W = \mathbb{R}^q$;

ii) $\mathcal{B}$ is a subspace of W^T, i.e. given any two system trajectories $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$, $\alpha_1 \mathbf{w}^{(1)} + \alpha_2 \mathbf{w}^{(2)}$ is a system trajectory too.

are too weak for guaranteeing the assignability of the system laws by means of a finite number of equations, and further restrictions on $\mathcal{B}$ have to be imposed. These correspond to the abstract assumption that $\mathcal{B}$ is *closed* in the pointwise convergence topology. An equivalent, more concrete assumption is the *completeness* of the system behavior, i.e. the possibility of deciding whether a signal $\mathbf{w}$ in W^T is a system trajectory just by looking at its restrictions to all finite time intervals. Linearity allows also to show that $\mathcal{B}$ is closed (or complete) if and only if the system trajectories are the solutions of a system of linear difference equations, and therefore constitute the kernel of a polynomial matrix.

The above picture becomes more complex when positive systems in behavior form, (*positive behaviors*, for sake of brevity) are considered.

Positive systems arise quite frequently in the applications, as the variables of many physical, biological and economical processes often involve quantities (such as pressures, densities, concentrations etc.) that may not have meaning unless they are nonnegative. State space theory of linear positive systems does not constitute a trivial extension of standard linear theory, and still represents an active field of research. In particular, noteworthy advances have been recently achieved in connection with the problem of obtaining positive state space realizations of input-output maps in transfer function form [4].

The investigation of positive systems in behavior form is still at a preliminary stage. Consequently, a first scope of this communication is to provide some basic definitions and to discuss to what extent they constitute a convenient setting for discrete time positive behavior theory. In particular, having in mind the highly efficient scheme of linear behaviors, it seems natural to investigate whether some implications between topological and recursive properties carry on to the positive behaviors context. As we shall see by resorting to suitable counterexamples, obtaining a recursive characterization of positive behaviors via a finite set of difference equations and inequalities does not reduce anymore neither to topological closedness, nor to a completeness assumption. Actually, the existence of a finite set of linear difference inequalities describing a positive system does not follow, in general, from a local characterization of its trajectories, i.e. from the possibility of checking whether a signal is a system trajectory by looking at the structure of finite sets of consecutive samples. On the other hand, even when a positive system is "well behaved" w.r.t. convergence processes, so that the limit of a sequence of systems trajectories still represents a system trajectory, there is no guarantee that the system laws can be expressed as a finite system of recursive linear inequalities.

2 Definitions and General Properties

Throughout the paper, $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

Definition 1 *A (discrete, time invariant) system in behavior form (1) is called positive if*

(i) the time set T is $\mathbb{Z}$;

(ii) the system alphabet is $W = \mathbb{R}_+^q$, i.e. only trajectories with nonnegative values are allowed;

(iii) the behavior $\mathcal{B} \subseteq (\mathbb{R}_+^q)^{\mathbb{Z}}$ is shift invariant, i.e. (2) holds or, equivalently, $\sigma\mathcal{B} \subseteq \mathcal{B}$ and $\sigma^{-1}\mathcal{B} \subseteq \mathcal{B}$;

(iv) $\mathcal{B}$ is closed w.r.t. linear combinations with nonnegative coefficients, namely

$$\mathbf{w}^{(1)}, \mathbf{w}^{(2)} \in \mathcal{B}, \quad \alpha_1, \alpha_2 \in \mathbb{R}_+ \Rightarrow \alpha_1 \mathbf{w}^{(1)} + \alpha_2 \mathbf{w}^{(2)} \in \mathcal{B}.$$

Note that condition (iv) implies that $\mathcal{B}$ is a convex cone in $(R^q)^{\mathbb{Z}}$, and corresponds the assumption that $\mathcal{B}$ is closed under (nonnegative) rescaling operations and finite superposition. We confine our investigation to this case, even though different hypotheses could be introduced, as the analytic aspects are simpler, and allow for a detailed comparison with the standard paradigm of linear behaviors.
Accordingly, extremely simple examples of positive systems in behavior form are, for instance,

- the set of all bilateral scalar sequences with nonnegative entries;
- the set of all bilateral sequences with compact support and nonnegative entries;
- the set of all bilateral ℓ_2 sequences with values in $\mathbb{R}_+^q$;
- the set of all bilateral sequences with nonnegative and nondecreasing entries.

Further, less trivial examples will be provided in the sequel.

