ALGEBRAIC REALIZATION THEORY OF BILINEAR DISCRETE TIME INPUT/OUTPUT MAPS

E. Fornasini - G. Marchesini

UPee - 75/03 April 1975

PRELIMINARY VERSION

E. Fornasini, G. Marchesini
Istituto di Elettrotecnica e di Elettronica
Universitá di Padova

Foreword

This is a draft copy of a paper intended to present in unified way new results as well as already published results on the algebraic realization theory of bilinear input/output maps.

New results are in particular those relative to realizable series (Section 2), finite dimensional realization (Sections 3 and 4) and reachability of Nerode state space in bounded time (Section 7): they are summarized in Proposition 7.2. Section 8 includes some original contributions to clarify connections between zero state realization of bilinear i/o maps and systems with bilinear internal structure.

This is a faithful version of the content of some lectures given by the authors in December 74 at the Center for Mathematical System Theory of the University of Florida, Gainesville, where the drafting has also been made.

The authors are indebted with the Center for the hospitality.

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E.Fornasini , G.Marchesini
Dept.of Electrical Engineering,University of Padua,Italy

INTRODUCTION

The purpose of this paper is to discuss and unify previous results obtained by different author's related to abstract realization theory of bilinear input-output maps as well as to present some new results.

Bilinear (and,in general,multilinear) maps constitute a well defined class of special nonlinear maps. The first motivation for this study is the characteristic one for any realization problem;i.e. to understand the internal dynamics of systems which are known only from an external description (input-output map).

A further motivation comes from the empirical aspects of the abstract realization theory of multilinear maps in the sense that it could provide a key result in the design of nonlinear filters and of recursive algorithms.

The realization problem for multilinear i/o maps has been investigated in the past by Balakrishnan[1] and,from an heuristic point of view, by Schetzen[2] and Bush[3].

The Nerode equivalence [4] furnishes the canonical way of introducing the state in the realization problem. This natural approach has been introduced in the realization theory of multilinear discrete-time,stationary i/o maps by Kalman [5] and Arbib [6] .Their results consist substantially in the derivation of decompositions of multilinear i/o maps in terms of linear subsystems and multipliers and were obtained from a module theoretic approach and a machine theoretic approach respectively.

In general the set of Nerode equivalence classes is not directly endowed with the structure of linear space,what is peculiar of the linear case.

This work was partially supported by CNR-GNAS and by Center For Mathematical System Theory University of Florida,under NSF-CNR joint research program.

Nevertheless some structure properties of the state set have been related to those of the i/o map by Fornasini and Marchesini [7].

The same approach introduced in [5] and [6] has been adopted and further developed by Marchesini and Picci [8] for the realization of non-stationary, continuous time, multilinear i/o maps.

The new results obtained in this paper are connected with the introduction of the class of realizable series whose structure is strictly related to the possibility of embedding the Nerode state space in a finite dimensional linear space and of getting updating equations for the state in the discrete time case. The relationships among realizations of bilinear i/o maps and systems described by bilinear equations [9, 10] are discussed at the end.

1. BILINEAR INPUT/OUTPUT MAPS

Let K be a field, $\mathbb{Z}$ the ring of integers and assume

i) $\mathcal{U}_1 = \mathcal{U}_2 = \{ u : u \in K^{\mathbb{Z}}$, supp u bounded on the left

ii) $\mathcal{Y} = \{ y : y \in K^{\mathbb{Z}} \}$

A map F

$$F : \mathcal{U}_1 \times \mathcal{U}_2 \longrightarrow \mathcal{Y}$$

is called a bilinear discrete time, i/o map iff it satisfies the following conditions:

1) bilinearity

a) $F(k u_1, u_2) = k F(u_1, u_2)$, $k \in K$, $u_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2$

b) $F(u_1, h u_2) = h F(u_1, u_2)$, $h \in K$, $u_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2$

c) $F(u_1 + v_1, u_2) = F(u_1, u_2) + F(v_1, u_2)$, $u_1, v_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2$

d) $F(u_1, u_2 + v_2) = F(u_1, u_2) + F(u_1, v_2)$, $u_1 \in \mathcal{U}_1$, $u_2, v_2 \in \mathcal{U}_2$

2) proper causality:

$$F(u_1, u_2)(r) = F(T_r u_1, T_r u_2)(r) \qquad r \in \mathbb{Z}$$

$$T_r\, u_i(t) \triangleq \begin{cases} u_i(t) & , t < r \\ 0 & , t \geq r \end{cases} \qquad i = 1, 2$$

Denoting by σ the backward shift operator in $K^{\mathbb{Z}}$, the bilinear i/o map F is stationary when:

$$\sigma F(u_1, u_2) = F(\sigma u_1, \sigma u_2) \qquad u_1 \in \mathcal{U}_1 \ , u_2 \in \mathcal{U}_2$$

The bilinear i/o map can be represented via a sequence of infinite matrices. In fact F(.,.)(r) can be considered as a bilinear functional acting from $T_r(\mathcal{U}_1) \times T_r(\mathcal{U}_2)$ into K.

Assume then as a basis in $T_r(\mathcal{U}_1)$ and $T_r(\mathcal{U}_2)$ the set $\{\beta_{r-1}, \beta_{r-2}, \dots\}$ where β_i is the sequence $(\delta_{ik})_{k=r-1, r-2 \dots}$ (δ_{ik} Kronecker symbol) and define the infinite matrix $W(r) = (w_{ij}(r))$ by:

$$w_{ij}(r) \triangleq F(\beta_i, \beta_j)(r) \ , \quad i, j = r-1, r-2, \dots$$

It results

$$F(u_1, u_2)(r) = \sum_{-\infty}^{r-1}{}_{ij}\, w_{ij}(r)\, u_1(i)\, u_2(j) \ , \quad u_1 \in \mathcal{U}_1, \ u_2 \in \mathcal{U}_2$$

Obviously if the i/o map is assumed to be stationary $w_{ij}(r)$, $i, j = r-1, r-2 \dots$ satisfy the relations $w_{ij}(r) = w_{i-r, j-r}(0)$, $i, j, r \in \mathbb{Z}$ and

$$(1.1) \qquad F(u_1, u_2)(r) = \sum_{-\infty}^{r-1}{}_{ij}\, w_{i-r, j-r}(0) \cdot u_1(i)\, u_2(j)$$

In the sequel F will be always assumed to be stationary so that the evaluation of the output can be carried out by restricting $\mathcal{U}_1$, $\mathcal{U}_2$ to $\mathcal{U}_1 = \mathcal{U}_2 = \{ u : u \in K^{\mathbb{Z}}$, supp u bounded on the left$\}$, $\mathcal{Y} = \{u : u \in \mathcal{Y}, u : [1, \infty) \to K\}$ and introducing a bilinear map $f : \mathcal{U} \times \mathcal{U} \to \mathcal{Y}$ defined by

$$f(u_1, u_2)(t) = \begin{cases} 0 \ , & t \leq 0 \\ F(u_1, u_2)(t) \ , & t > 0 \end{cases}$$

$$\forall u_1 \in \mathcal{U}_1 \ , \ u_2 \in \mathcal{U}_2$$

Analogously to the linear case the elements of $\mathcal{U}_1$ and $\mathcal{U}_2$ are biuniquely represented by polynomials in $K[z_1^{-1}]$ and $K[z_2^{-1}]$ respectively and the elements of $\mathcal{Y}$ by elements of $zK[[z]]$. The symbols u_1, u_2 and y will be used indifferently to designate elements of $\mathcal{U}_1, \mathcal{U}_2, \mathcal{Y}$ and elements of $K[z_1^{-1}]$, $K[z_2^{-1}]$ and $zK[[z]]$.

By (1.1) the output $\sum_r^{\infty} y(r)z^r$ can be expressed as (*):

$$\sum_{ij}^{\infty} s_{ij} z_1^i z_2^j u_1(z_1^{-1}) u_2(z_2^{-1}) \odot \sum_k^{\infty} (z_1 z_2)^k$$

where $s_{ij} = w_{-i,-j}(0)$. Consequently there exists a biunuque correspondence between bilinear, stationary i/o maps and formal power series belonging to $(z_1 z_2)K[[z_1, z_2]]$

The natural way to attack the realization problem is by introducing the Nerode equivalence [4] :

$$(u_1, u_2) \underset{N}{\sim} (u_1', u_2') \iff$$
$$\iff f(u_1 \circ \sigma_1 \ , \ u_2 \circ \sigma_2) = f(u_1' \circ \sigma_1 \ , \ u_2' \circ \sigma_2) \ , \ \forall \sigma_1 \in K[z_1^{-1}] \ , \ \forall \sigma_2 \in K[z_2^{-1}]$$

where

$$u_1(z_1^{-1}) \circ \sigma_1(z_1^{-1}) = u_1(z_1^{-1}) z_1^{-c} + \sigma_1(z_1^{-1})$$
$$u_2(z_2^{-1}) \circ \sigma_2(z_2^{-1}) = u_2(z_2^{-1}) z_2^{-c} + \sigma_2(z_2^{-1}) \ , \quad c = \max\{\text{lenght}\,\sigma_1, \text{lenght}\,\sigma_2\}$$

and similar expressions for $u_1' \circ \sigma_1$ and $u_2' \circ \sigma_2$.

It will result useful adopting the following notation:

$$f\begin{pmatrix} u_1 & \sigma_1 \\ u_2 & \sigma_2 \end{pmatrix} \triangleq f(u_1 \circ \sigma_1 \ , \ u_2 \circ \sigma_2)$$

(*) The symbol $\odot$ denotes the Hadamard product:

$$\sum a_{ij} z_1^i z_2^j \odot \sum b_{ij} z_1^i z_2^j \triangleq \sum a_{ij} b_{ij} z_1^i z_2^j$$

Let introduce the following equivalences:

$$(1)\quad u_1 \underset{1}{\sim} u_1' \iff f\begin{pmatrix} u_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = f\begin{pmatrix} u_1' & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \forall \sigma_2 \in \mathcal{U}_2$$

$$(2)\quad u_2 \underset{2}{\sim} u_2' \iff f\begin{pmatrix} 0 & \sigma_1 \\ u_2 & 0 \end{pmatrix} = f\begin{pmatrix} 0 & \sigma_1 \\ u_2' & 0 \end{pmatrix} \quad \forall \sigma_1 \in \mathcal{U}_1$$

$$(3)\quad (u_1, u_2) \underset{3}{\sim} (u_1', u_2') \iff f\begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix} = f\begin{pmatrix} u_1' & 0 \\ u_2' & 0 \end{pmatrix}$$

which result to be defined on $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_1 \times \mathcal{U}_2$ respectively.

It is known that [6] :

$$(u_1, u_2) \underset{N}{\sim} (u_1', u_2') \iff u_1 \underset{1}{\sim} u_1' \;,\; u_2 \underset{2}{\sim} u_2' \;,\; (u_1, u_2) \underset{3}{\sim} (u_1', u_2')$$

The Nerode state space is the set of Nerode equivalence classes:

$$X_N \triangleq (\mathcal{U}_1 \times \mathcal{U}_2)/\underset{N}{\sim} = \{[u_1, u_2] : (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2\}$$

In the linear case the Nerode state space X_N can be endowed with the linear vector space structure. If then the linear i/o map is represented by a formal power series $s \in \mathbb{Z}K[[z]]$, the following facts are equivalent [11] :

(1.2).
i) $\dim X_N < \infty$
ii) $\mathrm{card}\,([0]) > 1$
iii) $s \in \mathbb{Z}K^{rat}[[z]]$
iv) rank of Hankel matrix finite

In the bilinear case the Nerode state space does not have in general a K-module structure and consequently there is no way of extending some of the previous statements. In the following sections some similar facts will be introduced for the bilinear case and mutual implications will be proved with

special reference to the structure of the series s associated with the bilinear map f. A precise statement of this will be given in Section 4.

2. REALIZABLE SERIES

As it is known (and also apparent from (12)), in the realization of i/o linear maps the ring of rational power series has fundamental importance. A similar rule will be shown to be played in the bilinear case by the ring of realizable series $K^{real}[[z_1,z_2]]$.

