# On the spectral and combinatorial structure of 2D positive systems

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**Abstract**

The dynamics of a 2D positive system depends on the pair of nonnegative square matrices that provide the updating of its local states. In this paper, several spectral properties, like finite memory, separability and property L, which depend on the characteristic polynomial of the pair, are investigated under the nonnegativity constraint and in connection with the combinatorial structure of the matrices.
Some aspects of the Perron-Frobenius theory are extended to the 2D case; in particular, conditions are provided guaranteeing the existence of a common maximal eigenvector for two nonnegative matrices with irreducible sum. Finally, some results on 2D positive realizations are presented.

**Running title**: 2D positive systems

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# 1 Introduction

A (1D) discrete time linear system

$$\begin{aligned} \mathbf{x}(h+1) &= A\mathbf{x}(h) + C\mathbf{u}(h) \\ \mathbf{y}(h) &= H\mathbf{x}(h) + \mathbf{u}(h) \end{aligned} \tag{1.1}$$

is *positive* if its state, input and output variables are always nonnegative in value. Positive systems arise quite frequently [18] since the internal and the external variables of many real systems represent quantities (such as pressure, concentration, population levels, etc.) that may not have meaning unless they are nonnegative.

A fairly complete description of their dynamical behavior relies on a family of results, such as the celebrated Perron-Frobenius and König-Frobenius theorems [3,21], dealing with the spectral and combinatorial structure of nonnegative matrices. Interestingly enough, several new problems arising in a system theoretic context stimulate the research and open new vistas over the field of positive matrices. Just to mention a few of them, we recall the reachability and observability analysis, and the state space realization of one-dimensional (1D) positive systems [19,23].

Linear systems depending on two independent discrete variables (2D systems) appeared in the literature nearly twenty years ago [1,6,25,17]. At the very beginning they have been introduced to investigate recursive structures for processing two-dimensional data. This processing has been performed for a long time using an input-output description of the algorithms via ratio of polynomials in two indeterminates. The new idea that originated research on 2D systems consisted in considering these algorithms as external representations of dynamical systems and hence in introducing for such systems the concept of state and the updating equations, given by [7]

$$\begin{aligned} \mathbf{x}(h+1,k+1) &= A\mathbf{x}(h,k+1) + B\mathbf{x}(h+1,k) + C\mathbf{u}(h,k+1) + D\mathbf{u}(h+1,k) \\ \mathbf{y}(h,k) &= H\mathbf{x}(h,k) + J\mathbf{u}(h,k). \end{aligned} \tag{1.2}$$

It turns out that these models, which evolve according to a quarter plane causality law, are suitable for providing state space descriptions of a large class of processes. Typically they apply to two-dimensional data processing in various fields, as seismology, X-ray image enhancement, image deblurring, digital picture processing, etc.

Quite recently some contributions dealing with river pollution modelling [5] and the discretization of PDE's that describe gas absorption and water stream heating [20], naturally introduced the nonnegativity constraint in equation (1.2). The same constraint appears in examples of discretized biological processes involving diffusion and advection phaenomena, whose description relies on 2D compartmental models (see, for instance, the dynamics of a tracer injected in a blood vessel [29]).

As in the 1D case, positivity is expected to endow a 2D system with very special properties, that find no counterpart in the general case. As a consequence, it seems both useful and appealing to look for a more general setting where all such properties can be framed.

A 2D positive (linear) system is a state model whose variables take positive (or at least nonnegative) values. Here we restrict our investigation to unforced 2D state motions, as given by the following updating equation

$$\mathbf{x}(h+1,k+1) = A\mathbf{x}(h,k+1) + B\mathbf{x}(h+1,k), \tag{1.3}$$

where the doubly indexed *local state* sequence $\mathbf{x}(\cdot,\cdot)$ takes values in the positive cone $\mathbf{R}^n_+ := \{\mathbf{x} \in \mathbf{R}^n : x_i \geq 0,\ i = 1,2,...,n\}$, $A$ and $B$ are $n \times n$ nonnegative matrices. The initial conditions are assigned by specifying the (nonnegative) values of the local states on the *separation set* $\mathcal{C}_0 := \{(i,-i) : i \in \mathbf{Z}\}$. However, different choices for the support of the set of initial conditions are possible [5], (for instance, by assuming initial conditions on $\mathrm{S} = \{(i,0) : i > 0\} \cup \{(0,j) : j > 0\}$) and they do not affect the content of the paper.