As already mentioned in the previous section, for any linear time invariant system $(\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$ the following properties are equivalent [2], and are usually assumed as standard constraints on the structure of its trajectory set $\mathcal{B}$:

1 *Topological closure*: $\mathcal{B}$ is closed in the topology of the pointwise convergence, namely if $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots$ are trajectories in $\mathcal{B}$ and the sequence of trajectories $\mathbf{w}^{(i)}$ pointwise converges to a signal $\mathbf{w}^{(\infty)}$ in $(\mathbb{R}^q)^{\mathbb{Z}}$, then $\mathbf{w}^{(\infty)}$ must belong to $\mathcal{B}$.

2 *Completeness*: $\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}}$ is a system trajectory if and only if, for all integers t and nonnegative integers ν, the finite sequences $\mathbf{w}(t), \mathbf{w}(t+1), \dots, \mathbf{w}(t+\nu)$ are restrictions to the interval $[t, t+\nu]$ of trajectories in $\mathcal{B}$:

$$\mathbf{w}|_{[t,t+\nu]} \in \mathcal{B}|_{[t,t+\nu]}, \forall t \in \mathbb{Z}, \forall \nu \in \mathbb{N} \quad \Leftrightarrow \quad \mathbf{w} \in \mathcal{B}$$

3 *Local completeness*: there exists a nonnegative integer ν such that $\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}}$ is a system trajectory if and only if, for all integers t, the finite sequences $\mathbf{w}(t), \mathbf{w}(t+1), \dots, \mathbf{w}(t+\nu)$ are restrictions to the interval $[t, t+\nu]$ of trajectories in $\mathcal{B}$:

$$\mathbf{w}|_{[t,t+\nu]} \in \mathcal{B}|_{[t,t+\nu]}, \ \forall t \in \mathbb{Z} \quad \Leftrightarrow \quad \mathbf{w} \in \mathcal{B}$$

4 *Kernel structure*: there exists a Laurent polynomial matrix

$$H(\sigma,\sigma^{-1}) = [h_{ij}(\sigma,\sigma^{-1})] \in \mathbb{R}[\sigma,\sigma^{-1}]^{r\times q}$$

such that

$$\mathbf{w} \in \mathcal{B} \quad \Leftrightarrow \quad H(\sigma,\sigma^{-1})\mathbf{w} = 0$$

or, equivalently,

$$\mathbf{w} \in \mathcal{B} \quad \Leftrightarrow \quad \mathbf{w} \in \ker H(\sigma,\sigma^{-1})$$

This amounts to say that $\mathbf{w}(\cdot) = [\, w_1(\cdot) \quad w_2(\cdot) \quad \dots \quad w_q(\cdot)\,]^T$ satifies the following system of difference equations

$$\begin{array}{rcl} h_{11}(\sigma,\sigma^{-1})w_1 + h_{12}(\sigma,\sigma^{-1})w_2 + \ldots + h_{1q}(\sigma,\sigma^{-1})w_q &=& 0 \\ h_{21}(\sigma,\sigma^{-1})w_1 + h_{22}(\sigma,\sigma^{-1})w_2 + \ldots + h_{2q}(\sigma,\sigma^{-1})w_q &=& 0 \\ \dots\dots & & \dots \\ h_{r1}(\sigma,\sigma^{-1})w_1 + h_{r2}(\sigma,\sigma^{-1})w_2 + \ldots + h_{rq}(\sigma,\sigma^{-1})w_q &=& 0 \end{array}$$

Property 2 implies that belonging to $\mathcal{B}$ is not a global property of the trajectory, but only depends on the features of its finite restrictions. Property 3, on the other hand, shows that only restrictions to intervals of a suitable length ν have to be considered, and therefore the system “laws” only involve $\nu+1$ consecutive samples of any trajectory (ν-completeness). Finally, property 4 states that only a finite number of running parity checks must be applied for deciding whether some signal is a legal trajectory.
When positive behaviors are considered, the equivalences among properties 1, 2 and 3 do not hold any longer. On the other hand, the kernel characterization of linear behaviors via systems of difference equations naturally corresponds to a characterization of positive behaviors via systems of difference inequalities. However, positive behaviors recursively described by a system of linear difference inequalities constitute a narrower class w.r.t. complete or closed positive behaviors, as we shall see.