Definition. A series $s \in K[[z_1,z_2]]$ is said realizable if $s \in K^{rat}[[z_1]] \otimes K^{rat}[[z_2]] \otimes K^{rat}[[z_1 z_2]] \triangleq K^{real}[[z_1, z_2]]$

The following inclusions are obvious:

$$K^{rat}[[z_1,z_2]] \supset K^{real}[[z_1,z_2]] \supset K^{rec}[[z_1,z_2]]$$

where $K^{rat}[[z_1,z_2]]$ is the ring of rational series and $K^{rec}[[z_1,z_2]]$ is the ring of recognizable series [10].

With any $s \in K[[z_1,z_2]]$, $s = \sum_{0}^{\infty}{}_{ij}\, s_{ij}\, z_1^i z_2^j$ we can associate the following three families of series in one variables:

$$r_i(z) = \sum_{0}^{\infty}{}_k\, s_{i,i+k}\, z^k \qquad ,i = 1,2,\ldots\ldots$$

$$c_j(z) = \sum_{0}^{\infty}{}_h\, s_{j+h,j}\, z^h, \qquad j=1,2,\ldots\ldots$$

$$d_{ij}(z) = \sum_{0}^{\infty}{}_h\, s_{i+h,j+h}\, z^h, \qquad i,j = 0,1,\ldots\ldots\ldots$$

The series r_i, c_j and d_{ij} will be called "row series", "column series" and "diagonal series" respectively.

Lemma 2.1. Let $s \in K^{rec}[[z_1,z_2]]$. Then the series d_{ij} associated with s are formal power series of rational functions having the same denominator.

proof. Let $s \in K^{rec}[[z_1,z_2]]$ and $Z^*/\mathcal{E}$ the commutative monoid ge-

nerated by $\{z_1, z_2\}$. Then there exist an integer N, $\lambda \in K^{1\times N}$, $\gamma \in K^{N\times 1}$ and a monoid morphism $\mu : Z^*/\mathcal{C} \longrightarrow K^{N\times N}$ such that [10] :

$$(s, z_1^h z_2^k) = \lambda\, \mu^h(z_1)\, \mu^k(z_2)\, \gamma$$

The series d_{ij} can then be written in the form

$$\begin{aligned} d_{ij} &= \sum_0^\infty {}_k\, s_{i+k,\, j+k}\, (z_1 z_2)^k = \\ &= \sum_0^\infty {}_k\, \lambda\, \mu^{i+k}(z_1)\, \mu^{j+k}(z_2)\, \gamma\, (z_1 z_2)^k = \\ &= \sum_0^\infty {}_k\, (\lambda\, \mu^i(z_1))\, \mu^k(z_1 z_2)\, (\mu^j(z_2)\gamma)(z_1 z_2)^k = \\ &= (\lambda\, \mu^i(z_1))\, (I - z_1 z_2\, \mu(z_1 z_2))^{-1} (\mu^j(z_2)\gamma) = \\ &= (z_1 z_2)^{-1} (\lambda\, \mu^i(z_1))\, ((z_1 z_2)^{-1} I - \mu(z_1 z_2))^{-1} (\mu^j(z_2)\gamma) \end{aligned}$$

from which it is clear that each of the series d_{ij} is rational. A common denominator for all d_{ij} is given by $\det((z_1 z_2)^{-1} I - \mu(z_1 z_2))$.

<u>Proposition 2.1</u>. Let $s \in (z_1 z_2) K[[z_1, z_2]]$. Then $s \in (z_1 z_2)\, K^{real}[[z_1, z_2]]$ iff $r_i \in K^{rat}[[z]]$, $c_j \in K^{rat}[[z]]$, $d_{ij} \in K^{rat}[[z]]$ and there exist $\omega_1, \omega_2, p \in K[z^{-1}]$ such that i) $r_i = N_i(z^{-1})/\omega_1(z^{-1})$, ii) $c_j = M_j(z^{-1})/\omega_2(z^{-1})$, iii) $d_{ij} = Q_{ij}(z^{-1})/p(z^{-1})$.

proof. Let i), ii) and iii) hold and assume $\omega_1(z_1^{-1}) = \sum_0^m {}_r\, a_r z_1^{-r}$, $\omega_2(z_2^{-1}) = \sum_0^n {}_s\, b_s z_2^{-s}$, $p((z_1 z_2)^{-1}) = \sum_0^q {}_t\, c_t\, (z_1 z_2)^{-t}$.

Consider now the product

$$(2.1)\ \sum_{1}^{\infty}{}_{i,j}\, s_{ij} z_1^i z_2^j \sum_{0}^{m}{}_{r}\, a_r z_1^{-r} \sum_{0}^{n}{}_{s}\, b_s z_2^{-s} \sum_{0}^{q}{}_{t}\, c_t z_1^{-t} z_2^{-t} =$$

$$= \sum_{i,j,r,s,t} s_{ij}\, a_r b_s c_t\, z_1^{i-r-t} z_2^{j-s-t} =$$

$$= \sum_{h,k,r,s,t} s_{h+r+t,\, k+s+t}\ a_r b_s c_t\, z_1^h z_2^k$$

It is direct to verify that :

by i)

$$\sum_{r,s} a_r b_s \sum_{0}^{q}{}_{t}\, s_{h+r+t,\, k+s+t}\ c_t = 0$$

$$h+r>0,\ k+s>0$$

by ii)

$$\sum_{r,t} a_r c_t \sum_{0}^{n}{}_{s}\ b_s s_{h+r+t,\, k+s+t} = 0$$

$$k+s > h+r$$

by iii)

$$\sum_{s,t} b_s c_t \sum a_r s_{h+r+t,\, k+s+t} = 0$$

$$h+r > k+s$$

so that in (2.1) the coefficients of the monomials in $z_1^h z_2^k$ are zero for a) $h>0$, $k>0$,b) $k-h>m$, c) $k-h<-n$. These coefficients are also zero when d) $h<-m-q$, e) $k<-n-q$. This implies that in (2.1) there is only a finite number of nonzero coefficients. Consequently there exists an integer ν such that

$$s\, \omega_1(z_1^{-1})\, \omega_2(z_2^{-1})\, p(z_1 z_2^{-1})\, (z_1 z_2)^{-\nu} \in K\left[z_1^{-1}, z_2^{-1}\right]$$

and

$$s = N(z_1^{-1}, z_2^{-1}) \Big/ \omega_1(z_1^{-1})\, \omega_2(z_2^{-1})\, p((z_1 z_2)^{-1})\, (z_1 z_2)^{-\nu}$$

so that $s \in (z_1 z_2) K^{real} [[z_1, z_2]]$.

Conversely assume $s = N(z_1^{-1}, z_2^{-1}) / \omega_1(z_1^{-1})\, \omega_2(z_2^{-1}) p((z_1 z_2)^{-1})$ and $\deg_{z_1^{-1}}(N) < \deg_{z_1^{-1}}(\omega_1 p)$, $\deg_{z_2^{-1}}(N) < \deg_{z_2^{-1}}(\omega_2 p)$. Let $q = \deg(p)$ and consider the rational function $N/\omega_1 \omega_2 (z_1 z_2)^{-q}$ whose series expansion σ is recognizable.

By lemma 2.1 the series d_{ij} associated with σ are series expansions of rational functions with commom denominator g. So that the product $p((z_1 z_2)^{-1}) g((z_1 z_2)^{-1}) z_1^{-q} z_2^{-q} N / p \omega_1 \omega_2 z_1^{-q} z_2^{-q}$ contains monomials with non positive powers. This gives that pg is the common denominator of the series d_{ij} associated with the series s .

Let ν be an integer such that in $z_1^{-\nu} N(z_1^{-1}, z_2^{-1})$ the coefficients a_{ij} of $z_1^{-i} z_2^{-j}$ are zero if $i \leq j$. Then, in the series expansion $\sum b_{ij} z_1^{i} z_2^{j}$ of $\frac{z_1^{-\nu} N}{p \omega_2} = z_1^{-\nu} \omega_1 \frac{N}{p \omega_1 \omega_2}$, $b_{ij} = 0$, $i \leq j$. This implies that $z_1^{-\nu} \omega_1$ is the common denominator of r_i , $i = 1, 2, \ldots$.

In an analogous way it results that there exist an integer $\mu \geq 0$ such that $z_2^{-\mu} \omega_2$ is the common denominator of c_j , $j = 1, 2, \ldots$.

The following results constitute a first insight in clarifying the connection between the structure of a realizable series and the structure of the zero state defined as equivalence classes under Nerode equivalence.

By pursuing the idea of evidentiating the connection existing between the structure of the series operator and the properties of the state set, some results presented in [7] will be reformulated.

<u>Lemma 2.2.</u> Let $s \in (z_1 z_2) K[[z_1, z_2]]$, $u_1 \in K[z_1^{-1}]$, $u_1 \neq 0$. Then $(u_1, 0) \in [0,0]$ iff the row-series r_i associated with s satisfy $r_i = N_i(z_1^{-1}) / u_1(z_1^{-1})$, $i = 1, 2, \ldots$

<u>Lemma 2.3.</u> Let $s \in (z_1 z_2) K[[z_1, z_2]]$, $u_2 \in K[z_2^{-1}]$, $u_2 \neq 0$. Then $(0, u_2) \in [0,0]$ iff the column-series c_j associated with s satisfy $c_j = M_j(z_2^{-1}) / u_2(z_2^{-1})$, $j = 1, 2, \ldots$

<u>Proposition 2.2.</u> Let $s \in (z_1 z_2) K[[z_1, z_2]]$. Then

$$(2.2) \qquad s = \frac{N(z_1^{-1}, z_2^{-1})}{p_1(z_1^{-1}) p_2(z_2^{-1})} + \frac{(z_1 z_2)^{-1}}{p_1(z_1^{-1}) p_2(z_2^{-1})} \sum_{\substack{h,k \\ |h-k| < m}} s'_{h,k} z_1^h z_2^k$$

with $p_1 \in K[z_1^{-1}]$, $p_2 \in K[z_2^{-1}]$, iff there exist $u_1 \in K[z_1^{-1}]$, $u_2 \in K[z_2^{-1}]$ such that

i) $u_1 \in (p_1)$, $u_2 \in (p_2)$, $u_1, u_2 \neq 0$

ii) $(u_1, u_2) \in [0,0]$

<u>Proposition 2.3.</u> Let $s \in (z_1 z_2) K[[z_1, z_2]]$. Then $s \in (z_1 z_2) K^{real}[[z_1, z_2]]$ iff

i) there exist $u_1 \in K[z_1^{-1}]$, $u_2 \in K[z_2^{-1}]$, $u_1, u_2 \neq 0$ such that $(u_1, u_2) \in [0,0]$

ii) $s \in (z_1 z_2) K^{rat}[[z_1, z_2]]$

<u>Remark</u> The results presented in lemma 2.2 and lemma 2.3 involve the cardinality of the zero state $[0,0]$ in the sense that $\mathrm{card}\{(u_1, u_2) : (u_1, u_2) \in [0,0]\} > 1$ iff either the row series r_i satisfy $r_i = N_i(z_1^{-1}) / u_1(z_1^{-1})$ or the column series s_j satisfy $s_j = M_j(z_2^{-1}) / u_2(z_2^{-1})$.

Proposition 2.2 and Proposition 2.3 deal with the existence of a pair u_1, u_2, both different from zero, belonging to $[0,0]$. This does not involve $\mathrm{card}\{(u_1, u_2) : (u_1, u_2) \in [0,0]\}$ but $\mathrm{card}\{u_1 u_2 : (u_1, u_2) \in [0,0]\}$ and it is the condition $\{\mathrm{card}\, u_1 u_2 : (u_1, u_2) \in [0,0]\} > 1$ wich appears in Proposition 2.2 and Proposition 2.3 .

3. EMBEDDING NERODE STATE SPACE IN A LINEAR SPACE

The equivalences $\underset{1}{\sim}$ and $\underset{2}{\sim}$ lead to define two linear spaces $X_1 = \mathcal{U}_1/\underset{1}{\sim}$ and $X_2 = \mathcal{U}_2/\underset{2}{\sim}$. In general the set $\mathcal{U}_1 \times \mathcal{U}_2/\underset{3}{\sim}$ cannot be endowed with the structure of linear space. As known [5], it is possible to deal with a linear structure by embedding $\mathcal{U}_1 \times \mathcal{U}_2$ in the tensor space $\mathcal{U}_1 \otimes \mathcal{U}_2 \simeq K[z_1^{-1}, z_2^{-1}]$.