The aim of this contribution is to explore some mathematical issues, coming under the heading of nonnegative matrix theory, that entail important consequences on the pattern of the state evolution and on the internal structure of 2D positive systems.
The results we are going to present involve both the spectral and the combinatorial description of some classes of nonnegative matrix pairs, which occur quite frequently in the applications.
In section 2, we investigate in detail finite memory and separable pairs $(A,B)$. The main tools we resort to are the traces of the Hurwitz products and the (1D) characteristic polynomial of $A+B$, which allow for a complete picture of the spectral properties of $(A,B)$. On the other hand, using row-column permutations, we obtain canonical forms for finite memory and separable pairs, which provide good insights in the combinatorial structure of the corresponding 2D positive systems and, consequently, in the patterns of their evolutions. A more general class of matrix pairs, i.e., nonnegative pairs $(A,B)$ with property L, is considered in section 3. In general this property does not introduce obvious constraints on the zero pattern of the pair. However, if we require that property L is preserved for all pairs obtained from $(A,B)$ by modifying, or possibly zeroing, only its nonzero elements, we get a complete combinatorial characterization of the pair. A similar, yet not completely equivalent, point of view is that of giving an element of the pair, say $A$, and investigating what are the nonnegative matrices $B$ such that $(A,B)$ is endowed with property L. A complete solution in the case when $A$ is diagonal is provided.

A further relevant feature of nonnegative pairs with property L turns out to be the coupling of the Perron-Frobenius eigenvalues of $A$ and $B$ when $A+B$ is irreducible. The related question of the existence of a common maximal eigenvector for both $A$ and $B$ is addressed in section 4, and positively answered when $A$ and $B$ constitute a quasi-commutative pair. In general, however, a nonnegative pair $(A,B)$ with property L does not admit a common maximal eigenvector, and the maximal eigenvector of $\alpha A + (1-\alpha)B$ can only be expressed as a polynomial function of $\alpha$. On the other hand, a different approach, which gets rid of property L, allows to completely characterize nonnegative pairs with a common maximal eigenvector in terms of row stochastic matrices.

In section 5 we present some results on the 2D inverse spectral problem, namely on the construction of a nonnegative pair that exhibits a prescribed characteristic polynomial. As a byproduct, we obtain a counterexample showing that property L does not imply simultaneous triangularizability (the so-called "property P") even when nonnegative matrices are considered.

Some extensions to matrix pairs endowed with 1-linearity conclude the paper.

Before proceeding, we introduce some notation. If $M = [m_{ij}]$ is a matrix (in particular, a vector), we write

i) $M \gg 0$ ($M$ *strictly positive*), if $m_{ij} > 0$ for all $i, j$;

ii) $M > 0$ ($M$ *positive*), if $m_{ij} \geq 0$ for all $i, j$, and $m_{hk} > 0$ for at least one pair $(h, k)$;

iii) $M \geq 0$ ($M$ *nonnegative*), if $m_{ij} \geq 0$ for all $i, j$.

The positive matrix whose $(i, j)$-th entry is 1, while all others are 0, is denoted by $E_{ij}$.

To every $n \times n$ nonnegative matrix $M$ we associate [3] a *digraph* (directed graph) $D(M)$ of order $n$, with vertices indexed by $1, 2, ..., n$. There is an arc $\alpha = (i, j)$ from $i$ to $j$ if and only if $m_{ij} > 0$.

Two $n \times n$ nonnegative matrices $M = [m_{ij}]$ and $N = [n_{ij}]$ have the same *zero pattern* if $m_{ij} = 0$ implies $n_{ij} = 0$ and vice versa. $M$ and $N$ have the same zero pattern if and only if $D(M) = D(N)$.

In some cases, it will be convenient to denote the $(i, j)$-th entry of a matrix $M$ as $[M]_{ij}$.