Proposition 1 *A locally complete positive behavior is complete. In general, the converse is not true.*

Proof The first part is obvious. For the second statement, consider the positive system $\Sigma = (\mathbb{Z}, \mathbb{R}_+, \mathcal{B})$, where $\mathcal{B}$ is the set of all nonnegative, nondecreasing scalar sequences $\mathbf{w}$ such that

$$w(t+k) \leq 2w(t), \ \forall t \in \mathbb{Z}, \forall k \in \mathbb{N}.$$

It is easy to check that $\mathcal{B}$ satisfies conditions (ii) and (iii) of Definition 1, and therefore Σ is a positive system. Moreover, Σ is complete, as $\mathcal{B}|_{[t,t+k]}$ consists of all nonnegative, nondecreasing $k+1$-tuples such that the last term does not exceed the

first by more than twice, and an infinite sequence is in $\mathcal{B}$ if and only if its restrictions to $[t, t+k]$ fulfill the above property for all $k \in \mathbb{N}$ and $t \in \mathbb{Z}$.
On the other hand, Σ cannot be ν-complete, as the restrictions to $[t, t+\nu]$ of any positive geometric sequence $\mathbf{w}$ that increases with increasing rate $0 < \alpha \leq 2^{1/\nu}$ belongs to $\mathcal{B}|_{[t,t+\nu]}$ for all t, but $\mathbf{w}$ does not belong to $\mathcal{B}$. ■

Proposition 2 *If a positive behavior $\Sigma = (\mathbb{Z}, \mathbb{R}_+^q, \mathcal{B})$ is topologically closed, then it is complete. In general, however, complete (and even locally complete) positive behaviors need not be topologically closed.*

PROOF Suppose that $\mathcal{B}$ is topologically closed, and consider any signal $\mathbf{v} \in (\mathbb{R}_+^q)^{\mathbb{Z}}$ whose restrictions satisfy

$$\mathbf{v}|_{[-k,k]} \in \mathcal{B}|_{[-k,k]}, \ \forall k \in \mathbb{N}.$$

As a consequence, for each $k \in \mathbb{N}$ there exist a pair of signals $\mathbf{p}^{(k)}$ and $\mathbf{f}^{(k)}$ in $(\mathbb{R}_+^q)^{\mathbb{Z}}$, such that

$$\mathbf{w}^{(k)} := \mathbf{p}^{(k)} \ \circ_{-k-1} \ \mathbf{v}|_{[-k,k]} \ \circ_k \ \mathbf{f}^{(k)} \in \mathcal{B}.$$

where $\circ_k$ denotes the concatenation at time k. Clearly, the sequence of trajectories $\mathbf{w}^{(k)}, k = 1, 2, \dots,$ pointwise converges to the signal $\mathbf{v}$. Since the behavior is topologically closed, $\mathbf{v}$ is a trajectory of $\mathcal{B}$.

On the other hand, consider $\Sigma = (\mathbb{Z}, \mathbb{R}_+^2, \mathcal{B})$ and suppose that the unique constraint on its trajectories is that, for all $t \in \mathbb{Z}$,

$$w(t) \in \{(0, +\infty) \times (0, +\infty)\} \cup \{(0, 0)\}.$$

Then Σ is locally complete (actually, it is 0-complete, as only single samples have to be checked for deciding whether a signal is in $\mathcal{B}$).
Consider, however, the sequence of time constant trajectories $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots$ in $\mathcal{B}$, defined by

$$\mathbf{w}^{(k)}(t) = \begin{bmatrix} 1 \\ 1/k \end{bmatrix}, \forall t \in \mathbb{Z}.$$

It pointwise converges to the time constant sequence $\mathbf{w}^{(\infty)}$ given by

$$\mathbf{w}^{(\infty)}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \forall t \in \mathbb{Z},$$

and $\mathbf{w}^{(\infty)}$ is not in $\mathcal{B}$. ■

Remark 1 The behavior considered in the proof of Proposition 1 provides the example of a system that is topologically closed (and hence complete, by Proposition 2) but not locally complete.