The maps f and $f_\otimes$ can then be factorized as in diagram:

$$\begin{tikzcd}
\mathcal{U}_1 \times \mathcal{U}_2 \arrow[r, "f"] \arrow[d, "\otimes"'] & \mathcal{Y} \\
\mathcal{U}_1 \otimes \mathcal{U}_2 \arrow[ur, "f_\otimes"] \arrow[d, "\nu_3"', "\text{onto}"] & \\
\mathcal{U}_1 \otimes \mathcal{U}_2 / \mathrm{Ker}\, f_\otimes \arrow[uur, "\bar{f}_\otimes", "1:1"'] &
\end{tikzcd}$$

The map $f_\otimes$ induces an equivalence relation on $\mathcal{U}_1 \otimes \mathcal{U}_2$ and $(u_1, u_2) \underset{3}{\sim} (u_1', u_2')$ iff $(u_1 \otimes u_2) \equiv (u_1' \otimes u_2')$ mod $f_\otimes$. Consequently the equivalence classes under $\underset{3}{\sim}$ can be naturally embedded in the linear space $X_3 = \mathcal{U}_1 \otimes \mathcal{U}_2/\ker f_\otimes$,and X_N in $X_1 \oplus X_2 \oplus X_3$.

The aim now is to investigate how the dimensions of X_1 , X_2 , X_3 are connected with the structure of the series s. If the structure of this series fulfills the requirements for $X_1 \oplus X_2 \oplus X_3$ having finite dimension, then it will be possible to give explicitely a zero state realization of the i/o map in terms of difference equations.

Finite dimensionality of X_3 and consequences

From the above diagram dim Im $f_\otimes = n < \infty$ iff dim $X_3 = n < \infty$. Depending on the representation adopted for elements of $\mathcal{U}_1 \times \mathcal{U}_2$ and of $\mathcal{Y}$ it is possible to give different expressions for the action of $f_\otimes$.

In fact, if $\mathcal{U}_1 \otimes \mathcal{U}_2$ is considered as $K[z_1^{-1}, z_2^{-1}]$ and $\varphi(z_1^{-1}, z_2^{-1}) =$
$= \sum_{\tau,\sigma} \varphi_{\tau,\sigma} z_1^{-\tau} z_2^{-\sigma} \in K[z_1^{-1}, z_2^{-1}]$,then

$$\sum_{1}^{\infty}{}_t\, y(t)(z_1 z_2)^t = f_{\otimes}\left(\varphi(z_1^{-1}, z_2^{-1})\right) = \varphi(z_1^{-1}, z_2^{-1}) \sum_{ij} s_{ij} z_1^i z_2^j \odot \sum_{h=1}^{\infty} (z_1 z_2)^h$$

from which

$$y(t) = \sum_{0}^{-\infty}{}_{\tau,\sigma}\, \varphi_{\tau\sigma}\, s_{t-\tau,\, t-\sigma}$$

If we characterize the elements of $\mathcal{U}_1 \otimes \mathcal{U}_2$ and of $\mathcal{Y}$ by suitably ordered sequences, the map $f_{\otimes}$ can be also represented by an infinite matrix

$$\tilde{\mathcal{H}} = \begin{bmatrix} s_{11} & s_{12} & s_{21} & s_{22} & s_{13} & s_{31} & s_{23} & s_{32} & s_{33} & \cdots \\ s_{22} & s_{23} & s_{32} & s_{33} & s_{24} & s_{42} & s_{34} & s_{43} & s_{44} & \cdots \\ s_{33} & s_{34} & s_{43} & s_{44} & s_{35} & s_{53} & s_{45} & s_{54} & s_{55} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

so that

$$\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{21} & s_{22} & s_{13} & \cdots \\ s_{22} & s_{23} & s_{32} & s_{33} & s_{24} & \cdots \\ s_{33} & s_{34} & s_{43} & s_{44} & s_{35} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \varphi_{0,0} \\ \varphi_{0,-1} \\ \varphi_{-1,0} \\ \varphi_{-1,-1} \\ \vdots \end{bmatrix}$$

$$\bar{y} = \tilde{\mathcal{H}}\, \bar{\varphi}$$

The matrix $\tilde{\mathcal{H}}$ results to be a submatrix of the Hankel matrix $\mathcal{H}$ associated with the series s :

$$\mathcal{H} = \begin{bmatrix} s_{11} & s_{12} & s_{21} & s_{22} & s_{13} & s_{31} & s_{23} & s_{32} & s_{33} & \cdots \\ s_{12} & s_{13} & s_{22} & s_{23} & s_{14} & s_{32} & \cdots & & & \\ s_{21} & s_{22} & s_{31} & s_{32} & s_{23} & s_{33} & \cdots & & & \\ s_{22} & s_{23} & s_{32} & s_{33} & s_{24} & s_{42} & \cdots & & & \\ \cdots & \cdots & \cdots & & & & & & & \end{bmatrix}$$

It is obvious that dim $X_3 = n < \infty$ is equivalent to rank $\tilde{\mathcal{H}} = n < \infty$. If rank $\tilde{\mathcal{H}} = n$, then there exist n linearly independent rows of $\tilde{\mathcal{H}}$ indexed by $t_1, t_2, \ldots\ldots t_n$ so that

$$s_{t-\tau,t-\sigma} = \sum_1^n{}_i \; \alpha_i(t)\, s_{t_i-\tau,t_i-\sigma}$$

The n linearly independent rows in $\tilde{\mathcal{H}}$ can always be assumed to be the first n. In fact if in $\tilde{\mathcal{H}}$ the n-th row is a linear combination of the n-1 previous rows then any row of $\tilde{\mathcal{H}}$ is a linear combination of the first n-1 rows.

This is easily proved noting that if

$$s_{n+r,n+t} = \sum_1^{n-1}{}_i \; R_i\, s_{n+r-i,n+t-i}$$

then because of the structure of $\tilde{\mathcal{H}}$

$$s_{n+1+r,n+1+t} = \sum_1^{n-1}{}_i \; R_i\, s_{n+1+r-i,n+1+t-i} =$$
$$= \sum_2^{n-1}{}_i \; R_i\, s_{n+1+r-i,n+1+t-i} + R_1\, s_{n+r,n+t} =$$
$$= \sum_1^{n-2}{}_j \; R_{j+1}\, s_{n+r-j,n+t-j} + R_1 \sum_1^{n-1}{}_i \; s_{n+r-i,n+t-i}$$

As a consequence if in the expression $s_{t-\tau,t-\sigma} = \sum_1^n{}_i \; \alpha_i(t)\, s_{t_i-\tau,t_i-\sigma}$ $t_i = i$, $i = 1,2,\ldots n$ is assumed, then it is easily proved that $\alpha_i(j) = \delta_{ij}$, $i,j = 1,2,\ldots,n$.

<u>Remark</u> This property of $\tilde{\mathcal{H}}$ induces a fact analogous to the linear case, i.e. if an input φ gives an output which is zero for $t = 1,2,\ldots,n$, this output is zero also for $t > n$.

In the sequel it will be convenient to denote $s_{t_i-\tau,t_i-\sigma}$ by $\beta_i(\tau,\sigma)$. Using this notation the output y(t) is expressed as

$$(3.1) \qquad y(t) = \sum_0^{-\infty}{}_{\tau,\sigma} \; \varphi_{\tau,\sigma} \sum_1^n{}_i \; \alpha_i(t)\, \beta_i(\tau,\sigma) =$$
$$= \sum_1^n{}_i \; \alpha_i(t) \sum_0^{-\infty}{}_{\tau,\sigma} \; \varphi_{\tau,\sigma}\, \beta_i(\tau,\sigma) = \sum_{i=1}^n \alpha_i(t)\, (\varphi \times \beta_i)$$

where

$$\varphi \times \beta_i \triangleq \sum_0^{-\infty}{}_{\tau\sigma} \; \varphi_{\tau,\sigma}\, \beta_i(\tau,\sigma)$$

Consequently Im $f_{\otimes}$ = span($\sum_t \alpha_i(t)\, z^t$, i = 1,2,...,n)

It is possible to give a procedure for constructing inputs φ_i such that $\varphi_i \times \beta_j = \delta_{ij}$. The outputs corresponding to φ_i are of course the series α_i. In fact consider the $n \times \infty$ matrix whose rows are the sequences β_i, $i = 1,2,\ldots n$

$$\begin{bmatrix} \beta_1(0,0) & \beta_1(0,-1) & \beta_1(-1,0) & \beta_1(-1,-1) & \cdots \\ \beta_2(0,0) & \beta_2(0,-1) & \beta_2(-1,0) & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ \beta_n(0,0) & \beta_n(0,-1) & \beta_n(-1,0) & \cdots & \end{bmatrix}$$

This matrix has rank n, so that there exists a full rank $n \times n$ submatrix M

$$M = \begin{bmatrix} \beta_1(\tau_1,\sigma_1) & \beta_1(\tau_2,\sigma_2) & \cdots & \beta_1(\tau_n,\sigma_n) \\ \cdots & \cdots & \cdots & \cdots \\ \beta_n(\tau_1,\sigma_1) & \beta_n(\tau_2,\sigma_2) & \cdots & \beta_n(\tau_n,\sigma_n) \end{bmatrix}$$

and the inputs $\varphi_i = \sum_1^n{}_k \bar{\varphi}_i(\tau_k,\sigma_k)\, z_1^{\tau_k} z_2^{\sigma_k}$, $i = 1,2,\ldots,n$, where

$$\begin{bmatrix} \bar{\varphi}_1(\tau_1,\sigma_1) & \bar{\varphi}_1(\tau_2,\sigma_2) & \cdots & \bar{\varphi}_1(\tau_n,\sigma_n) \\ \cdots & \cdots & \cdots & \cdots \\ \bar{\varphi}_n(\tau_1,\sigma_1) & \bar{\varphi}_n(\tau_2,\sigma_2) & \cdots & \bar{\varphi}_n(\tau_n,\sigma_n) \end{bmatrix} \triangleq (M^{-1})^T$$

satisfy the conditions

(3.2) $\quad \varphi_i \times \beta_j = \delta_{ij}$

From (3.1) the series $\sum_1^\infty{}_t \alpha_i(t) z^t$, $i = 1,2,\ldots,n$ are the outputs when $\varphi = \varphi_i$, $i = 1,2,\ldots,n$.

<u>Proposition 3.1.</u> Let $\{\varphi_i\}$ satisfy (3.2). Then $\mathrm{span}(\nu_3(\varphi_1) \cdots \nu_3(\varphi_n)) = X_3$

proof. Assume

$$\sum_1^n{}_i\, \rho_i\, \nu_3(\varphi_i) = 0 \quad , \quad \rho_i \in K \ , \quad i = 1,2,\ldots,n .$$

Then

$$0 = \bar{f}_\otimes \sum_i \rho_i\, \nu_3(\varphi_i) = \sum_i \rho_i\, f_\otimes(\varphi_i) = \sum_i \rho_i \sum_t \alpha_i(t)\, z^t$$

This implies $\rho_i = 0$, $i = 1,2,\ldots,n$

<u>Proposition 3.2.</u> Assume $\nu_3(\varphi_1) \cdots \nu_3(\varphi_n)$ as a basis in X_3. Then for any $\varphi \in \mathcal{U}_1 \otimes \mathcal{U}_2$, $\nu_3(\varphi) = \sum_1^n{}_i (\varphi \times \beta_i)\, \nu_3(\varphi_i)$.

proof. Let $\varphi \in \mathcal{U}_1 \otimes \mathcal{U}_2$ and $\nu_3(\varphi) = \sum_i \rho_i\, \nu_3(\varphi_i)$.

It results

$$f_{\otimes}(\varphi) = \sum_{i}^{n} \sum_{t}^{\infty} \alpha_i(t)\,(\varphi \times \beta_i)\, z^t$$

$$f_{\otimes}(\varphi) = \bar{f}_{\otimes}(\nu_3(\varphi)) = \sum_{i}^{n} \rho_i\, \bar{f}_{\otimes}(\nu(\varphi_i)) = \sum_{i}^{n} \rho_i\, f_{\otimes}(\varphi_i) = \sum_{i}^{n} \sum_{t}^{\infty} \rho_i\, \alpha_i(t)\, z^t$$

The series α_i are linearly independent, so that $\rho_i = \varphi \times \beta_i$

<u>Remark</u> Consider the subspace $\mathrm{span}(\varphi_1, \ldots, \varphi_n) \subset \mathcal{U}_1 \otimes \mathcal{U}_2$.

