ON SOME CONNECTIONS BETWEEN BILINEAR INPUT/OUTPUT MAPS AND 2D SYSTEMS

ETTORE FORNASINI and MARIA ELENA VALCHER
Dip. di Elettronica ed Informatica, Univ. di Padova,
via Gradenigo 6a, 35131 Padova, Italy

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1. INTRODUCTION

In the current literature by "bilinear system" one generally denotes a state space model where the state updating involves products of input and state variables [1,2]. These systems have been extensively analysed both in the continuous and in the discrete case, and several results are presently available, both clarifying their internal structure and suggesting how to design dynamic controllers. In this contribution, we assume a different point of view and consider (discrete time scalar-inputs/scalar-output) bilinear systems in input/output form. More precisely, we assume that the input and output sequences, which are defined on $\mathbb{N}$, take values over the real field $\mathbb{R}$, and the output y causally depends on the inputs u_1 and u_2 according to a map $f(\cdot,\cdot)$ satisfying the following properties:

$$\begin{array}{ccc} f(\alpha u_1, u_2) = \alpha f(u_1, u_2) & & f(u_1, \beta u_2) = \beta f(u_1, u_2) \\ f(u_1 + v_1, u_2) & = & f(u_1, u_2) + f(v_1, u_2) \\ f(u_1, u_2 + v_2) & = & f(u_1, u_2) + f(u_1, v_2), \end{array}$$

where α and β are in $\mathbb{R}$ and u_1, u_2, v_1 and v_2 arbitrary real sequences.
In this setting it is convenient to represent system signals and input/output maps by means of formal power series in suitable indeterminates. Throughout the paper, the rings of formal power series in a single indeterminate z and in two indeterminates z_1 and z_2 will be denoted by $\mathbb{R}[[z]]$ and $\mathbb{R}[[z_1, z_2]]$, respectively. Accordingly, $\mathbb{R}[[z_1 \cdot z_2]]$ will be the ring of formal power series in the "product indeterminate" $z = z_1 \cdot z_2$. Moreover, the Hadamard product of two series $r(z_1, z_2) = \sum_{i,j\in\mathbb{N}} r_{ij} z_1^i z_2^j$ and $s(z_1, z_2) = \sum_{i,j\in\mathbb{N}} s_{ij} z_1^i z_2^j$ is defined as follows

$$r(z_1, z_2) \odot s(z_1, z_2) := \sum_{i,j\in\mathbb{N}} (r_{ij} s_{ij}) z_1^i z_2^j$$

If we associate with two input sequences $\{u_1(t)\}_{t\in\mathbb{N}}$ and $\{u_2(t)\}_{t\in\mathbb{N}}$ the series

$$U_1(z_1) := \sum_{t=0}^{+\infty} u_1(t) z_1^t \qquad \text{and} \qquad U_2(z_2) := \sum_{t=0}^{+\infty} u_2(t) z_2^t,$$

any discrete bilinear input/output (i/o) map $f : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}} : (u_1, u_2) \mapsto y$ can be described as follows [3]

$$Y(z_1 \cdot z_2) = \Big(F(z_1, z_2)U_1(z_1)U_2(z_2)\Big) \odot \sum_{t=0}^{+\infty}(z_1 \cdot z_2)^t, \tag{1}$$

where $Y(z_1 \cdot z_2) := \sum_t y(t)(z_1 \cdot z_2)^t$ is the formal power series associated with the output y and $F(z_1, z_2)$ is a suitable power series in z_1 and z_2 that characterizes the map f.
From a computational point of view this amounts to saying that the output sequence $\{y(t)\}_{t\in\mathbb{N}}$ coincides with the sequence of "diagonal coefficients" of the two-dimensional series

$$Z(z_1, z_2) := F(z_1, z_2)U_1(z_1)U_2(z_2),$$

which is the output generated by the 2D system with transfer function $F(z_1, z_2)$, when excited by the input $U_1(z_1)U_2(z_2)$. In other words, for every $t \in \mathbb{N}$, $y(t)$ coincides with the diagonal element $z(t, t)$.
Notice that every diagonal $\{f(t, t+h)\}_t$ of the series expansion of $F(z_1, z_2)$, $h \in \mathbb{N}$, can be identified as the output of (1) corresponding to the canonical inputs $U_1(z_1) = z_1^h$ and $U_2(z_2) = 1$, and similarly $\{f(t+h, t)\}_t$, $h \in \mathbb{N}$, is the output sequence produced by $U_1(z_1) = 1$ and $U_2(z_2) = z_2^h$. In the sequel, as no confusion can arise, the symbol "$\cdot$" appearing in the product of z_1 and z_2 will be omitted.
A bilinear map (1) admits a (causal) finite-dimensional state-space realization if and only if [3] $F(z_1, z_2)$ is a rational power series with the following structure

$$F(z_1, z_2) = \frac{n(z_1, z_2)}{h_0(z_1z_2)h_1(z_1)h_2(z_2)}, \tag{2}$$

where $h_0(z_1z_2), h_1(z_1)$ and $h_2(z_2)$ are 1D polynomials with nonzero constant terms, while $n(z_1, z_2)$ is a 2D polynomial. *Throughout the paper we will assume, without loss of generality, that $h_0(z_1z_2)$, $h_1(z_1)$ and $h_2(z_2)$ have unitary constant terms and no common factor with $n(z_1, z_2)$.*