The symbol $*$ represents the *Hadamard product*: if $A$ and $B$ are $n \times n$ (nonnegative) matrices, then $A * B$ is the $n \times n$ matrix whose entries are given by

$$[A * B]_{ij} = [A]_{ij}[B]_{ij}.$$

We shall use some terminology borrowed from semigroup theory [26]. Given the alphabet $\Xi = \{\xi_1, \xi_2\}$, the free monoid $\Xi^*$ with base $\Xi$ is the set of all words

$$w = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}, \qquad m \in \mathbf{Z}, \quad \xi_{i_h} \in \Xi.$$

The integer $m$ is called the length of the word $w$ and denoted by $|w|$, while $|w|_i$ represents the number of occurencies of $\xi_i$ in $w$, $i = 1, 2$. If

$$v = \xi_{j_1}\xi_{j_2}\cdots\xi_{j_p}$$

is another element of $\Xi^*$, the product is defined by concatenation

$$wv = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}\xi_{j_1}\xi_{j_2}\cdots\xi_{j_p}.$$

This produces a monoid with $1 = \emptyset$, the empty word, as unit element. Clearly, $|wv| = |w| + |v|$ and $|1| = 0$. For each pair of matrices $A, B \in \mathbf{C}^{n\times n}$, the map $\psi$ defined on $\{1, \xi_1, \xi_2\}$ by the assignements $\psi(1) = I_n$, $\psi(\xi_1) = A$ and $\psi(\xi_2) = B$, uniquely extends to a monoid morphism of $\Xi^*$ into $\mathbf{C}^{n\times n}$. The $\psi$-image of a word $w \in \Xi^*$ is denoted by $w(A, B)$.

The Hurwitz products of two square matrices $A$ and $B$ are inductively defined [10] as

$$A^i \sqcup\!\sqcup^0 B = A^i, \ i > 0 \quad \text{and } A^0 \sqcup\!\sqcup^j B = B^j \ j > 0 \tag{1.4}$$

and, when $i$ and $j$ are both greater than zero,

$$A^i \sqcup\!\sqcup^j B = A(A^{i-1} \sqcup\!\sqcup^j B) + B(A^i \sqcup\!\sqcup^{j-1} B). \tag{1.5}$$

One easily sees that

$$A^i \sqcup\!\sqcup^j B = \sum_{\substack{|w|_1=i, \ |w|_2=j \\ w \ \in \ \Xi^*}} w(A, B),$$

namely, the $(i, j)$-th Hurwitz product is the sum of all matrix products that include the factors $A$ and $B$, $i$ and $j$ times respectively. Assuming zero initial conditions on $\mathcal{C}_0$, except at $(0, 0)$, then $\mathbf{x}(h, k)$ can be expressed as

$$\mathbf{x}(h, k) = A^h \sqcup\!\sqcup^k B \ \mathbf{x}(0, 0), \qquad \forall h, k \geq 0.$$

## 2 Finite memory and separability

It is clear that the dynamics of a 2D system (1.3) is essentially determined by the matrix pair $(A, B)$. However, the algebraic tools we have at our disposal for studying a pair of linear transformations are not as simple and effective as those available for the investigation of a single linear transformation. Actually, no decomposition of the state space into $\{A, B\}$-invariant subspaces can be given, allowing for an effective representation of the system behavior as a superposition of elementary modes with simple structure.

Consequently, the modal decomposition approach to the unforced dynamics does not extend to 2D systems, and serious difficulties arise even when an approximate analysis is attempted, basing on some generalized version of the Perron-Frobenius theorem [8].

Interestingly enough, however, some natural assumptions on the structure of the pair $(A, B)$ allow to single out important classes of positive systems, whose spectral and combinatorial properties are easily investigated. The characteristic polynomial

$$\Delta_{A,B}(z_1, z_2) := \det(I - Az_1 - Bz_2), \tag{2.1}$$

is probably the most useful tool we can resort to when analyzing and classifying the matrix pairs. Like the characteristic polynomial of a single matrix, which in general does not capture the underlying Jordan structure, in the same way $\Delta_{A,B}$ does not identify the similarity orbit of the pair $(A, B)$. Nevertheless, several aspects of the 2D motion completely rely on it. There is, first of all, the internal stability of system (1.3), which depends [2,7] only on the variety of the zeros of $\Delta_{A,B}$. Moreover, as an immediate consequence of the 2D Cayley-Hamilton theorem [8], the state evolution of (1.3) satisfies an autoregressive equation which involves the coefficients of $\Delta_{A,B}$.