Let $\varphi \in \mathcal{U}_1 \otimes \mathcal{U}_2$ and denote by $[\varphi]_3$ its equivalence class under $\underset{3}{\sim}$, then $[\varphi]_3$ contains one and only one element which belongs to $\mathrm{span}(\varphi_1 \ldots \varphi_n)$, and this element is given by $\pi(\psi) = \sum_{i}^{n} (\psi \times \beta_i)\varphi_i$, any $\psi \in [\varphi]_3$.

The linear map $\pi : \mathcal{U}_1 \otimes \mathcal{U}_2 \to \mathrm{span}(\varphi_1 \ldots \varphi_n)$, characterized by the projection of ψ on the rows β_i, $i = 1,2,\ldots,n$ of $\tilde{\mathcal{H}}$, plays a rule analogous to the map which associates in the linear system theory any input $u \in K[z^{-1}]$ with its remainder under division by the denominator of the transfer funtion.

These facts are summarized in the following diagram:

$$\begin{tikzcd}
\mathbb{R}^n & \mathrm{span}(\varphi_1, \ldots, \varphi_n) \arrow[l, "\{\varphi_i\}"'] \arrow[dr, "f_{\otimes}", "\simeq"'] & \\
 & \mathcal{U}_1 \otimes \mathcal{U}_2 \arrow[u, "\pi"'] \arrow[r, "f_{\otimes}"] \arrow[d, "\nu_3"] & \mathrm{Im}\, f_{\otimes} \\
\mathbb{R}^n \arrow[uu, leftrightarrow, "Id"] & \mathcal{U}_1 \otimes \mathcal{U}_2 / \mathrm{Ker}\, f_{\otimes} \arrow[l, "\{\nu_3(\varphi_i)\}"] \arrow[ur, "\simeq", "\bar{f}_{\otimes}"'] &
\end{tikzcd}$$

Since $d_{ij} = f_{\otimes}(z_1^{-i} z_2^{-j})$, it is clear that $\mathrm{Im}\, f_{\otimes} = \mathrm{span}(d_{ij})$.

<u>Proposition 3.3.</u> Let $s \in (z_1 z_2) K[[z_1, z_2]]$. Then $\dim X_3 < \infty$ iff there exist $N_{ij}, p \in K[z_1^{-1} z_2^{-1}]$ such that $d_{ij} = N_{ij}((z_1 z_2)^{-1})/p((z_1 z_2)^{-1})$, $i,j = 1,2,\ldots$

proof. Let $d_1, d_2, \ldots, d_n$ be a basis for $\mathrm{span}(d_{ij})$ and observe that for any $q \in K[z_1^{-1} z_2^{-1}]$ and $d \in \mathrm{span}(d_{ij})$

$$q d \odot \sum_{i}^{\infty} (z_1 z_2)^i \in \mathrm{span}(d_{ij})$$

Define the map $\psi : K[z_1^{-1} z_2^{-1}] \longrightarrow K^{n \times 1}[[z]]$ by the assignement

$$\psi : q((z_1z_2)^{-1}) \longmapsto \begin{bmatrix} qd_1 \odot \sum_{i=1}^{\infty}(z_1z_2)^i \\ qd_2 \odot \sum_{i=1}^{\infty}(z_1z_2)^i \\ \cdots\cdots\cdots\cdots \\ qd_n \odot \sum_{i=1}^{\infty}(z_1z_2)^i \end{bmatrix}$$

The map ψ is linear and $\dim \operatorname{Im}\psi \leq n^2$, since $qd_h \odot \sum_{i=1}^{\infty}(z_1z_2)^i \in \operatorname{span}(d_i)$, $\forall h, \forall q$.

The following diagram

$$\begin{tikzcd} K[(z_1z_2)^{-1}] \arrow[r, "\psi"] \arrow[d, "g"'] & \operatorname{Im}\psi \\ K[(z_1z_2)^{-1}]/\operatorname{Ker}\psi \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

commutes. $\operatorname{Ker}\psi$ is a non zero ideal in $K[z_1^{-1}z_2^{-1}]$ so that there exists a non zero polynomial $p \in K[z_1^{-1}z_2^{-1}]$ such that $\ker = (p)$.

This implies that

$$p d_1 \odot \sum_1^{\infty} {}_i (z_1z_2)^i = \cdots\cdots = p d_n \odot \sum_1^{\infty} {}_i (z_1z_2)^i = 0$$

which means that $d_1, d_2, \ldots, d_n$ are series expansions of rational functions with common denominator p.

Viceversa, assume $d_{ij} \in (z_1z_2)K^{rat}[[z_1z_2]]$, $d_{ij} = N_{ij}(z_1^{-1}, z_2^{-1})/p((z_1z_2)^{-1})$ and $\deg p = m$. Then $m \geqslant \dim \operatorname{span}(d_{ij}) = \dim \operatorname{Im} f_{\otimes} = \dim X_3$.

<u>Finite dimensionality of</u> X_1 <u>and</u> X_2

The linear space X_1 has been introduced as $X_1 = \mathcal{U}_1/\sim_1$ and

$$u_1 \sim_1 u_1' \iff f\begin{pmatrix} u_1 & 0 \\ 0 & \sigma \end{pmatrix} = f\begin{pmatrix} u_1' & 0 \\ 0 & \sigma \end{pmatrix}, \ \forall \sigma \in \mathcal{U}_1$$

Define now a map $f_1 : \mathcal{U}_1 \otimes 1 \longrightarrow K[[z]]^{1\times\infty}$ by the assignement:

$$f_1(u_1) = \left[f\begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}, f\begin{pmatrix} u_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \cdots\cdots \right]$$

where $f\begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}, f\begin{pmatrix} u_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ldots \in K[[z]]$. The space $K[[z]]^{1\times\infty}$ is endowed with the structure of $K[z_1^{-1}]$-module in the following way: let $[s_1, s_2, \ldots] \in K[[z]]^{1\times\infty}$ the product $z_1^{-1}[s_1, s_2, \ldots] \triangleq [s_2, s_3, \ldots]$.

The map f_1 is then a $K[z_1^{-1}]$-morphism. In fact:

$$f_1(z_1^{-1}u_1) = \left[f\begin{pmatrix} u_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, f\begin{pmatrix} u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \dots \right]$$

is equal to

$$z_1^{-1}\left[f\begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}, f\begin{pmatrix} u_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \dots \right]$$

by definition of module operation on $K[[z]]^{1\times\infty}$.

Consequently the diagram

$$\begin{tikzcd} K[z_1^{-1}] \simeq \mathcal{U}_1 \otimes 1 \arrow[r, "f_1"] \arrow[d, "\nu_1"'] & K[[z]]^{1\times\infty} \\ \mathcal{U}_1 \otimes 1 / \operatorname{Ker} f_1 \arrow[ur, "\bar{f}_1"'] & \end{tikzcd}$$

commutes. It is immediate that $u_1 \underset{1}{\sim} u_1'$ iff $u_1 - u_1' \in \ker f_1$. As a consequence $\mathcal{U}_1 \otimes 1/\ker f_1 = X_1$. Since $\ker f_1$ is an ideal in $K[z_1^{-1}]$, X_1 has finite dimension iff there exist a non zero polynomial $\omega_1 \in K[z_1^{-1}]$ such that $\ker f_1 = (\omega_1)$. Moreover $\dim X_1 = \deg \omega_1$.

The elements of $f_1(1) \in K[[z]]^{1\times\infty}$ are the series $d_{j,1}$:

$$f_1(1) = \left[f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, f\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \dots \right] = \left[f(z_1^{-1}), f(z_1^{-2}) \dots \right] =$$

$$= \left[d_{21}, d_{31}, d_{41} \dots \right]$$

<u>Proposition 3.4.</u> Let $s = \sum_{1}^{\infty}{}_{ij}\, s_{ij} z_1^i z_2^j$. Then the following facts are equivalent:

i) $\dim X_1 < \infty$

ii) $\ker f_1 \neq 0$

iii) there exist $N_j, \omega_1 \in K[z_1^{-1}]$ such that $c_j = N_j(z_1^{-1})/\omega_1(z_1^{-1})$, $j = 1,2\dots$

proof. i) $\Longleftrightarrow$ ii) has already been proved.

ii) $\Longrightarrow$ iii) Let $\omega_1 \in \ker f_1$, $\omega_1 \neq 0$, and $\omega_1(z_1^{-1}) = k_0 + k_1 z_1^{-1} + \dots + z_1^{-n}$.

Clearly $\omega_1(z_1^{-1})f_1(1) = f_1(\omega_1(z_1^{-1})) = 0$ implies

$$0 = k_0 f_1(1) + k_1 f_1(z_1^{-1}) + \ldots\ldots + f_1(z_1^{-n}) =$$

$$= k_0\left[d_{21}, d_{31} \ldots\ldots\right] + k_1\left[d_{31}, d_{41} \ldots\ldots\right] + \ldots + \left[d_{n+2,1}, d_{n+3,1} \ldots\ldots\right] =$$

$$= \left[k_0 d_{21} + k_1 d_{31} + \ldots + d_{n+2,1} \;\vdots\; k_0 d_{31} + k_1 d_{41} + \ldots d_{n+3,1} \;\vdots\; \ldots\ldots\right] =$$

$$= \left[k_0 \sum_1^\infty {}_k\, s_{2+k,1+k}(z_1 z_2)^k + k_1 \sum_1^\infty {}_k\, s_{3+k,1+k}(z_1 z_2)^k + \ldots + \sum_1^\infty {}_k\, s_{n+2+k,1+k}(z_1 z_2)^k \;\vdots\right.$$

$$k_0 \sum_1^\infty {}_k\, s_{3+k,1+k}(z_1 z_2)^k + k_1 \sum_1^\infty {}_k\, s_{4+k,1+k}(z_1 z_2)^k + \ldots + \sum_1^\infty {}_k\, s_{n+3+k,1+k}(z_1 z_2)^k \;\vdots$$

$$\left.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$= \left[\sum_1^\infty {}_k (k_0 s_{2+k,1+k} + k_1 s_{3+k,1+k} + \ldots + s_{n+2+k,1+k})(z_1 z_2)^k \;\vdots\; \sum_1^\infty {}_k (k_0 s_{3+k,1+k} + \right.$$

$$\left. + k_1 s_{4+k,1+k} + \ldots + s_{n+3+k,1+k})(z_1 z_2)^k \;\vdots\; \ldots\ldots\ldots\right]$$

This implies that:

$$k_0 s_{2+k,1+k} + k_1 s_{3+k,1+k} + \ldots + s_{n+2+k,1+k} = 0 \qquad k = 1,2,\ldots$$

$$k_0 s_{3+k,1+k} + k_1 s_{4+k,1+k} + \ldots + s_{n+3+k,1+k} = 0 \qquad k = 1,2,\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

or equivalently that all the column series are series expansions of rational functions with the same denominator ω_1.

iii) $\Rightarrow$ ii) Let all column series be described by rational functions N_i/ω_1 $\omega_1 \neq 0$. Then $\omega_1 \in \ker f_1$ and $\ker f_1 \neq 0$.

Using similar techniques one can define a map f_2 and an analogous result can be proved for X_2.

Proposition 3.5. Let $s = \sum_{ij}^{\infty} s_{ij} z_1^i z_2^j$. Then the following facts are equivalent:

i) $\dim X_2 < \infty$

ii) $\ker f_2 \neq 0$

iii) there exist $M_i, \omega_2 \in K[z_2^{-1}]$ such that $r_i = M_i(z_2^{-1}) / \omega_2(z_2^{-1})$, $i = 1,2\ldots$

Remark 1 Suppose $\ker f_1 = (\omega_1) \neq 0$, and $c_j = N_j(z_1^{-1})/D_j(z_1^{-1})$, $\text{g.c.d.}(N_j, D_j) = 1$, $j = 1,2,\ldots$ Then $\text{l.c.m.}(D_1, D_2, \ldots\ldots)$ is ω_1 , and viceversa. A similar remark holds for f_2.