To prove that $\mathcal{B}$ is closed in the topology of the pointwise convergence, take a sequence $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots$ of trajectories in $\mathcal{B}$ that pointwise converges to some (nonnegative) signal $\mathbf{w}^{(\infty)} \in (\mathbb{R}_+)^{\mathbb{Z}}$, and consider any integer pair $t, t+k$, with $k > 0$. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $i > N$ implies

$$|w^{(i)}(t+k) - w^{(\infty)}(t+k)| < \epsilon \quad \text{and} \quad |w^{(i)}(t) - w^{(\infty)}(t)| < \epsilon.$$

We therefore have

$$\begin{array}{rl} & w^{(\infty)}(t+k) - w^{(\infty)}(t) \\ = & [w^{(\infty)}(t+k) - w^{(i)}(t+k)] + [w^{(i)}(t+k) - w^{(i)}(t)] + [w^{(i)}(t) - w^{(\infty)}(t)] \\ \geq & [w^{(i)}(t+k) - w^{(i)}(t)] - 2\epsilon \geq -2\epsilon \end{array} \tag{3}$$

and

$$\begin{array}{rl} & w^{(\infty)}(t+k) - 2w^{(\infty)}(t) \\ = & [w^{(\infty)}(t+k) - w^{(i)}(t+k)] + [w^{(i)}(t+k) - 2w^{(i)}(t)] + [2w^{(i)}(t) - 2w^{(\infty)}(t)] \\ \leq & [w^{(i)}(t+k) - 2w^{(i)}(t)] + 3\epsilon \leq 3\epsilon \end{array} \tag{4}$$

which imply that the sequence $\mathbf{w}^{(\infty)}$ is non decreasing, satisfies $w^{(\infty)}(t+k) \leq 2w^{(\infty)}(t)$ for all $k \in \mathbb{N}$ and $t \in \mathbb{Z}$, and therefore is in $\mathcal{B}$.
On the other hand, the behavior considered in the proof of Proposition 2 shows that a locally complete behavior needs not be topologically closed.

Consequently the only implications existing among topological closure, completeness and L-completeness of a positive system are the following:

Topological Closure $\Rightarrow$ Completeness $\Leftarrow$ Local completeness

Concerning the topological structure of positive behaviors, it is worthwhile to notice that the topological closure of $\mathcal{B}$ does not imply, in general, the closure of the set $\mathcal{B}|_{[0,0]} = \{\mathbf{a} \in \mathbb{R}_+^q \, : \mathbf{a} = \mathbf{w}(0), \exists \mathbf{w} \in \mathcal{B}\}$, that consists of all instantaneous values of system trajectories.
Consider, for instance, the following

Example Let

$$\mathcal{B} = \{\mathbf{w} \in (\mathbb{R}_+^2)^{\mathbb{Z}} : w_1(t)w_1(t+1) \geq w_2^2(t), \forall t \in \mathbb{Z}\} \tag{5}$$

Clearly $\mathbf{w} \in \mathcal{B}$ implies $\alpha\mathbf{w} \in \mathcal{B}$ for any nonnegative real α. Moreover, if $\mathbf{v}$ and $\mathbf{w}$ are in $\mathcal{B}$, we have

$$\begin{array}{rcl} v_1(t)v_1(t+1) & \geq & v_2^2(t) \\ w_1(t)w_1(t+1) & \geq & w_2^2(t) \end{array}$$

$$\begin{aligned} v_1(t)w_1(t+1) + w_1(t)v_1(t+1) &\geq 2\Big[v_1(t)w_1(t+1)w_1(t)v_1(t+1)\Big]^{1/2} \\ &\geq 2v_2(t)w_2(t) \end{aligned}$$

which gives

$$[v_1(t) + w_1(t)][v_1(t+1) + w_1(t+1)] \geq [v_2(t) + w_2(t)]^2.$$

Consequently $\mathcal{B}$ is a positive behavior. Moreover, if a sequence $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \ldots$ in $\mathcal{B}$ converges to some signal $\mathbf{w}^{(\infty)}$, the inequalities of definition (5) hold in the limit. So $\mathbf{w}^{(\infty)}$ is in $\mathcal{B}$ and the behavior is topologically closed. However $w_2(0) > 0$ implies $w_1(0) > 0$, $\mathcal{B}|_{[0,0]}$ cannot include the axis $\{(0, w_2), w_2 > 0\}$. As any other point of the positive orthant $\mathbb{R}^2_+$ is the instantaneous value of some trajectory in $\mathcal{B}$, the set $\mathcal{B}|_{[0,0]}$ fails to be closed.

Consider, finally, the possibility of describing a positive system in behavior form via a finite set of linear difference inequalities.