The aim of this contribution is to discuss some basic issues connected with bilinear i/o maps as described by (1) and (2). We consider, first, stability problems and derive necessary and sufficient conditions guaranteeing that bounded inputs always produce bounded outputs (BIBO stability). These conditions refer to the singularities of $F(z_1, z_2)$ (i.e. to the zeros of the polynomials h_0, h_1 and h_2) and enlighten some interesting connections between BIBO stability of 2D transfer functions and that of bilinear i/o maps.
We analyse, next, the free system dynamics, namely the output evolutions determined by finite support input sequences. As we shall see, these evolutions can be expressed as linear combinations of elementary modes associated with the zeros of $h_0(z)$ and with the products of zeros of $h_1(z)$ and of $h_2(z)$.

Finally, assuming BIBO stability, we consider the asymptotic behavior of the output y corresponding to periodic inputs u_1 and u_2. It turns out that, except for very special cases, y is eventually periodic and its permanent component can be determined by resorting to suitable diophantine equations.

2. BIBO STABILITY

Consider the bilinear i/o map

$$\begin{aligned} f \; &: \; \mathbb{R}[[z_1]] \times \mathbb{R}[[z_2]] \to \mathbb{R}[[z_1z_2]] \\ &: \; (U_1(z_1), U_2(z_2)) \mapsto \left(\frac{n(z_1, z_2)U_1(z_1)U_2(z_2)}{h_0(z_1z_2)h_1(z_1)h_2(z_2)}\right) \odot \sum_{t=0}^{\infty}(z_1z_2)^t. \end{aligned} \tag{3}$$

The map (3) is said to be *bounded input-bounded output* (BIBO) *stable* if for every pair of series, $U_1(z_1)$ and $U_2(z_2)$, with bounded coefficients, $f(U_1, U_2)$ has bounded coefficients, too. BIBO stability of (3) is strictly related to the polar structure of $F(z_1, z_2) = n(z_1, z_2)/[h_0(z_1 z_2)h_1(z_1)h_2(z_2)]$, and, furthermore, it turns out that the above map is BIBO stable if and only if the 2D i/o map associated with $F(z_1, z_2)$ is endowed with this property.

Proposition 1 *The bilinear map (3) is BIBO stable if and only if $h_0(z), h_1(z)$ and $h_2(z)$ have no zero in the closed unit disk $D_1 := \{z \in \mathbb{C} : |z| \leq 1\}$.*

Proof If $h_0(z), h_1(z)$ and $h_2(z)$ have no zero in D_1, the 2D transfer function $F(z_1, z_2)$ has no singularities in the closed unit polydisk $D_2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\}$ and hence is BIBO stable [4]. Consequently, every pair of bounded inputs $(U_1(z_1), U_2(z_2))$ produces a 2D bounded output $F(z_1, z_2)U_1(z_1)U_2(z_2)$, whose diagonal, i.e., the output y, is obviously bounded.

Suppose, now, that (3) is BIBO stable.

- We prove, first, that $h_2(z)$ has no zero in D_1. If not, we would have $(1 - \alpha z_2) \mid h_2(z_2)$, for some $\alpha \in \mathbb{C}$, $|\alpha| \geq 1$. Consider the following power series

$$\begin{aligned} U_1(z_1) &= h_1(z_1)\bar{U}_1(z_1) \\ U_2(z_2) &= \begin{cases} 1 & \text{if } |\alpha| > 1, \\ \dfrac{1}{1 - \alpha z_2} & \text{if } \alpha = \pm 1 \\ \dfrac{1}{(1 - \alpha z_2)(1 - \alpha^* z_2)} & \text{if } \alpha \in \mathbb{C} \setminus \mathbb{R} \text{ and } |\alpha| = 1 \end{cases}, \end{aligned} \tag{4}$$

where α^* is the conjugate of α and $\bar{U}_1(z_1)$ is a series, with bounded coefficients, to be determined. Both $U_1(z_1)$ and $U_2(z_2)$ have bounded coefficients and the corresponding output is given by

$$Y(z_1 z_2) = \left(\frac{n(z_1, z_2)}{h_0(z_1 z_2)} \bar{U}_1(z_1) \frac{U_2(z_2)}{h_2(z_2)} \right) \odot \sum_{t=0}^{\infty} (z_1 z_2)^t. \tag{5}$$