Remark 2 If $\dim X_3 < \infty$ all the columns of $\widetilde{\mathcal{H}}$ are coefficients of series expansions of rational functions having the same denominator. The viceversa is also true. The condition $X_1 < \infty$ has also a counterpart on $\widetilde{\mathcal{H}}$. In fact, let denote by A_j the $(1,j)$ indexed column of $\widetilde{\mathcal{H}}$. Then

$$f_1(1) = \left[A_2, A_3, A_4, \ldots\ldots\right]$$

$$f_1(z_1^{-1}) = \left[A_3, A_4, A_5, \ldots\ldots\right] \quad \ldots\ldots$$

Consider now the Hankel matrix whose elements are the sequences A_j :

$$\mathcal{H}_1 = \begin{bmatrix} A_2 & A_3 & A_4 & A_5 & \ldots\ldots \\ A_3 & A_4 & A_5 & A_6 & \ldots\ldots \\ A_4 & A_5 & \ldots\ldots & & \\ \ldots\ldots & & & & \end{bmatrix}$$

Then $n_1 = \dim X_1$ is also the maximal number of K-independent rows of $\mathcal{H}_1$.

Proposition 3.6. Let $s \in (z_1 z_2) K[[z_1, z_2]]$ be the series operator associated with a bilinear i/o map f , and X_1, X_2, X_3 be defined as in Sec.3 . Then $\dim(X_1 \oplus X_2 \oplus X_3) < \infty$ iff $s \in (z_1 z_2) K^{real}[[z_1, z_2]]$.

proof. direct consequence of Propositions 2.1, 3.3, 3.4, 3.5 .

4. FINITE DIMENSIONAL REALIZATION

This section is concerned with the derivation of updating equations in $X_1 \oplus X_2 \oplus X_3$ when dim ($X_1 \oplus X_2 \oplus X_3$) is finite.

4.1 Updating equation in X_1 (X_2)

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{U}_1 \otimes 1 \arrow[rr, "z_1^{-1} \otimes 1"] \arrow[dr, "\gamma_1"] \arrow[dd, "f_1"'] & & \mathcal{U}_1 \otimes 1 \arrow[dl, "\gamma_1"'] \arrow[dd, "f_1"] \\
& X_1 \arrow[r, dashed, "\psi_1"] \arrow[dl, "\bar{f}_1"] & X_1 \arrow[dr, "\bar{f}_1"'] \\
\mathrm{Im}\, f_1 \arrow[rr, "z_1^{-1}"'] & & \mathrm{Im}\, f_1
\end{tikzcd}$$

Because of invariance of f_1 this diagram commutes along the external arrows and Zeiger Lemma can be applied to the following subdiagram:

$$\begin{tikzcd}
\mathcal{U}_1 \otimes 1 \arrow[r, "\gamma_1 \circ (z_1^{-1} \otimes 1)"] \arrow[d, "\gamma_1"'] & X_1 \arrow[d, "\bar{f}_1"] \\
X_1 \arrow[r, "z_1^{-1} \circ \bar{f}_1"'] \arrow[ur, dashed, "\psi_1"] & \mathrm{Im}\, f_1
\end{tikzcd}$$

to prove the existence of a $K[z^{-1}]$-morphism $\psi_1 : X_1 \longrightarrow X_1$ such that $\gamma_1 \circ (z_1^{-1} \otimes 1) = \psi_1 \circ \gamma_1$

The updating equation for X_1 is then:

$$x_1(t+1) = A_1\, x_1(t) + B_1\, u_1(t)$$

where $A_1 \in K^{n_1 \times n_1}$, $B_1 \in K^{n_1 \times 1}$, $n_1 = \dim X_1$, once a basis has been introduced in X_1.

A similar equation is obtained for X_2:

$$x_2(t+1) = A_2\, x_2(t) + B_2\, u_2(t)$$

where $A_2 \in K^{n_2 \times n_2}$, $B_2 \in K^{n_2 \times 1}$, $n_2 = \dim X_2$.

4.2 Updating equation in X_3

The following equality descends direcly from bilinearity hypothesis on f:

$$f\begin{pmatrix} u_1 & k_1 \\ u_2 & k_2 \end{pmatrix} = f\begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix} + k_2 f\begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} + k_1 f\begin{pmatrix} 0 & 1 \\ u_2 & 0 \end{pmatrix} +$$

$$+ k_1 k_2 f\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad k_1, k_2 \in K$$

Consider first the term $f\begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix}$. The following diagram

$$\begin{tikzcd}
\mathcal{U}_1 \otimes \mathcal{U}_2 \arrow[rr, "z_1^{-1} \otimes z_2^{-1}"] \arrow[dr, "\gamma_3"] \arrow[dd, "f_\otimes"'] & & \mathcal{U}_1 \otimes \mathcal{U}_2 \arrow[dl, "\gamma_3"'] \arrow[dd, "f_\otimes"] \\
& X_3 \arrow[r, dashed] \arrow[dl, "\bar{f}_\otimes"] & X_3 \arrow[dl, "\bar{f}_\otimes"'] \\
\operatorname{Im} f_\otimes \arrow[rr, "z^{-1}"'] & & \operatorname{Im} f_\otimes
\end{tikzcd}$$

commutes along the external arrows by invariance of f. Direct application of Zeiger Lemma gives

$$\gamma_3 \circ (z_1^{-1} \otimes z_2^{-1}) = \psi_3 \circ \gamma_3$$

so that

$$\gamma_3 \begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix} = \varphi_3 \circ \nu_3 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} .$$

Let

$$\pi : K[[z]]^{\infty \times 1} \longrightarrow K[[z]]$$

defined by

$$\pi[s_1 : s_2 : s_3 : \ldots\ldots] = s_1$$

Since the series $f\begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}$ is the first component of $f_1(u_1) \in K[[z]]^{\infty \times 1}$ one gets:

$$\pi(f_1(u_1)) = f\begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} = f_{\otimes}(u_1 z_1^{-1} \otimes 1)$$

In a similar way it results

$$\pi(f_2(u_2)) = f\begin{pmatrix} 0 & 1 \\ u_2 & 0 \end{pmatrix} = f_{\otimes}(1 \otimes u_2 z_2^{-1})$$

Finally

$$f\begin{pmatrix} u_1 & k_1 \\ u_2 & k_2 \end{pmatrix} = \bar{f}_{\otimes} \circ \gamma_3 \begin{pmatrix} u_1 & k_1 \\ u_2 & k_2 \end{pmatrix} = \bar{f}_{\otimes} \circ \gamma_3 \begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix} +$$

$$+ k_2 \bar{f}_{\otimes} \circ \bar{f}_{\otimes}^{-1} \circ \pi \circ f_1(u_1) + k_1 \bar{f}_{\otimes} \circ \bar{f}_{\otimes}^{-1} \circ \pi \circ f_2(u_2) +$$

$$+ k_1 k_2 \bar{f}_{\otimes} \circ \gamma_3 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} =$$

$$= \bar{f}_{\otimes} \left[\psi_3 \circ \gamma_3 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + k_2 \tau_1 \circ \gamma_1(u_1) + k_1 \tau_2 \circ \gamma_2(u_2) + \right.$$

$$\left. + k_1 k_2 \gamma_3 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right]$$

where $\tau_1 = \bar{f}_{\otimes}^{-1} \circ \pi \circ \bar{f}_1 : X_1 \to X_3$; $\tau_2 = \bar{f}_{\otimes}^{-1} \circ \pi \circ \bar{f}_2 : X_2 \to X_3$

Consequently

$$\bar{f}_{\otimes} \left[\gamma_3 \begin{pmatrix} u_1 & k_1 \\ u_2 & k_2 \end{pmatrix} - \psi_3 \circ \gamma_3 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - k_2 \tau_1 \circ \gamma_1(u_1) - \right.$$

$$\left. - k_1 \tau_2 \circ \gamma_2(u_2) - k_2 k_2 \gamma_3 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right] = 0$$

and since $\bar{f}_{\otimes}$ is an isomorphism, one gets:

$$x_3(1) = A_3 x_3(0) + k_2 T_1 x_1(0) + k_1 T_2 x_2(0) + k_2 k_1 B_3$$

where $A_3 \in K^{n \times n}$, $T_1 \in K^{n \times n_1}$, $T_2 \in K^{n \times n_2}$, $B_3 \in K^{n \times 1}$, once a basis has been introduced in X_1, X_2, X_3.

So that

$$x_3(t+1) = A_3\, x_3(t) + T_1\, x_1(t)\, u_2(t) + T_2\, x_2(t)\, u_1(t) + B_3\, u_1(t)\, u_2(t)$$

$$y(t) = \bar{f}_{\otimes}\, x_3(t) = C\, x_3(t) \quad ; C \in K^{1 \times n_3}$$

In sections 5 and 6 two different procedures will be presented for evaluating A_3, T_1, T_2. The first method is based on the knowledge of the series β_i, $i = 1, \ldots, n$ and of the inputs φ_j, $j = 1, \ldots, n$ defined in Section 3. The second one will require that the series operator s has to be given in the form of rational function.

5. EVALUATING A_3, T_1, T_2 VIA $\{\beta_i\}$ AND $\{\varphi_j\}$

Let $\nu(\varphi_1), \ldots, \nu(\varphi_n)$ be a basis in X_3 and φ belong to $\mathcal{U}_1 \otimes \mathcal{U}_2$. Then

$$\nu_3(\varphi) = (\beta_1 \times \varphi)\,\nu_3(\varphi_1) + \cdots + (\beta_n \times \varphi)\,\nu_3(\varphi_n)$$

and

$$\nu_3((z_1 z_2)^{-1}\varphi) = (\beta_1 \times (z_1 z_2)^{-1}\varphi)\,\nu_3(\varphi_1) + \cdots + (\beta_n \times (z_1 z_2)^{-1}\varphi)\,\nu_3(\varphi_n) =$$
$$= [\nu_3(\varphi_1) \cdots \nu_3(\varphi_n)]\, A_3\, [(\beta_1 \times \varphi) \cdots (\beta_n \times \varphi)]^T$$

In particular if $\varphi = \varphi_j$, $j = 1, 2, \ldots\ldots$ where

$$\beta_j \times \varphi_i = \delta_{ij}$$

one gets

$$[(\beta_1 \times (z_1 z_2)^{-1}\varphi_j) \cdots (\beta_n \times (z_1 z_2)^{-1}\varphi_j)] = \nu_3((z_1 z_2)^{-1}\varphi_j) =$$
$$= [\nu_3(\varphi_1) \cdots \nu_3(\varphi_n)]\, A_3\, [(\beta_1 \times \varphi_j) \cdots (\beta_n \times \varphi_j)]^T =$$
$$= [\nu_3(\varphi_1) \cdots \nu_3(\varphi_n)]\, [a_{1j} \cdots a_{nj}]^T$$

and then

$$a_{ij} = (z_1 z_2)^{-1}\, \varphi_j \times \beta_i$$

$$A_3 = (a_{ij})$$

<u>Remark</u>. The matrix A_3 is nilpotent iff the sequences $\beta_1, \ldots, \beta_n$ have compact support.

Consider now the construction of T_1 and T_2. Let $\gamma_1(1)$, $\gamma_1(z_1^{-1}), \ldots\ldots\ldots \gamma_1(z_1^{-n_1+1})$ be a basis in X_1 and $\gamma_3(\varphi_1), \ldots\ldots \gamma_3(\varphi_n)$ a basis in X_3. Then:

$$\tau_1 \circ \gamma_1(z_1^{-k}) = \bar{f}_{\otimes}^{-1} \circ \pi \circ \bar{f}_1 \circ \gamma_1(z_1^{-k}) = \bar{f}_{\otimes}^{-1} \circ f \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \underbrace{0 \cdots 0}_{k} & & 1 \end{pmatrix} =$$

$$= \bar{f}_{\otimes}^{-1} \sum_{t=1}^{\infty} \sum_{i=1}^{n} \alpha_i(t) \left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \times \beta_i \right) z^t =$$

$$= \sum_{i=1}^{n} \gamma_3(\varphi_i) \left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \times \beta_i \right), \quad k = 0, 1, \ldots\ldots, n-1$$

so that

$$T_1 = (t_{ki}^{(1)}) \quad , \quad t_{ki}^{(1)} = \begin{pmatrix} 1 & 0 & \cdots\cdots & 0 \\ 0 & \underbrace{0 \cdots\cdots 0}_{k} & & 1 \end{pmatrix} \times \beta_i$$

Similarly:

$$T_2 = (t_{ki}^{(2)}) \quad , \quad t_{ki}^{(2)} = \begin{pmatrix} 0 & \overbrace{0 \cdots\cdots 0}^{k} & 1 \\ 1 & 0 \cdots\cdots & 0 \end{pmatrix} \times \beta_i$$

if in X_2 the basis $\gamma_2(1), \ldots\ldots, \gamma_2(z_2^{-n_2+1})$ has been taken.

