Duality analysis of 2D convolutional codes

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Abstract **The behavioral approach, developed by J.C.Willems [1] for the analysis of dynamical systems provides an appropriate framework for investigating convolutional codes. In the two-dimensional (2D) case, this approach is very effective, as it allows to avoid apriori assumptions on the ordering of two dimensional data and artificial notions of causality in Z × Z. Moreover, no restriction on the support of the signals is needed, and the only reasonable constraint one might introduce on 2D sequences is that of having finite support, as extra requirements on their shape seem questionable in many practical applications. In this communication we analyse the algebraic structure of 2D convolutional codes over finite fields, and discuss how 2D finite support convolutional codes are related to infinite support codes via algebraic duality.**

1. Finite codes and their encoders

A *2D code of length* n over a finite field $\mathbf{F}$ is a subset of

$$\mathcal{F}_\infty^n := \{ \sum_{h,k \,\in\, \mathbf{Z}} w(h,k) z_1^h z_2^k : w(h,k) \in \mathbf{F}^n \}.$$

In particular, a 2D *finite code* C of length n can be identified with a subset of $\mathcal{F}_\pm^n$, the module of n-dimensional row vectors with entries in the ring of Laurent polynomials $\mathcal{F}_\pm := \mathbf{F}[z_1, z_2, z_1^{-1}, z_2^{-1}]$. Finite codes have a convolutional structure, once endowed with the following mathematical properties:

• LINEARITY and SHIFT INVARIANCE A linear shift-invariant code $\mathcal{C}$ (*modular code*) can be represented as $\mathcal{C} = \mathrm{Im}_\pm G := \{\mathbf{u}G : \mathbf{u} \in \mathcal{F}_\pm^k\}$, for a suitable $k \times n$ polynomial matrix $G(z_1, z_2)$ (the *encoder*).

• INJECTIVITY Different information sequences in $\mathcal{F}_\pm^k$ need not produce different codewords unless G has full row rank over the field of rational functions $\mathbf{F}(z_1, z_2)$. Finite codes which admit injective encoders are free $\mathcal{F}_\pm$-modules, and are called *free modular codes*

• SYNDROME DECODER EXISTENCE The codewords of $\mathcal{C}$ are the solutions of an autoregressive system of equations, or, equivalently, $\mathcal{C} = \ker_\pm H^T := \{\mathbf{w} \in \mathcal{F}_\pm^n : \mathbf{w}H^T(z_1, z_2) = 0\}$, $H^T(z_1, z_2)$ a polynomial matrix (*syndrome decoder*). Each column of H^T provides a *parity check*, which can be applied to a received sequence for testing whether it belongs to the code. Codes having a syndrome decoder are called *(finite) convolutional codes* and are ALSO characterized by the existence of a left factor prime (ℓFP) encoder.

The reliability requirement of preventing catastrophic decoding errors makes it imperative to further restrict the class of codes we consider and to use codes which admit a polynomial decoder (*basic code*). Such a decoder exists if and only if $\mathcal{C}$ admits a left zero prime (ℓZP) encoder. 2D finite basic codes constitute a proper subclass of the convolutional ones, and are equivalently characterized by the existence of a rZP decoder $\bar{H}^T(z_1, z_2)$

As the nested subclasses of modular codes previously introduced are defined in terms of rank and primeness properties of their encoders, an important issue is to decide whether a code $\mathcal{C}$, given through the assignment of an arbitrary encoder $G \in \mathcal{F}_\pm^{k \times n}$ with rank $\bar{k}$ over $\mathbf{F}(z_1, z_2)$, admits an encoder enjoying the aforementioned properties. To answer this question, we first factorize G into $T\bar{G}$, where $\bar{G} \in \mathcal{F}_\pm^{\bar{k} \times n}$ is ℓFP and $T \in \mathcal{F}_\pm^{k \times \bar{k}}$is full column rank. Then $\mathcal{C}$ is
* free modular iff T factorizes into the product $T = \bar{T}L$, $\bar{T}$ rZP and L square nonsingular
* finite convolutional iff T is rZP
* finite basic iff T is rZP and $\bar{G}$ is ℓZP.

Two finite convolutional codes $\mathcal{C}_i := \mathrm{Im}G_i = \ker_\pm H_i^T,\ i = 1, 2$ coincide (and their encoders G_1 and G_2 are *equivalent* iff there exist two full column rank L-polynomial matrices P_1 and P_2

s.t. $P_1G_1 = P_2G_2$ or, equivalently, there exist two full row rank L-polynomial matrices Q_1 and Q_2 such that $H_1^TQ_1 = H_2^TQ_2$.

2. Infinite codes and duality

The connections between finite and infinite codes are very clear in the general framework of algebraic duality. Every complete (infinite) code can be seen as the set of all parity checks that can be applied to decide whether an arbitrary sequence of $\mathcal{F}_\pm^n$ belongs to a modular code $\mathcal{C}$. From an algebraic point of view, this requires to identify $\mathcal{F}_\infty^n$ with the space of the linear functionals on $\mathcal{F}_\pm^n$ and to regard an infinite convolutional code as the algebraic dual of some modular code. In particular, if $\mathcal{C}$ is a modular code with encoder G, we have

$$\mathrm{Im}_\pm G = (\ker G^T)^\perp \quad \text{and} \quad \ker G^T = (\mathrm{Im}_\pm G)^\perp,$$

where $\ker G^T = \{\mathbf{v} \in \mathcal{F}_\infty^n : \mathbf{v}G^T = 0\} =: \mathcal{D}$ represents the set of all linear functions we are allowed to apply when deciding whether $\mathbf{w} \in \mathcal{F}_\pm^n$ belongs to $\mathcal{C}$. $\mathcal{D}$ is called the *dual code* of $\mathcal{C}$. The above relations induce a bijection between modular codes and those $\mathcal{F}_\pm$-submodules of $\mathcal{F}_\infty^n$ that can be described as kernels of polynomial operators. As $\mathcal{D}$ includes infinite support sequences, which cannot provide feasible parity checks, it is interesting to determine when the submodule $\mathcal{D}_f := \mathcal{D} \cap \mathcal{F}_\pm^n$ constitutes a set of parity checks sufficient to decide whether $\mathbf{w}$ is in $\mathcal{C}$. This happens if and only if $\mathcal{C}$ is convolutional.

Finally, the algebraic duality leads to a characterization of finite basic codes as modular codes whose duals can be described as kernels of right zero prime matrices.

References

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