If we express $n(z_1, z_2)$ as

$$n(z_1, z_2) = z_1^{T-1}\, \tilde{n}_{T-1}(z_2) + z_1^{T-2}\, \tilde{n}_{T-2}(z_2) + \ldots + \tilde{n}_0(z_2), \tag{6}$$

with $T \in \mathbb{N}$, $\tilde{n}_i(z_2) \in \mathbb{R}[z_2]$, $i = 0, 1, \ldots, T-1$, by the coprimality assumption on the pair $(n(z_1, z_2), h_2(z_2))$, it follows that there exists j such that $(1 - \alpha z_2)$ does not divide $\tilde{n}_j(z_2)$. Consequently, the coefficients of the power series expansion of

$$W(z_2) := \frac{\tilde{n}_j(z_2)U_2(z_2)}{h_2(z_2)}$$

constitute an unbounded sequence, and one at least of the subsequences

$$\{w(kT)\}_k,\ \{w(kT+1)\}_k,\ \ldots,\ \{w(kT+T-1)\}_k$$

diverges. So, there exists $P \in \mathbb{N}$ such that

$$\left(\frac{z_1^P}{1 - z_1^T} \frac{n(z_1, z_2)U_2(z_2)}{h_2(z_2)} \right) \odot \sum_t (z_1 z_2)^t$$

corresponds to an unbounded sequence. It is now clear that, if we assume in (4)

$$\bar{U}_1(z_1) := \frac{z_1^P}{1 - z_1^T},$$

the output series in (5) represents an unbounded output.

- The proof that $h_1(z)$ has no zero in D_1 follows the same lines.
- To show that $h_0(z)$ has no zero in D_1, suppose, by contradiction, that there exists $\alpha \in \mathbb{C}$, $|\alpha| \geq 1$, such that $(1-\alpha z) \mid h_0(z)$ and express $n(z_1, z_2)$ as follows

$$n(z_1, z_2) = z_1^N\ n_N(z_1 z_2) + z_1^{N-1}\ n_{N-1}(z_1 z_2) + \ldots + n_0(z_1 z_2) + \ldots + z_2^M\ n_{-M}(z_1 z_2), \tag{7}$$

with $n_i(z_1 z_2) \in \mathbb{R}[z_1 z_2]$ and $N, M \in \mathbb{N}$. As $n(z_1, z_2)$ and $h_0(z_1 z_2)$ have no common factor, there exists $i \in \{-M, -M+1, \ldots, N\}$ such that $n_i(z_1 z_2)$ is not a multiple of $1 - \alpha z_1 z_2$.
If $|\alpha| > 1$ or $|\alpha| = 1$ and its multiplicity is greater than 1, the power series associated with $n_i(z_1 z_2)/h_0(z_1 z_2)$ corresponds to an unbounded sequence, and the output sequence corresponding to the bounded inputs

$$\begin{cases} U_1(z_1) &= h_1(z_1) \\ U_2(z_2) &= h_2(z_2) z_2^i \end{cases} \quad \text{for } i \geq 0, \qquad \text{or} \quad \begin{cases} U_1(z_1) &= h_1(z_1) z_1^{-i} \\ U_2(z_2) &= h_2(z_2) \end{cases} \quad \text{for } i < 0$$

is unbounded.
Assume, now, that the only zeros of $h_0(z)$ in D_1 are simple and of unitary modulus, and let $1/\alpha$ be one of them. By the coprimality of $n(z_1, z_2)$ and $h_0(z_1 z_2)$, it follows that there exists j such that $(1 - \alpha z_1 z_2)$ does not divide $n_j(z_1 z_2)$. Consequently, the coefficients of the series

$$\sum_t w(t)(z_1 z_2)^t = \frac{n_j(z_1 z_2)}{h_0(z_1 z_2)}$$

do not constitute an ℓ_1 sequence.
Set $T := N + M + 1$ and assume, for instance, $j \geq 0$. If we consider the pair of (possibly complex) bounded inputs associated with

$$U_1(z_1) = \frac{h_1(z_1)}{1 - (\alpha z_1)^T} \qquad U_2(z_2) = \frac{h_2(z_2) z_2^j}{1 - z_2^T}, \tag{8}$$

we have that the series

$$\begin{aligned} Y(z_1 z_2) &= \left(\frac{n(z_1, z_2)}{h_0(z_1 z_2) h_1(z_1) h_2(z_2)}\ \frac{h_1(z_1)}{1 - (\alpha z_1)^T}\ \frac{h_2(z_2) z_2^j}{1 - z_2^T} \right) \odot \sum_t (z_1 z_2)^t \\ &= \left(\frac{n(z_1, z_2)}{h_0(z_1 z_2)}\ \frac{1}{1 - (\alpha z_1)^T}\ \frac{z_2^j}{1 - z_2^T} \right) \odot \sum_t (z_1 z_2)^t \\ &= \left(\frac{n_j(z_1, z_2) z_1^j z_2^j}{h_0(z_1 z_2)}\ \frac{1}{1 - (\alpha z_1)^T}\ \frac{1}{1 - z_2^T} \right) \odot \sum_t (z_1 z_2)^t = \frac{n_j(z_1, z_2) z_1^j z_2^j}{h_0(z_1 z_2)[1 - (\alpha z_1 z_2)^T]} \end{aligned}$$