6. CONSTRUCTION OF A REALIZATION OF $s \in (z_1 z_2)\, K^{real}[[z_1, z_2]]$

In section 4 it has been proved that under the hypothesis $\dim(X_1 \oplus X_2 \oplus X_3) < \infty$ the following zero-state, finite dimensional realization $(A_1, A_2, A_3, B_1, B_2, B_3, T_1, T_2, C)$ exists:

(6.1) $x_1(t+1) = A_1\, x_1(t) + B_1\, u_1(t)$

(6.2) $x_2(t+1) = A_2\, x_2(t) + B_2\, u_2(t)$

(6.3) $x_3(t+1) = A_3\, x_3(t) + u_2(t)\, T_1\, x_1(t) + u_1(t)\, T_2\, x_2(t) + B_3\, u_1(t)\, u_2(t)$

(6.4) $y(t) = C\, x_3(t)$

Starting from (6.1-4) the structure of the rational function associated with the input/output operator s will be now derived and it will be proved that $s \in (z_1 z_2)\, K^{rat}[[z_1]] \otimes K^{rat}[[z_2]] \otimes K^{rat}[[z_1 z_2]]$. This will constitute also an alternative proof of the necessary part of Proposition 3.6 .

Let introduce the following generalized series:

$$x_1(z_1) \triangleq \sum_t x_1(t)\, z_1^t$$

$$x_2(z_2) \triangleq \sum_t x_2(t)\, z_2^t$$

$$x_3(z_1 z_2) \triangleq \sum_t x_3(t)\, (z_1 z_2)^t$$

$$u_1(z_1) \triangleq \sum_t u_1(t)\, z_1^t$$

$$u_2(t) \triangleq \sum_t u_2(t)\, z_2^t$$

$$y(z_1 z_2) \triangleq \sum_t y(t)\, (z_1 z_2)^t$$

Then from (6.1) and (6.2) one gets:

$$x_1(z_1) = (z_1^{-1} I - A_1)^{-1}\, B_1\, u_1(z_1)$$

$$x_2(z_2) = (z_2^{-1} I - A_2)^{-1}\, B_2 u_2(z_2)$$

and from (6.3) it results:

$$(z_1z_2)^{-1}x_3(z_1z_2) = A_3\, x_3(z_1z_2) + u_2(z_2)\, T_1\, x_1(z_1) \odot \sum_t (z_1z_2)^t +$$
$$+ u_1(z_1)\, T_2\, x_2(z_2) \odot \sum_t (z_1z_2)^t + B_3\, u_1(z_1)\, u_2(z_2) \odot \sum_t (z_1z_2)^t$$

$$x_3(z_1z_2) = ((z_1z_2)^{-1}I - A_3)^{-1} \Big(T_1(z_1^{-1}I - A_1)^{-1} + T_2(z_2^{-1}I - A_2)^{-1}B_2 +$$
$$+ B_o\Big)\, u_1(z_1)\, u_2(z_2) \odot \sum_t (z_1z_2)^t$$

Since the output is given by

$$y(z_1z_2) = s\, u_1\, u_2 \odot \sum_t (z_1z_2)^t =$$
$$= C\, (\, (z_1z_2)^{-1}I - A_3)^{-1}(T_1(z_1^{-1}I - A_1)^{-1}B_1 + T_2(z_2^{-1}I - A_2)^{-1}B_2 +$$
$$+ B_o)\, u_1(z_1)\, u_2(z_2) \odot \sum_t (z_1z_2)^t$$

this implies that the series operator s is given by:

$$(6.5) \qquad s = C\, ((z_1z_2)^{-1}I - A_3)^{-1}\, (\, T_1(z_1^{-1}I - A_1)^{-1}B_1 + T_2(z_1^{-1}I - A_2)^{-1}B_2 + B_0)$$

An immediate consequence of (6.5) is that s has the structure

$$N(z_1^{-1},z_2^{-1})\, /p((z_1z_2)^{-1})\, \omega_1(z_1^{-1})\, \omega_2(z_2^{-1})$$

Starting from a series s in the form $N / p\omega_1\, \omega_2$ an effective procedure will be now given for getting s in the form (6.5) and hence to obtain the matrices of a realization.

Let $s \in (z_1z_2)\, K^{real}[[z_1,z_2]]$ and $s = N(z_1^{-1},z_2^{-1}) / p((z_1z_2)^{-1})\, \omega_1(z_1^{-1})\omega_2(z_2^{-1})$ with deg p = q. Consider $\sigma \in K^{rec}[[z_1,z_2]]$:

$$\sigma = N(z_1^{-1},z_2^{-1}) / \omega_1(z_1^{-1})\, z_1^{-q+1}\, \omega_2(z_2^{-1})\, z_2^{-q+1}$$

so that there exist an integer ν and $\lambda \in K^{1 \times \nu}$, $\mu_1, \mu_2 \in K^{\nu \times \nu}$ $\gamma \in K^{\nu \times 1}$ such that

$$\sigma = \sum_{ij} \lambda \mu_1^i \mu_2^j \gamma\, z_1^i z_2^j$$

and $\mu_1 \mu_2 = \mu_2 \mu_1$. Consequently

$$\sigma = \sum_{1}^{\infty}{}_i \lambda \mu_1^i z_1^i \left(\sum_{0}^{\infty}{}_k (\mu_1\mu_2)^k \gamma (z_1 z_2)^k \right) +$$

$$+ \sum_{1}^{\infty}{}_j \lambda \mu_2^j z_2^j \left(\sum_{0}^{\infty}{}_k (\mu_1\mu_2)^k \gamma (z_1 z_2)^k \right) + \lambda \sum_{0}^{\infty}{}_k (\mu_1\mu_2)^k \gamma (z_1 z_2)^k =$$

$$= \lambda \sum_{0}^{\infty}{}_k (\mu_1\mu_2)^k (z_1 z_2)^k \left(\sum_{1}^{\infty}{}_i \mu_1^i \gamma z_1^i + \sum_{1}^{\infty}{}_j \mu_2^j \gamma z_2^j + \gamma \right) =$$

$$= \lambda \frac{(z_1 z_2)^{-1}}{(z_1 z_2)^{-1} I - (\mu_1\mu_2)} \left(\frac{1}{I z_1^{-1} - \mu_1} \gamma + \frac{1}{I z_2^{-1} - \mu_2} \gamma + \gamma \right)$$

and

$$s = \left(\lambda \frac{(z_1 z_2)^{-q}}{p(z_1^{-1} z_2^{-1})} \frac{1}{(z_1 z_2) I - \mu_1\mu_2} \right) \left(\frac{1}{z_1^{-1} I - \mu_1} \gamma + \frac{1}{z_2^{-1} I - \mu_2} \gamma + \gamma \right)$$

The first member in the right is a row matrix $1 \times Y$ whose elements belong to $(z_1 z_2)\, K^{rat}[[z_1 z_2]]$. Consequently it can be linearly realized in some dimension m. This means that one can construct three matrices $A_o \in K^{m \times m}$, $B_o \in K^{m \times Y}$, $C_o \in K^{1 \times m}$ such that

$$C_o (I (z_1 z_2)^{-1} - A_o)^{-1} B_o = \frac{(z_1 z_2)^{-q}}{p((z_1 z_2)^{-1}} \frac{1}{(z_1 z_2)\, I - \mu_1 \mu_2}$$

The series operator s is then rewritten as

$$s = C_0\,(I(z_1z_2)^{-1} - A_0)^{-1}\left(B_0(z_1^{-1}I - \mu_1)^{-1} + B_0(z_2^{-1}I - \mu_2)^{-1}\gamma + B_0\gamma\right)$$

and the realization of s is characterized by $(\mu_1, \mu_2, A_0, \gamma, \gamma, B_0\gamma, B_0, B_0, C_0)$.

A technique for constructing $\mu_1, \mu_2, \gamma, \lambda$ for recognizable series can be found in [12] .

7. REACHABILITY OF NERODE STATE SPACE IN BOUNDED TIME

A further characteristic property of bilinear i/o maps represented by realizable series is the reachability (from zero state) of Nerode state space in a bounded time interval.

Lemma 7.1. Let $s \in (z_1z_2)K^{real}[[z_1,z_2]]$ be the series operator of the bilinear i/o map f, $\ker f_1 = (\omega_1)$ and $\ker f_2 = (\omega_2)$. Then there exists an integer M such that for any pair (u_1,u_2) , $u_1 \in (\omega_1)$, $u_2 \in (\omega_2)$

$$(u_1,u_2) \underset{3}{\sim} (u_1',u_2')$$

some $u_1' \in (\omega_1)$, $u_2' \in (\omega_2)$, $\deg u_1'$, $\deg u_2' \leq M$.

proof. It is immediate that $(\omega_1, 0), (\omega_2, 0) \in [0,0]$. Then, by Proposition 2.2.,

$$0 = \frac{N(z_1^{-1}, z_2^{-1})}{\omega_1(z_1^{-1})\,\omega_2(z_2^{-1})} + \frac{(z_1z_2)^{-1}}{\omega_1(z_1^{-1})\,\omega_2(z_2^{-1})} \sum_{\substack{h,k \\ |h-k| \leq m}} s'_{hk}\, z_1^{-h} z_2^{-k}$$

Assume $\dim X_3 = n_3$ and consider the equivalence $\underset{3}{\sim}$ restricted to $\omega_1\mathcal{U}_1 \times \omega_2\mathcal{U}_2$, so that

$$(q_1\omega_1, q_2\omega_2) \underset{3}{\sim} (q_1'\omega_1, q_2'\omega_2) \quad , \quad q_1, q_1' \in K[z_1^{-1}],\ q_2, q_2' \in K[z_2^{-1}]$$

iff

$$\sum_{1}^{n_3} {}_i\,(z_1z_2)^i \odot \left((q_1q_2 - q_1'q_2')\,(z_1z_2)^{-1} \sum_{\substack{h,k \\ |h-k| \leq m}} s'_{hk}\, z_1^h z_2^k\right) = 0$$

Introduce the map $\mu : \mathcal{U}_1 \otimes \mathcal{U}_2 \longrightarrow R^{n_3}$,

$$\mu(\varphi) = \sum_{1}^{n_3}{}_i (z_1 z_2)^i \odot \varphi(z_1^{-1}, z_2^{-1})(z_1 z_2)^{-1} \sum_{\substack{h,k \\ |h-k| \le m}} s^i_{hk} z_1^h z_2^k$$

and define the subspaces $\mathcal{S}_i$ of $\mathcal{U}_1 \otimes \mathcal{U}_2$ by

$$\mathcal{S}_i = \mathcal{U}_1 \otimes \mathcal{U}_2 - \mathcal{R}_i \cup \{0\} \quad , \qquad i = 0,1,2\ldots\ldots$$

where

$$\mathcal{R}_i = \{\varphi : \varphi \in \mathcal{U}_1 \otimes \mathcal{U}_2,\ \deg_{z_1^{-1}} \varphi \le i,\ \deg_{z_2^{-1}} \varphi \le i\}$$

Then $n_3 \geqslant \dim \mu(\mathcal{S}_0) \geqslant \dim \mu(\mathcal{S}_1) \geqslant \ldots\ldots$.

Let L be an integer such that $\dim \mu(\mathcal{S}_L) = \dim \mu(\mathcal{S}_{L+1}) = \ldots\ldots = r$

Define the operator $T^L : K[z^{-1}] \to K[z^{-1}]$ by

$$T^L : \sum_{0}^{\infty}{}_i a_i z^{-i} \longmapsto \sum_{0}^{L}{}_i a_i z^{-i}$$

Then if $r = 0$, $(\omega_1 q_1, \omega_2 q_2) \underset{3}{\sim} (\omega_1 T^L q_1, \omega_2 T^L q_2)$ and the Lemma would be proved. Let $r \neq 0$ and assume $\deg \omega_1 = n_1$, $\deg \omega_2 = n_2$. Then there exist $(h_1, k_1), \ldots\ldots (h_r, k_r)$ such that

$$h_1, k_1 > L + \max(n_1, n_2)$$

$$h_2, k_2 > \max(h_1, k_1) + \max(n_1, n_2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$h_r, k_r > \max(h_{r-1}, k_{r-1}) + \max(n_1, n_2)$$

and $\operatorname{span}(\mu(z_1^{-h_1} z_2^{-k_1}), \ldots\ldots, \mu(z_1^{-h_2} z_2^{-k_2}) = \mu(\mathcal{S}_L)$.