Definition 2 *A behavior $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$ is a* positive behavior with recursive structure *if there exist an integer p and a $(p+q) \times q$ Laurent polynomial matrix*

$$H(\sigma, \sigma^{-1}) = \begin{bmatrix} I_q \\ M(\sigma, \sigma^{-1}) \end{bmatrix}$$

such that $\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}}$ belongs to $\mathcal{B}$ if and only if $H(\sigma, \sigma^{-1})\mathbf{w}$ is a nonnegative sequence in $(\mathbb{R}^p_+)^{\mathbb{Z}}$, and therefore

$$H(\sigma, \sigma^{-1})\mathbf{w} \geq 0. \tag{6}$$

Note that a trajectory $\mathbf{w}(\cdot) = [\, w_1(\cdot) \quad w_2(\cdot) \quad \ldots \quad w_q(\cdot)\,]^T$ satifies (6) if all its samples are nonnegative, i.e.

$$w_i(t) \geq 0, \quad \forall t \in \mathbb{Z}, \quad \forall i \in \{1, 2, \ldots, q\},$$

and it satisfies the following difference inequalities

$$\sum_{i=1}^{q} m_{ji}(\sigma, \sigma^{-1}) w_i \geq 0, \quad \forall j \in \{1, 2, \ldots, p\}.$$

A positive behavior with recursive structure is locally complete, as the condition expressed by (6) only involves a finite window of data on the trajectory $\mathbf{w}$, and closed, as every converging sequence of trajectories $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \ldots$ in $\mathcal{B}$ produces a signal $\mathbf{w}^{(\infty)}$ that still satisfies (6). However a locally complete and topologically closed positive system needs not exhibit a recursive structure. As an example, the behavior $\mathcal{B} := W^{\mathbb{Z}}$, where W is a nonempty closed, pointed, convex circular cone in $\mathbb{R}^3_+$, is 0-complete and topologically closed, but cannot be represented as a positive behavior with recursive structure.

The following figure provides a picture of all implications, as resulting from our previos discussion.
(i) "locally complete" $\Rightarrow$ "complete" : obvious;
(ii) "complete" $\not\Rightarrow$ "locally complete" : Proposition 1
(iii) "topologically closed" $\Rightarrow$ "complete" : Proposition 2
(iv) "locally complete" $\not\Rightarrow$ "topologically closed" : Proposition 2
(v) "topologically closed" $\not\Rightarrow$ "locally complete" : Remark 1
(vi) "recursive structure" $\Rightarrow$ "locally complete and topologically closed" : last paragraph
(vii) "locally complete and topologically closed" $\not\Rightarrow$ "recursive structure" : last paragraph

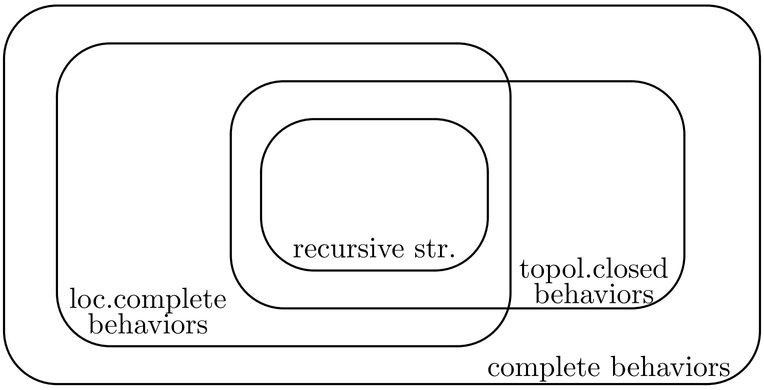

fig. 1

3 Linear embedding of positive behaviors

A natural way to investigate the properties of a positive system $\Sigma = (\mathbb{Z}, \mathbb{R}^q_+, \mathcal{B})$ is to embed $\mathcal{B}$ into the smallest linear behavior $\mathcal{L} \subseteq (\mathbb{R}^q)^{\mathbb{Z}}$ generated by $\mathcal{B}$. It turns our that the elements of $\mathcal{B} - \mathcal{B} := \{\mathbf{v} = \mathbf{w}^{(1)} - \mathbf{w}^{(2)},\ \mathbf{w}^{(1)}, \mathbf{w}^{(2)} \in \mathcal{B}\}$ already constitute a linear space, and therefore provide the linear behavior $\mathcal{L}$ we are looking for.

In general, the local completeness (and, a fortiori, the completeness) of a positive behavior $\mathcal{B}$ does not imply that of $\mathcal{L}$.