represents an unbounded output. If α belongs to $\mathbb{R}$ we have obtained in this way a pair of real valued bounded inputs producing an unbounded real output. If α is complex, it is sufficient to assume in (8)

$$U_1(z_1) = \frac{h_1(z_1)}{[1 - (\alpha z_1)^T][1 - (\alpha^* z_1)^T]},$$

and proceed as before. ■

Remark Given a 2D transfer function $F(z_1, z_2) = n(z_1, z_2)/h(z_1, z_2)$, with $n(z_1, z_2)$ and $h(z_1, z_2)$ factor coprime, if $h(z_1, z_2)$ is devoid of zeros in D_2 the function $F(z_1, z_2)$ is 2D BIBO stable. As it has been proved by D.Goodman [4], however, the converse is not true. Indeed, $F(z_1, z_2)$ can be BIBO stable even if it has nonessential singularities of the second kind on the distinguished boundary $T_2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = 1, |z_2| = 1\}$ of D_2, namely if $n(\alpha, \beta) = h(\alpha, \beta) = 0$ for some $(\alpha, \beta) \in T_2$.

When restricting ourselves to the class of 2D transfer functions described as in (2), namely to rational functions which are adopted for representing i/o bilinear maps, 2D BIBO stability is equivalent to assuming that $h_0(z), h_1(z)$ and $h_2(z)$ have no zeros in the closed unit disk D_1. In fact, if $h_1(\alpha) = 0$, $|\alpha| = 1$, then (α, β) is a zero of the denominator of $F(z_1, z_2)$ for every $\beta \in \mathbb{C}$ and hence $F(z_1, z_2)$ has nonessential singularities in $D_2 \setminus T_2$, which rules out 2D BIBO stability. The same reasoning obviously applies to $h_2(z)$. On the other hand, if $h_0(\alpha) = 0$, $|\alpha| = 1$, then all pairs $(\alpha e^{i\theta}, e^{-i\theta}) \in T_2$, $\theta \in \mathbb{R}$, are zeros of $h_0(z_1 z_2)$, viewed as a 2D polynomial. As $n(z_1, z_2)$ and $h_0(z_1 z_2)$ are factor coprime, and hence have a finite number of common zeros, one at least of the pairs $(\alpha e^{i\theta}, e^{-i\theta})$ is a pole of $F(z_1, z_2)$.

Consequently, the rational function (2) is 2D BIBO stable if and only if it represents a BIBO stable bilinear i/o map.

3. MODAL STRUCTURE

Once a bilinear i/o map f is given, one can exploit shift-invariance to extend f to pairs of inputs with left compact supports, or, equivalently, to pairs of Laurent series. If we consider the space $\mathcal{U}_1 \times \mathcal{U}_2$ of all pairs of inputs whose supports are finite subsets of $(-\infty, 0]$, i.e. the set of trajectories whose power series are elements of $\mathbb{R}[z_1^{-1}] \times \mathbb{R}[z_2^{-1}]$, we can introduce the Nerode equivalence as follows. Two elements (u_1, u_2) and (v_1, v_2) of $\mathcal{U}_1 \times \mathcal{U}_2$ are said to be *Nerode equivalent* if for every positive integer N and every pair (w_1, w_2), whose support is included in $[1, N]$, the output sequences $f(u_1 + w_1, u_2 + w_2)$ and $f(v_1 + w_1, v_2 + w_2)$ coincide in $[N + 1, +\infty)$. The classes induced by this equivalence relation are naturally viewed as the states of a "canonical realization", and we can identify the outputs of (3) corresponding to finite support input sequences as "free evolutions" corresponding to initial states of the canonical realization.

Let $F(z_1, z_2)$ be given as in (2) and suppose that $h_0(z), h_1(z)$ and $h_2(z)$ factorize over the complex field in the following way

$$h_0(z) = \prod_{i=1}^{r_0} (1 - \gamma_i z)^{\rho_i} \qquad h_1(z) = \prod_{i=1}^{r_1} (1 - \alpha_i z)^{\mu_i} \qquad h_2(z) = \prod_{i=1}^{r_2} (1 - \beta_i z)^{\nu_i}, \tag{9}$$

with $\alpha_i \neq \alpha_j, \beta_i \neq \beta_j, \gamma_i \neq \gamma_j$ for $i \neq j$, and $r_0, r_1, r_2, \rho_i, \mu_i, \nu_i$ positive integers. We aim to explicitly relate the elementary modes describing the free evolution of (3) to the parameters α_i, β_i and γ_i appearing in (9). To reach this goal, we need a couple of technical lemmas.