For any (q_1, q_2) one has $\mu(q_1, q_2) = \mu(T^L q_1 T^L q_2 + (q_1 q_2 - T^L q_1 T^L q_2))$

and $\mu(q_1 q_2 - T^L q_1 T^L q_2) \in \mu(\mathcal{S}_L)$, so that

$$\mu(q_1 q_2 - T^L q_1 T^L q_2) = \mu\Big(\sum_{1}^{r}{}_i \alpha_i z_1^{-h_i} z_2^{-k_i}\Big), \quad \alpha_i \in K$$

Consequently $\mu(q_1 q_2) = \mu((T^L q_1 + \sum_{1}^{r}{}_i \alpha_i z_1^{-h_i})(T^L q_2 + \sum_{1}^{r}{}_i z_2^{-k_i}))$

and then

$$(\omega_1 q_1, \omega_2 q_2) \underset{3}{\sim} (\omega_1(T^L q_1 + \sum_1^r{}_i \alpha_i z^{-h_i}), \omega_2(T^L q_2 + \sum_1^r{}_i z_2^{-k_i}))$$

so that

$$M = \begin{cases} \max(n_1,n_2) + L & , r = 0 \\ \max(n_1,n_2) + \max(h_r,k_r) & , r > 0 \end{cases}$$

<u>Proposition 7.1.</u> Let f be a bilinear, discrete time, stationary i/o map and $s \in (z_1 z_2)K[[z_1,z_2]]$ the associated series. Then s is a realizable series iff there exists an integer M such that each Nerode equivalence class contains at least an input of length less than M.

proof. Let $s \in (z_1 z_2)K^{real}[[z_1,z_2]]$. Assume $(\omega_1) = \ker f_1$, $(\omega_2) = \ker f_2$ as in Lemma 7.1 and consider a generic Nerode state $[u_1,u_2] = [\rho_1+\omega_1 q_1, \rho_2+\omega_2 q_2]$, $\deg \rho_1 < \deg \omega_1 = n_1$, $\deg \rho_2 < \deg \omega_2 = n_2$ -

Note that $(\rho_1+\omega_1 g_1, \rho_2+\omega_2 g_2) \underset{1,2}{\sim} (\rho_1+\omega_1 q_1, \rho_2+\omega_2 q_2)$ for any (g_1,g_2). Consequently $(\rho_1+\omega_1 g_1, \rho_2+\omega_2 g_2) \in [u_1,u_2]$ iff $(\rho_1+\omega_1 g_1, \rho_2+\omega_2 g_2) \underset{3}{\sim} (u_1,u_2)$ i.e. iff

$$(7.1) \qquad f(\rho_1+\omega_1 g_1, \rho_2+\omega_2 g_2) = f(\rho_1+\omega_1 q_1, \rho_2+\omega_2 q_2)$$

It is sufficient to show that the pair $(g_1,g_2) = (T^L q_1 + \sum_1^r{}_i z_1^{-h_i}, T^L q_2 + \sum_1^r{}_i z_2^{-k_i})$, determined in Lemma 7.1., with $L > \max(n_1,n_2)$, satisfy (7.1).

In fact by Lemma 7.1.

$$f(\omega_1 q_1, \omega_2 q_2) = f(\omega_1 g_1, \omega_2 g_2)$$

Moreover since the row-series r_i are rational functions with common denominator ω_1 , one has

$$f(\omega_1 q_1, \rho_2) = f(\omega_1(T^L q_1 + \sum_1^2 {}_i \alpha_i z_1^{-h_i}), \rho_2) = f(\omega_1 T^L q_1, \rho_2) = f(\omega_1 q_1, \rho_2)$$

Similarly $f(\rho_1, \omega_2 q_2) = f(\rho_1, \omega_2 q_2)$.

Consequently

$$f(\rho_1 + \omega_1 q_1, \rho_2 + \omega_2 q_2) = f(\rho_1, \rho_2) + f(\rho_1, \omega_2 q_2) + f(\omega_1 q_1, \rho_2) + f(\omega_1 q_1, \omega_2 q_2) =$$
$$= f(\rho_1, \rho_2) + f(\rho_1, \omega_2 g_2) + f(\omega_1 g_1, \rho_2) + f(\omega_1 g_1, \omega_2 g_2) = f(\rho_1 + \omega_1 g_1, \rho_2 + \omega_2 g_2)$$

and M can be assumed as in Lemma 7.1.

Conversely let X_N be reachable in bounded time M and assume that s is not realizable. By Proposition 3.6. this implies $\dim X_1 \oplus X_2 \oplus X_3 = \infty$.

$\dim X_1 = \infty$ implies $\ker f_1 = (0)$, so that :

$$u_1 \underset{1}{\sim} u_1' \quad \text{iff} \quad u_1 = u_1' \; ; \; u_1, u_1' \in \mathcal{U}_1 .$$

This contradicts the hypothesis of reachability in bounded time.

Analogous considerations hold for X_2 and $\mathcal{U}_2$.

Assume now $\dim X_3 = \infty$, i.e. $\dim \operatorname{Im} f_\otimes = \infty$.

Then $\dim f_\otimes(\mathcal{R}_M) \leq (M+1)^2$ implies that there exist integers r,s such that $f_\otimes(z_1^{-r}, z_2^{-s}) \notin f_\otimes(\mathcal{R}_M)$.

Consequently there is no element of $\mathcal{R}_M$ equivalent under $\underset{3}{\sim}$ (and hence under $\underset{N}{\sim}$) to (z_1^{-r}, z_2^{-s}), which contradicts the hypothesis that $[z_1^{-r}, z_2^{-s}]$ is reachable in at most M steps.

It is interesting at this point to evidentiate the rule that realizable series have in the realization theory of bilinear i/o maps.

These series seem to have here the same importance as the rational series in the linear system theory and allow to clarify some structural implications as it appears from the following Proposition.

<u>Proposition 7.2.</u> Let f be a bilinear, discrete time, stationary i/o map and $s \in (z_1 z_2) K[[z_1, z_2]]$ the associated series. Then the following facts are equivalent:

(i) $\dim (X_1 \oplus X_2 \oplus X_3) < \infty$

(ii) $s \in (z_1 z_2) K^{real}[[z_1, z_2]]$

(iii) $s \in (z_1 z_2) K^{rat}[[z_1, z_2]]$ and the zero state $[0,0]$ contains at least one element (p_1, p_2) , $p_1 \neq 0$, $p_2 \neq 0$.

(iv) X_N is reachable in bounded time.

(v) There exist three integers n_1, n_2, n_3 and $A_1 \in K^{n_1 \times n_1}$, $A_2 \in K^{n_2 \times n_2}$, $A_3 \in K^{n_3 \times n_3}$, $T_1 \in K^{n_3 \times n_1}$, $T_2 \in K^{n_3 \times n_2}$, $B_1 \in K^{n_1 \times 1}$, $B_2 \in K^{n_2 \times 1}$, $B_3 \in K^{n_3 \times 1}$, $C \in K^{1 \times n_3}$ such that

$$x_1(t+1) = A_1 x_1(t) + B_1 u_1(t)$$
$$x_2(t+1) = A_2 x_2(t) + B_2 u_2(t)$$
$$x_3(t+1) = A_3 x_3(t) + T_1 x_1(t) u_2(t) + T_2 x_2(t) u_1(t) + B_3 u_1(t) u_2(t)$$
$$y(t) = C x_3(t)$$

proof. (i) $\Longleftrightarrow$ (ii) by Proposition 3.6. (ii) $\Longleftrightarrow$ (iii) by Proposition 2.3. (ii) $\Longleftrightarrow$ (iv) by Proposition 7.1. (ii) $\Longleftrightarrow$ (v) proved in Section 5 .

8. CONNECTION BETWEEN REALIZATIONS (ZERO STATE) OF BILINEAR INPUT/OUTPUT MAPS AND SYSTEMS WITH BILINEAR INTERNAL STRUCTURE

In the previous sections the structure of finite dimensional realizations ($A_1, A_2, A_3, T_1, T_2, B_1, B_2, B_3, C$) for a bilinear i/o map have been derived and the characteristic updating equations are given in (5.1-4).

If one assumes $x = [x_3\ x_2\ x_1]$ these equations can be put in the form:

$$x(t+1) = \bar{F}_0 x(t) + \bar{F}_1 x(t) u_1(t) + \bar{F}_2 x(t) u_2(t) + \bar{G}_1 u_1(t) + \bar{G}_2 u_2(t) + \bar{G}_0 u_1(t) u_2(t)$$

$$y(t) = \bar{H} x(t)$$

where :

$$\bar{F}_0 = \begin{bmatrix} A_3 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_1 \end{bmatrix}, \quad \bar{F}_1 = \begin{bmatrix} 0 & T_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{F}_2 = \begin{bmatrix} 0 & 0 & T_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\bar{G}_1 = \begin{bmatrix} 0 \\ 0 \\ B_1 \end{bmatrix}, \quad \bar{G}_2 = \begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix}, \quad \bar{G}_0 = \begin{bmatrix} B_3 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{H} = [C\ 0\ 0]$$

A natural question which arises, concerns whether a generic dynamical system $\sum_B = (F_0, F_1, F_2, G_1, G_2, G_0, H)$ (*) :

(*) These systems generalize the systems with "bilinear internal structure" [9] because of the term $G_0 u_1(t) u_2(t)$.

$$\Sigma_B : \; x(t+1) = F_0 x(t) + F_1 x(t) u(t) + F_2 x(t) u_2(t) + G_1 u_1(t) + G_2 u_2(t) + \\ + G_0 u_1(t) u_2(t)$$

$$y(t) = H x(t)$$

could be a zero-state realization of a bilinear, discrete-time, stationary i/o map.

Let $Z = (\zeta_0, \zeta_1, \zeta_2)$ an alphabet and Z^* the free monoid generated. Define the three series $\sigma_0, \sigma_1, \sigma_2 \in K^{rec}\langle\langle \zeta_0, \zeta_1, \zeta_2 \rangle\rangle$ as

$$\sigma_0 = \sum_{\zeta_{i_1} \cdots \zeta_{i_n} \in Z^*} H F_{i_1} F_{i_2} \cdots F_{i_n} G_0 \zeta_{i_1} \cdots \zeta_{i_n}$$

$$\sigma_1 = \sum_{\zeta_{i_1} \cdots \zeta_{i_n} \in Z^*} H F_{i_1} F_{i_2} \cdots F_{i_n} G_1 \zeta_{i_1} \cdots \zeta_{i_n}$$

$$\sigma_2 = \sum_{\zeta_{i_1} \cdots \zeta_{i_n} \in Z^*} H F_{i_1} F_{i_2} \cdots F_{i_n} G_2 \zeta_{i_1} \cdots \zeta_{i_n}$$

A necessary and sufficient condition for the bilinearity of the i/o map defined by $(\Sigma_B, x(0)=0)$ is that:

$$\text{supp}\, \sigma_0 \subseteq \zeta_0^*$$

$$\text{(8.1)} \qquad \text{supp}\, \sigma_1 \subseteq \zeta_0^* \zeta_2 \zeta_0^*$$

$$\text{supp}\, \sigma_2 \subseteq \zeta_0^* \zeta_1 \zeta_0^*$$

If Σ_B is a zero-state realization of a bilinear i/o map f, then the three systems:

$$\Sigma_{B_0} = (F_0, 0, 0, G_0, 0, 0, H)$$

$$\Sigma_{B_1} = (F_0, F_1, 0, 0, 0, G_2, H)$$

$$\Sigma_{B_2} = (F_o, 0, F_2, 0, G_1, 0, H)$$

are zero-state realizations of three i/o maps f_o, f_1, f_2 and

$$f(u_1,u_2) = f_o(u_1,u_2) + f_1(u_1,u_2) + f_2(u_1,u_2) \quad , \quad (u_1,u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$$

The structural prperties of these systems are examined in Appendix.