Example Consider the positive system $\Sigma = (\mathbb{Z}, \mathbb{R}_+, \mathcal{B})$, where $\mathcal{B}$ is the set of all nonnegative scalar bilateral trajectories $\mathbf{w}$ satisfying

$$w(t-1) - w(t) \leq 0. \tag{7}$$

As $\mathcal{B}$ satisfies the requirement of Definition 2, it has recursive structure, and hence is locally complete. In fact, it is 2-complete, as a two-samples sliding window allows to check whether any signal is a trajectory of $\mathcal{B}$. In order to show that $\Sigma_{\text{lin}} := (\mathbb{Z}, \mathbb{R}, \mathcal{B} - \mathcal{B})$ is non complete, we introduce for any signal $\mathbf{x} \in \mathcal{B} - \mathcal{B}$ its *total variation up to time* k:

$$\mathcal{V}_{(-\infty,k]}(\mathbf{x}) := \sum_{h \leq k} |x(h) - x(h-1)|.$$

Upon writing $\mathbf{x}$ as $\mathbf{x} = \mathbf{a} - \mathbf{b}$, $\mathbf{a}, \mathbf{b} \in \mathcal{B}$, we have

$$\begin{aligned} \mathcal{V}_{(-\infty,k]}(\mathbf{x}) &= \sum_{h \leq k} |x(h) - x(h-1)| \\ &\leq \sum_{h \leq k} \{[a(h) - a(h-1)] + [b(h) - b(h-1)]\} \\ &= a(k) - a(-\infty) + b(k) - b(-\infty) \\ &= [a(k) + b(k)] - [a(-\infty) + b(-\infty)] < \infty, \ \forall k \in \mathbb{Z}. \end{aligned} \tag{8}$$

Consequently all signals in $\mathcal{B} - \mathcal{B}$ have finite total variation up to any time k.
We introduce next the following sequence of signals $\mathbf{x}^{(0)}, \mathbf{x}^{(2)}, \mathbf{x}^{(4)}, \ldots,$, where $\mathbf{x}^{(2\nu)}$ is defined by

$$x^{(2\nu)}(t) = \begin{cases} 1, & \text{if } t = -2\nu, -2\nu+2, \ldots, 2\nu; \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Each $\mathbf{x}^{(2\nu)}$ belongs to $\mathcal{B} - \mathcal{B}$. In fact, the signals $\mathbf{a}^{(2\nu)}$ and $\mathbf{b}^{(2\nu)}$, $\nu = 0, 1, , 2, \ldots$ defined by

$$a^{(2\nu)}(t) = \begin{cases} 1, & \text{if } t < -2\nu; \\ 2, & \text{if } t = -2\nu, -2\nu+1; \\ 3, & \text{if } t = -2\nu+2, -2\nu+3; \\ \ldots & \\ k, & \text{if } t = -2\nu+2k-4, -2\nu+2k-3; \\ \ldots & \end{cases} \tag{10}$$

and

$$b^{(2\nu)}(t) = \begin{cases} 1, & \text{if } t < -2\nu-1; \\ 2, & \text{if } t = -2\nu-1, -2\nu; \\ 3, & \text{if } t = -2\nu+1, -2\nu+2; \\ \ldots & \\ k, & \text{if } t = -2\nu+2k-5, -2\nu+2k-4; \\ \ldots & \end{cases} \tag{11}$$

are trajectories of $\mathcal{B}$, and $\mathbf{x}^{(2\nu)} = \mathbf{a}^{(2\nu)} - \mathbf{b}^{(2\nu)}, \forall \nu \in \mathbb{N}$. However, the limit signal

$$\mathbf{x}^{(\infty)} = \lim_{2\nu \to \infty} \mathbf{x}^{(2\nu)} \tag{12}$$

given by

$$x^{(\infty)}(t) = \begin{cases} 1, & \text{if } t \text{ is even;} \\ 0, & \text{if } t \text{ is odd.} \end{cases} \tag{13}$$

does not belong to $\mathcal{B}-\mathcal{B}$, as $\mathcal{V}_{(-\infty,k]}(\mathbf{x}^{(\infty)}) = +\infty$. Consequently the linear behavior $\mathcal{B}-\mathcal{B}$ fails to be topologically closed.

4 Conclusions

The paper makes a first attempt to introduce behavior systems methods in the analysis of positive systems. Several standard properties have been considered, and important differences with respect to the linear case have been highlighted. Further research [1] will, it is to be hoped, do much to clarify their mathematical structure, and to point out a set of axioms that constitute a reasonable framework for developing a complete theory.

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