Lemma 2 *For every choice of the nonnegative integers n, p and q the following identity holds*

$$\binom{n+p}{p}\binom{n+q}{q} = \sum_{k=0}^{p \wedge q} (-1)^k \binom{q}{k}\binom{p+q-k}{q}\binom{n+p+q-k}{p+q-k}. \tag{10}$$

PROOF Rewrite the Vandermonde convolutional formula [6]

$$\binom{x}{y} = \sum_{k=0} \binom{x-q}{y-k}\binom{q}{k}$$

in the alternate form

$$\binom{x}{y} = \sum_{k=0}(-1)^k \binom{x+q-k}{y-k}\binom{q}{k}, \tag{11}$$

and note that with x replaced by $n+p$ and y by p, (11) becomes

$$\binom{n+p}{p} = \sum_{k=0}(-1)^k \binom{q}{k}\binom{n+p+q-k}{p-k}.$$

Consequently, we have

$$\binom{n+p}{p}\binom{n+q}{q} = \sum_{k=0}(-1)^k \binom{q}{k}\binom{n+p+q-k}{p-k}\binom{n+q}{q}. \tag{12}$$

It is a matter of straightforward computation to check the identity

$$\binom{n+p+q-k}{p-k}\binom{n+q}{q} = \binom{p+q-k}{q}\binom{n+p+q-k}{p+q-k}, \tag{13}$$

and the proof of the lemma is complete upon replacing (13) in (12). ∎

As it is well-known, the power series expansions of the rational functions $1/(1-\delta z)^{k+1}, k \in \mathbb{N}$, are given by

$$\frac{1}{(1-\delta z)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} (\delta z)^n. \tag{14}$$

The following lemma shows that the Hadamard product of $\frac{1}{(1-\alpha z)^{p+1}}$ and $\frac{1}{(1-\beta z)^{q+1}}$ is a linear combination of power series expansions with similar structure, involving the powers of $\alpha\beta$.

Lemma 3 *For every α and β in $\mathbb{C}$ and every pair (p, q) of nonnegative integers, we have*

$$\frac{1}{(1-\alpha z)^{p+1}} \odot \frac{1}{(1-\beta z)^{q+1}} = \sum_{k=0}^{p\wedge q}(-1)^k \binom{q}{k}\binom{p+q-k}{q}\frac{1}{(1-\alpha\beta z)^{p+q-k+1}}, \tag{15}$$

or, in more compact form,

$$\frac{1}{(1-\alpha z)^{p+1}} \odot \frac{1}{(1-\beta z)^{q+1}} = \frac{1}{q!}\frac{d^q}{dz^q}\left[\frac{z^q}{(1-\alpha\beta z)^{p+1}}\right] = \frac{1}{p!}\frac{d^p}{dz^p}\left[\frac{z^p}{(1-\alpha\beta z)^{q+1}}\right]. \tag{16}$$

Proof By applying the previous lemma and the power series expansion in (14), we get

$$\begin{aligned}
\frac{1}{(1-\alpha z)^{p+1}} \odot \frac{1}{(1-\beta z)^{q+1}} &= \left(\sum_{n=0}^{\infty}\binom{n+p}{p}(\alpha z)^n\right) \odot \left(\sum_{n=0}^{\infty}\binom{n+q}{q}(\beta z)^n\right) \\
&= \sum_{n=0}^{\infty}\binom{n+p}{p}\binom{n+q}{q}(\alpha\beta z)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{p\wedge q}(-1)^k\binom{q}{k}\binom{p+q-k}{q}\binom{p+q+n-k}{p+q-k}(\alpha\beta z)^n \\
&= \sum_{k=0}^{p\wedge q}(-1)^k\binom{q}{k}\binom{p+q-k}{q}\sum_{n=0}^{\infty}\binom{p+q+n-k}{p+q-k}(\alpha\beta z)^n \\
&= \sum_{k=0}^{p\wedge q}(-1)^k\binom{q}{k}\binom{p+q-k}{q}\frac{1}{(1-\alpha\beta z)^{p+q-k+1}}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\sum_{k=0}^{p\wedge q}(-1)^k\binom{q}{k}\binom{p+q-k}{q}\frac{1}{(1-\alpha\beta z)^{p+q-k+1}} = \\
&= \frac{1}{q!}\sum_{k=0}^{p\wedge q}(-1)^k\binom{q}{k}\frac{(p-k+1)(p-k+2)\dots(p-k+q)}{(1-\alpha\beta z)^{p+q-k+1}} \\
&= \frac{1}{(\alpha\beta)^q q!}\,\frac{d^q}{dz^q}\left[\sum_{k=0}^{p\wedge q}(-1)^k\binom{q}{k}\frac{1}{(1-\alpha\beta z)^{p-k+1}}\right] \\
&= \frac{1}{(\alpha\beta)^q q!}\,\frac{d^q}{dz^q}\left[\frac{1}{(1-\alpha\beta z)^{p+1}}\sum_{k=0}^{p\wedge q}\binom{q}{k}(\alpha\beta z-1)^k\right]. \qquad (17)
\end{aligned}
$$