Consider now systems defined by [10]:

$$\Sigma_o \quad \begin{aligned} z(t+1) &= \hat{F}_o z(t) + \hat{F}_1 z(t)\, u_1(t) + \hat{F}_2 z(t)\, u_2(t) \\ y(t) &= \hat{H}\, z(t) \end{aligned}$$

The conditions under which a system Σ_o with initial state $z(0) = q_o$ realizes an i/o bilinear map have been derived in [10]. In effect the i/o map of a system (Σ_o, q_o) can be represented via the generating series $s_o \in K^{rec}\langle\langle \zeta_o, \zeta_1, \zeta_2 \rangle\rangle$:

$$s_o = \sum_{\zeta_{i_1} \cdots \zeta_{i_n} \in Z^*} \hat{H}\, \hat{F}_{i_1} \hat{F}_{i_2} \cdots\cdots \hat{F}_{i_n}\, q_o\, \zeta_{i_1} \zeta_{i_2} \cdots\cdots \zeta_{i_n}$$

and the condition for bilinearity with initial state q_o is then:

$$\operatorname{supp} s_o \subseteq \bigcup_{\substack{i_1 \neq i_2 \\ i_1, i_2 = 1,2}} \zeta_o^* \zeta_{i_1} \zeta_o^* \zeta_{i_2} \zeta_o^*$$

It is known that the i/o map of the system Σ_B with $G_o = 0$ and $x(0) = x_o$ can be obtained as the input /output map of a system Σ_o with:

$$\hat{F}_o = \begin{bmatrix} 1 & 0 \\ 0 & F_o \end{bmatrix} \quad , \quad \hat{F}_i = \begin{bmatrix} 0 & 0 \\ G_i & F_i \end{bmatrix} \quad , \quad i = 1,2$$

$$\hat{H} = [0 \quad H] \quad , \quad z(0) = [1 \quad x_o]^T$$

The conditions by which a system Σ_B with $G_o = 0$ is a zero--state realization of a bilinear i/o map are now known as a particular case of condition (8.1). What seems natural to evidentiate now is the connection existing between the class of Σ_o systems which are also i/o bilinear starting from a state q_o and the class of Σ_B systems which are zero-state realizations of bilinear i/o maps.

In other words the aim now is to identify the subclass of Σ_o systems, having an i/o bilinear map starting from q_o, which can be zero-state realized by a dynamical system ($\Sigma_B, G_o = 0$). The fact that G_o should result zero is at this point obvious since the i/o map given by a system Σ_o does not contain terms of $u_1(t)\, u_2(t)$ type.

<u>Proposition 8.1</u> . Let $\Sigma_o = (\hat{F}_o, \hat{F}_1, \hat{F}_2, \hat{H})$ and $z(0) = q_o$ realize a bilinear i/o map. Then this map can be realized by a system Σ_B with $x(0) = 0$ iff

$$\hat{H}\, \hat{F}_o W\, q_o = \hat{H}\, W\, q_o \tag{8.2}$$

for any matrix W in the multiplicative monoid generated by $\hat{F}_o, \hat{F}_1, \hat{F}_2$.

proof. Let $\Sigma_B = (F_o, F_1, F_2, G_1, G_2, H)$, $x(0) = 0$ realize the same i/o map as $\Sigma_o = (\hat{F}_o, \hat{F}_1, \hat{F}_2, \hat{H})$. This implies that:

$$\hat{H}\, \hat{F}_o^h\, \hat{F}_i\, \hat{F}_o^{k-h-1}\, \hat{F}_j\, \hat{F}_o^{t-h} q_o = H\, F_o^{t-h}\, F_o^{h-k-1}\, G_j$$

This relation and the bilinearity hypothesies give (.2).
Conversely let (8.2) hold. Represent the inputs u_1, u_2 of Σ_o by polynomials in $K[z_1]$ and $K[z_2]$ respectively and the output y of Σ_o, starting from q_o, via formal power series in $K[[z_1 z_2]]$. Recalling bilinearity hypothesies and condition (8.2), this series is explicitely written as:

$$y(z_1z_2) = s(z_1,z_2)\ u_1(z_1)\ u_2(z_2) \odot \sum_{1}{}_i\ (z_1z_2)^i$$

and

$$s = z_2(z_1z_2)\sum_{o}{}_{ij}\ \hat{H}\ \hat{F}_1\ \hat{F}_o^i\hat{F}_2\ \hat{F}_o^j\ q_o(z_1z_2)^j z_2^i + z_1(z_1z_2)\sum_{o}{}_{ij}\ \hat{H}\ \hat{F}_2\hat{F}_o^i\hat{F}_1\ \hat{F}_o^j\ q_o(z_1z_2)^i z_1^j$$

The series s can be expressed as

$$s = \hat{H}\ \hat{F}_1\ \frac{z_1z_2}{I-(z_1z_2)\hat{F}_o}\ \hat{F}_2\ \frac{z_2}{I - z_2\hat{F}_o}\ q_o + \hat{H}\ \hat{F}_2\ \frac{z_1z_2}{I - (z_1z_2)\hat{F}_o}\ \hat{F}_1\ \frac{z_1}{I - z_o\hat{F}_o}\ q_o =$$

$$= \frac{N_2(z_1^{-1},z_2^{-1})}{p((z_1z_2)^{-1})\ p(z_2^{-1})} + \frac{N_1(z_1^{-1},z_2^{-1})}{p((z_1z_2)^{-1})\ p(z_1^{-1})}$$

so that s belongs to $(z_1z_2)\ K^{real}[[z_1,z_2]]$.

<u>Remark</u>. The systems $(\sum_o, q_o)$ with $F_o = 0$ (homogeneous case) reduce to trivial cases when the bilinearity conditions are imposed. In fact the generating series s_o results to be composed by at most two monomials.

Appendix

This appendix is devoted to the investigation of the structural properties of the series $s^{(o)}$, $s^{(1)}$, $s^{(2)}$ associated with the maps f_o, f_1, f_2 introduced in Section 8.

It is clear that the series operator $s^{(o)}$ associated with $(\sum_{B_o}, x(0) = 0)$ is a rational series in $(z_1 z_2)$. The structural properties of $s^{(1)}$ and $s^{(2)}$ are summarized in the following Propositions.

Proposition A.1. There exists a zero-state realization of a bilinear i/o map f_1 which has structure $\sum_{B_1} = (F_o, F_1, 0, 0, 0, G_2, H)$ iff the series $s^{(1)}$ associated with f_1 satisfies $s^{(1)} = N/p\,\omega_2$, $N \in K[z_1^{-1}, z_2^{-1}]$, $p \in K[z_1^{-1} z_2^{-1}]$, $\omega_2 \in K[z_2^{-1}]$; $\deg_{(z_1 z_2)^{-1}} N < \deg p$,

$\deg_{z_2^{-1}} N < \deg \omega_2$.

proof. Assume $(\sum_{B_1}, x^*(0)=0)$ realizes a bilinear i/o map f_1. Then for any $u_1 \in K[z_1]$, $u_2 \in K[z_2]$ one has:

$$y^{(1)}(t+1) = \sum_{\substack{i \\ r+s+1=t-i}}^{t-1} H F_o^r F_1 F_o^s G_2 u_1(i+s+1) u_2(i)$$

or

$$\text{(A.1)} \quad y^{(1)}(t+1) = \sum_{2}^{t+1}{}_k \sum_{1}^{k-1}{}_h H F_o^{h-1} F_1 F_o^{k-h-1} G_2 u_1(t+1-h) u_2(t+1-k)$$

Let $s^{(1)} = \sum_{ij} s_{ij}^{(1)} z_1^i z_2^j$ be the series associated with f_1. Then it results also:

$$\text{(A.2)} \quad y(t+1) = \sum_{1}^{t+1}{}_{h,k} s_{hk}^{(1)} u_1(t+1-h) u_2(t+1-k)$$

Comparing (A.1) with (A.2) the following properties of the se-

ries operator $s^{(1)}$ appear evident:

$$s^{(1)}_{h,i+h} = H\, F_o^{h-1}\, F_1\, F_o^{i-1}\, G_2 \quad , \; i \geq 1, h \geq 1$$

$$s^{(1)}_{h,k} = 0 \; , \; h > k-1 \quad , \; k = 0,1,\ldots\ldots$$

The series operator $s^{(1)}$ can then be written as

$$s^{(1)} = \sum_{1}^{\infty}{}_{h,k}\, s^{(1)}_{h,k}\, z_1^{h}\, z_2^{k} = \sum_{k>h}{}_{h,k}\, s^{(1)}_{h,k}\, (z_1 z_2)^{h}\, z_2^{k-h} =$$

$$= \sum_{1}^{\infty}{}_{h,i}\, s^{(1)}_{h,i+h}\, (z_1 z_2)^{h}\, z_2^{i} =$$

$$= \sum_{1}^{\infty}{}_{h,i}\, H\, F_0^{h-1}\, F_1\, F_0^{i-1}\, G_2\, (z_1 z_2)^{h}\, z_2^{i} =$$

$$= (z_1 z_2)\, z_2 \sum_{1}^{\infty}{}_{h,i}\, H\, F_0^{h-1}\, F_1\, F_0^{i-1}\, G_2\, (z_1 z_2)^{h-1}\, z_2^{i-1} =$$

$$= (z_1 z_2)\, z_2 \sum_{0}^{\infty}{}_{k,j}\, H\, F_0^{k}\, F_1\, F_0^{j}\, G_2\, (z_1 z_2)^{k}\, z_2^{j} =$$

$$= (z_1 z_2)\, z_2\, H\, \frac{1}{I - F_0 (z_1 z_2)}\, F_1\, \frac{1}{I - F_0 z_2}\, G_2 =$$

$$= H\, \frac{1}{(z_1 z_2)^{-1} I - F_0}\, F_1\, \frac{1}{z_2^{-1} I - F_0}\, G_2$$

Consequently $s^{(1)}$ is the series expansion of a rational function of the form $N(z_1^{-1},z_2^{-1}) \,/\, p((z_1z_2)^{-1})\, p(z_2^{-1})$,where if $N(z_1^{-1},z_2^{-1})$ is considered as $N((z_1z_2)^{-1},z_2^{-1})$ (and there is a unique way to do that), $\deg\limits_{(z_1z_2)^{-1}} N < \deg p$, $\deg\limits_{z_2^{-1}} N < \deg p$.

Conversely let $s^{(1)} = N \,/\, p((z_1z_2)^{-1})\, \omega_2(z_2^{-1})$ with $\deg\limits_{(z_1z_2)^{-1}} N < \deg p = n$, $\deg\limits_{z_2^{-1}} \mathcal{N} < \deg \omega_2 = m$. Then

$$\frac{N(z_1^{-1},z_2^{-1})}{p((z_1z_2)^{-1})\,\omega_2(z_2^{-1})} = [0 \;\cdots\; 0\; 1]\; \frac{1}{(z_1z_2)^{-1}I - \begin{bmatrix} 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & & 0 & -p_1 \\ 0 & 1 & & & \vdots \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & -p_{n-1} \end{bmatrix}}$$

$$\cdot\; T\; \frac{1}{z_2^{-1}I - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \\ -\omega_0 & & \cdots & -\omega_{m-1} \end{bmatrix}} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

where T is the matrix associated with the coefficients of $N((z_1z_2)^{-1},z_2^{-1})$.

A zero-state realization is then:

$$\Sigma = (\ \mathrm{comp}^T(p), 0, \mathrm{comp}(\omega_2), 0, T, 0, 0, [0 \ldots\ldots 1]^T, [0 \ldots\ldots 1]\ )$$

which has the same i/o map as the following Σ_{B_1} system:

$$x(t+) = \begin{bmatrix} \mathrm{comp}^T(p) & 0 \\ 0 & \mathrm{comp}(\omega_2) \end{bmatrix} x(t) + \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} x(t)\, u_1(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u_2(t)$$

$$y(t) = [0 \cdots 1 \,\vdots\, 0 \cdots 0]\ x(t)$$

<u>Proposition A.2</u>. There exists a zero-state realization of a bilinear i/o map f_2 which has structure $\Sigma_{B_2} = (F_o, 0, F_2, 0, G_1, 0, H)$ iff the series $s^{(2)}$ associated with f_2 satisfies $s^{(2)} = N/p\ \omega_1$, $N \in K[z_1^{-1}, z_2^{-1}]$, $p \in K[z_1^{-1}\ z_2^{-1}]$, $\omega_1 \in K[z_1^{-1}]$; $\deg_{(z_1 z_2)^{-1}} N < \deg p$, $\deg_{z_1^{-1}} N < \deg \omega_1$.

The proof is analogous to Proposition A.1.

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