If $q = p \wedge q$, the summation in (17) gives $(\alpha\beta z)^q$, thus proving (16); otherwise, when $p = p \wedge q$, then

$$
\frac{1}{(1-\alpha\beta z)^{p+1}}\sum_{k=0}^{p\wedge q}\binom{q}{k}(\alpha\beta z-1)^k = \frac{(\alpha\beta z)^q}{(1-\alpha\beta z)^{p+1}} + m(z),
$$

where $m(z) \in \mathbb{R}[z]$ is a polynomial of degree smaller than q. So, (16) holds also in this case. ■

The above lemma has a significant system theoretic interpretation. Indeed, given a linear system whose transfer function has a pole in $1/\delta$, the coefficients of the power series expansion in (14) can be regarded as describing a mode associated with the pole. So, the identities (15) and (16) clarify that the Hadamard product of the modes corresponding to the poles $1/\alpha$ and $1/\beta$, is a combination of modes corresponding to a pole in $1/(\alpha\beta)$.
This result is extremely useful for analyzing the free evolution of the bilinear model (3). Actually, assume that $U_1(z_1)$ and $U_2(z_2)$ are polynomial inputs and consider the corresponding output series

$$
Y(z_1,z_2) = \frac{1}{h_0(z_1z_2)}\left(\frac{n(z_1,z_2)U_1(z_1)U_2(z_2)}{h_1(z_1)h_2(z_2)} \odot \sum_t (z_1z_2)^t\right).
$$

Set $n(z_1,z_2) := \sum_{i,j} n_{ij} z_1^i z_2^j$. By resorting to partial fraction expansions and to the factorizations of h_0, h_1 and h_2 given in (9), we get

$$
\begin{aligned}
\frac{n(z_1,z_2)U_1(z_1)U_2(z_2)}{h_1(z_1)h_2(z_2)} &= \sum_{i,j} n_{ij}\frac{U_1(z_1)z_1^i}{h_1(z_1)}\,\frac{U_2(z_2)z_2^j}{h_2(z_2)} \\
&= \sum_{i,j} n_{ij}\left[p_{1i}(z_1) + \sum_{r,s}\frac{\rho_{irs}}{(1-\alpha_r z_1)^s}\right]\left[p_{2j}(z_2) + \sum_{v,t}\frac{\tau_{jvt}}{(1-\beta_v z_2)^t}\right],
\end{aligned}
$$

with $p_{1i}(z_1) \in \mathbb{R}[z_1]$ and $p_{2j}(z_2) \in \mathbb{R}[z_2]$. It is not difficult to check that the Hadamard product of $\sum_t (z_1z_2)^t$ with the above expression gives

$$
\frac{n(z_1,z_2)U_1(z_1)U_2(z_2)}{h_1(z_1)h_2(z_2)} \odot \sum_t (z_1z_2)^t = p(z_1z_2) + \left[\sum_{r,s,v,t}\frac{n_{ij}\rho_{irs}\tau_{jvt}}{(1-\alpha_r z_1)^s(1-\beta_v z_2)^t}\right] \odot \sum_t (z_1z_2)^t,
$$

with $p(z_1z_2) \in \mathbb{R}[z_1z_2]$. As

$$
\frac{1}{(1-\alpha_r z_1)^s(1-\beta_v z_2)^t} \odot \sum_t (z_1z_2)^t \equiv \left.\frac{1}{(1-\alpha_r z)^s} \odot \frac{1}{(1-\beta_v z)^t}\right|_{z=z_1z_2},
$$

Lemma 3 applies and we get

$$Y(z_1, z_2) = \frac{1}{h_0(z_1 z_2)} \left[p(z_1 z_2) + \sum_{r,s,v,t} \frac{c_{rstvk}}{(1 - \alpha_r \beta_v z_1 z_2)^{s+t-1-k}} \right], \tag{18}$$

where $c_{rstvk} := (-1)^k \binom{s-1}{k} \binom{s+t-2-k}{s-1} \sum_{i,j} \rho_{irs} \tau_{jut} n_{ij}$. By expressing (18) as sum of partial fractions, we obtain the output sequence y as a linear combination of elementary modes associated with the poles $1/\gamma_i$ and $1/(\alpha_r \beta_v)$.

Interestingly enough, the case possibly occurs that, even though some zeros of $h_1(z)$ belong to the interior of D_1, all products $1/(\alpha_r \beta_v)$ belong to $\mathbb{C} \setminus D_1$. Under this assumption, when all zeros of $h_0(z)$ are in $\mathbb{C} \setminus D_1$, the bilinear system (3) exhibits only convergent modes, although it is not BIBO stable.

Example 1 Assume in (1)

$$F(z_1, z_2) = \frac{1}{(1 - 1/3\ z_1 z_2)(1 - 2z_1)(1 - 1/4\ z_2)}.$$

This i/o bilinear map is not BIBO stable, because $h_1(z_1) = 1 - 2z_1$ has a zero inside D_1. On the other hand, by resorting to the Euclidean algorithm, every pair of finite support inputs can be written as

$$(U_1(z_1), U_2(z_2)) = \Big(a(z_1)(1 - 2z_1) + c, b(z_2)(1 - 1/4\ z_2) + d \Big),$$

where $a(z_1) = \sum_i a_i z_1^i$, $b(z_2) = \sum_i b_i z_2^i$, and $a_i, b_i, c, d \in \mathbb{R}$. So, the corresponding output $Y(z_1 z_2)$ is given by

$$\begin{aligned} Y(z_1, z_2) &= \frac{1}{1 - 1/3\ z_1 z_2} \left(a(z_1) b(z_2) + \frac{c b(z_2)}{1 - 2z_1} + \frac{d a(z_1)}{1 - 1/4\ z_2} + \frac{cd}{(1 - 2z_1)(1 - 1/4\ z_2)} \right) \odot \sum_t (z_1 z_2)^t \\ &= \frac{1}{1 - 1/3\ z_1 z_2} \left(\sum_i a_i b_i (z_1 z_2)^i + \sum_i c b_i 2^i (z_1 z_2)^i + \sum_i \frac{d a_i}{4^i} (z_1 z_2)^i + \sum_i \frac{cd}{2^i} (z_1 z_2)^i \right), \end{aligned}$$

and hence is always convergent.

4. PERIODIC EVOLUTIONS

This section surveys, rather sketchily, some aspects of bilinear i/o maps which are connected with their limiting behavior, once a periodic excitation is applied. If we assume BIBO stability, it is quite easy to realize that (a part from an exceptional set of periodic inputs) any pair of inputs of period T eventually induces a nonzero output with the same period. (Notice that this assumption is not restrictive, since if $T_1, T_2 \in \mathbb{N}$ are the periods of u_1 and u_2, we can always set $T := \ell.\text{c.m}\{T_1, T_2\}$.) Actually, suppose that $U_1(z_1)$ and $U_2(z_2)$ are given by

$$U_1(z_1) = \frac{p_1(z_1)}{1 - z_1^T}, \qquad U_2(z_2) = \frac{p_2(z_2)}{1 - z_2^T}, \qquad \deg p_1, \deg p_2 < T.$$

If $n(z_1, z_2)$ is expressed as in (7), we have

$$\begin{aligned} Y(z_1z_2) &= \sum_{i=0}^{N} \frac{z_1^i\ n_i(z_1z_2)}{h_0(z_1z_2)}\Big[\frac{p_1(z_1)}{h_1(z_1)(1-z_1^T)}\ \frac{p_2(z_2)}{h_2(z_2)(1-z_2^T)}\Big] \odot \sum_t (z_1z_2)^t \\ &+ \sum_{i=1}^{M} \frac{z_2^i\ n_{-i}(z_1z_2)}{h_0(z_1z_2)}\Big[\frac{p_1(z_1)}{h_1(z_1)(1-z_1^T)}\ \frac{p_2(z_2)}{h_2(z_2)(1-z_2^T)}\Big] \odot \sum_t (z_1z_2)^t \\ &= \sum_{i=0}^{N} \frac{z_1^i\ n_i(z_1z_2)}{h_0(z_1z_2)}\Big[\Big(\frac{v_{10}(z_1)}{(1-z_1^T)} + \frac{t_{10}(z_1)}{h_1(z_1)}\Big)\Big(\frac{z_2^i v_{2i}(z_2)}{(1-z_2^T)} + \frac{t_{2i}(z_2)}{h_2(z_2)}\Big)\Big] \odot \sum_t (z_1z_2)^t \\ &+ \sum_{i=1}^{M} \frac{z_2^i\ n_{-i}(z_1z_2)}{h_0(z_1z_2)}\Big[\Big(\frac{z_1^i v_{1i}(z_1)}{(1-z_1^T)} + \frac{t_{1i}(z_1)}{h_1(z_1)}\Big)\Big(\frac{v_{20}(z_2)}{(1-z_2^T)} + \frac{t_{20}(z_2)}{h_2(z_2)}\Big)\Big] \odot \sum_t (z_1z_2)^t, \end{aligned}$$

where $v_{1i}, v_{2i}, t_{1i}, t_{2i}$ are polynomial solutions of the following diophantine equations [5]

$$\begin{aligned} (1-z_1^T)t_{1i}(z_1) + z_1^i h_1(z_1)v_{1i}(z_1) &= p_1(z_1), \qquad i = 0, 1, \ldots, M, \\ (1-z_2^T)t_{2i}(z_2) + z_2^i h_2(z_2)v_{2i}(z_2) &= p_2(z_2), \qquad i = 0, 1, \ldots, N. \end{aligned}$$

By the BIBO stability assumption, for large values of the time variable the behavior of the output sequence does not depend on the terms t_{1i}/h_1 and t_{2i}/h_2 and, consequently, the power series expansion of

$$\begin{aligned} &\Bigg(\sum_{i=0}^{N} \frac{z_1^i z_2^i\ n_i(z_1z_2)}{h_0(z_1z_2)}\Big[\frac{v_{10}(z_1)}{(1-z_1^T)}\frac{v_{2i}(z_2)}{(1-z_2^T)}\Big] + \sum_{i=1}^{M} \frac{z_1^i z_2^i\ n_{-i}(z_1z_2)}{h_0(z_1z_2)}\Big[\frac{v_{1i}(z_1)}{(1-z_1^T)}\frac{v_{20}(z_2)}{(1-z_2^T)}\Big]\Bigg) \odot \sum_t (z_1z_2)^t \\ = &\frac{1}{h_0(z_1z_2)(1-(z_1z_2)^T)}\{\sum_{i=0}^{N} n_i(z_1z_2)[v_{10}(z_1)v_{2i}(z_2) \odot \sum_t (z_1z_2)^t] \\ + &\sum_{i=1}^{M} n_{-i}(z_1z_2)[v_{1i}(z_1)v_{20}(z_2) \odot \sum_t (z_1z_2)^t]\} \end{aligned}$$

asymptotically fits the actual output of the system. Upon setting

$$p(z) := \sum_{i=0}^{N} n_i(z_1z_2)[v_{10}(z_1)v_{2i}(z_2) \odot \sum_t (z_1z_2)^t] + \sum_{i=1}^{M} n_{-i}(z_1z_2)[v_{1i}(z_1)v_{20}(z_2) \odot \sum_t (z_1z_2)^t]\Bigg|_{z=z_1z_2},$$

the output series can be rewritten as

$$Y(z) = \frac{p(z)}{h_0(z)(1-z^T)} = \frac{t(z)}{h_0(z)} + \frac{v(z)}{1-z^T}, \tag{19}$$

where $(v(z), t(z))$ is a polynomial solution of the diophantine equation

$$(1-z^T)t(z) + h_0(z)v(z) = p(z), \tag{20}$$

satisfying $\deg v < T$. Again, the BIBO stability assumption can be used to show that the term $t(z)/h_0(z)$ decays asymptotically to zero. Thus the output is eventually periodic, and its permanent evolution is given by the expansion of $v(z)/(1-z^T)$. Note that, as $v(z)$ has degree smaller than T, its coefficents give the restriction to a period of the permanent part of the output sequence.

Remark As mentioned at the beginning of the section, the case possibly occurs that a nonzero pair of periodic inputs produces a zero permanent output. This happens when in (19) the polynomial

$p(z)$ is a multiple of $1 - z^T$, and hence, by the BIBO stability assumption, the whole output sequence asymptotically decays to zero.
The following examples enlighten two possible situations when this phaenomenon arises. In the former, the denominators of the periodic inputs simplify with the polynomial $n(z_1, z_2)$; in the latter, the supports of the periodic inputs do not intersect.

Example 2 Consider the rational function

$$F(z_1, z_2) = \frac{1 - (z_1 z_2)^2}{h_0(z_1 z_2)(1 - \alpha z_1)(1 - \beta z_2)},$$

with $\alpha, \beta \in \mathbb{R}$ and all zeros of $h_0(z)$ outside D_1. Corresponding to the pair of periodic inputs

$$\Big(U_1(z_1), U_2(z_2)\Big) = \Big(\frac{1 - \alpha z_1}{1 - z_1^2}, \frac{1 - \beta z_2}{1 - z_2^2}\Big),$$

we get the output series

$$\begin{aligned} Y(z_1, z_2) &= \frac{1 - (z_1 z_2)^2}{h_0(z_1 z_2)(1 - z_1^2)(1 - z_2^2)} \\ &= \frac{1 - (z_1 z_2)^2}{h_0(z_1 z_2)} \left(\frac{1}{(1 - z_1^2)(1 - z_2^2)} \odot \sum_{t=0}^{\infty} (z_1 z_2)^t \right) \\ &= \frac{1 - (z_1 z_2)^2}{h_0(z_1 z_2)} \, \frac{1}{1 - (z_1 z_2)^2} = \frac{1}{h_0(z_1 z_2)}, \end{aligned}$$

thus proving that y asymptotically decays to zero.

Example 2 Consider the rational function

$$F(z_1, z_2) = \frac{1}{h_0(z_1 z_2)},$$

where $h_0(z)$ has all zeros out of D_1. The pair of periodic inputs

$$\Big(U_1(z_1), U_2(z_2)\Big) = \Big(\frac{1 + z_1^2}{1 - z_1^3}, \frac{2 z_2}{1 - z_2^3}\Big),$$

produces the output series

$$Y(z_1, z_2) = \frac{1}{h_0(z_1 z_2)} \left(\frac{1 + z_1^2}{1 - z_1^3} \frac{2 z_2}{1 - z_2^3} \odot \sum_{t=0}^{\infty} (z_1 z_2)^t \right) = 0. \tag{21}$$

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