Additional insights into the structure of 2D systems come from the factorization of the characteristic polynomial. In this section we consider two special cases, namely when the characteristic polynomial is a constant:

$$\Delta_{A,B}(z_1, z_2) = 1 \tag{2.2}$$

and when it factors into the product of two polynomials in one variable:

$$\Delta_{A,B}(z_1, z_2) = r(z_1)s(z_2). \tag{2.3}$$

Systems which satisfy condition (2.2) exhibit the so-called *finite memory* property, i.e. the zeroing of the unforced state evolution in a finite number of steps [2]. They constitute the natural framework for the state space synthesis of two-dimensional digital filters with finite impulse response (F.I.R. filters, for short) [12], and of convolutional encoders, decoders and syndrome formers [11]. On the other hand, in feedback control specifications commonly include a "dead-beat" performance of the controller [2,14], which implies that the resulting closed loop system exhibits once again the finite memory property.

*Separable* systems, which satisfy condition (2.3), are usually thought of as the simplest class of state models for realizing infinite impulse response (I.I.R.) 2D filters [9,10]. Indeed, just the knowledge that a 2D system is separable allows one to make strong statements about its behaviour; in particular, internal stability can be quickly deduced from the general theory of discrete time 1D systems, as the long term performance of separable systems is determined by the eigenvalues of $A$ and $B$ separately. The above properties motivate the widespread interest in these filters for image processing applications, and the existence of approximation techniques for reducing general I.I.R. filters to separable ones.

So far, finite memory and separable systems have been investigated in the literature without any constraint on the matrix pair. Introducing the nonnegativity assumption allows to strenghthen their properties and to obtain more penetrating characterizations of both classes of systems.

In the finite memory case, the spectral features of certain matrices associated with a given pair $(A, B)$, like the Hurwitz products and the elements of the multiplicative monoid generated by $A$ and $B$, are clarified by the following proposition.

**Proposition 2.1** For a pair of $n \times n$ nonnegative matrices $(A, B)$, the following statements are equivalent *i)* $\Delta_{A,B}(z_1, z_2) = 1$;

*ii)* $A + B$ is a nilpotent (and, a fortiori, a reducible) matrix;

*iii)* $A^i ⊔⊔^j B$ is nilpotent, for all $(i, j) \neq (0, 0)$;

*iv)* $w(A, B)$ is nilpotent, for all $w \in \Xi^* \setminus \{1\}$.

PROOF *i)* $\Rightarrow$ *ii)* Letting $z_1 = z_2 = z$ in $\Delta_{A,B}(z_1, z_2) = 1$, we get $\det(I-(A+B)z) = 1$, which implies the nilpotency of $A + B$.

*ii)* $\Rightarrow$ *iii)* For all $\nu \geq n$ we have $0 = (A+B)^\nu = \sum_{i+j=\nu} A^i ⊔⊔^j B$. The nonnegativity assumption further implies that $A^i ⊔⊔^j B$ is zero whenever $i + j \geq n$. Consequently, when $(i, j) \neq (0, 0)$, one gets $0 \leq (A^i ⊔⊔^j B)^n \leq A^{in} ⊔⊔^{jn} B = 0$ which proves the nilpotency of $A^i ⊔⊔^j B$.

*iii)* $\Rightarrow$ *iv)* Let $|w|_1 = i, |w|_2 = j$. As $[w(A, B)]^n \leq (A^i ⊔⊔^j B)^n = 0$, we see that $w(A, B)$ is nilpotent.

*iv)* $\Rightarrow$ *i)* By a classical theorem of Levitzki [16], assumption *iv)* corresponds to the existence of a similarity transformation that reduces both $A$ and $B$ to upper triangular form. Clearly, the characteristic polynomial of a pair of nilpotent upper triangular matrices is 1. ∎

**Remark** In the general case, when the matrix entries assume both positive and negative values, condition *ii)* is necessary, but not sufficient, for guaranteeing the finite memory property, which depends [9] on the nilpotency of all linear combinations $\alpha A + \beta B$, $\alpha, \beta \in \mathbf{C}$. On the contrary, conditions *iii)* and *iv)* are sufficient, but not necessary, for the finite memory property, as one easily checks with the pair

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.4}$$

Moreover, while for a general finite memory pair $(A, B)$ we can only guarantee that the Hurwitz products $A^i ⊔⊔^j B$ are zero when $i + j \geq n$, in the nonnegative case this property extends to all matrix products $w(A, B), w \in \Xi^*$ and $|w| \geq n$.

The above results make it clear that, for a 2D positive system, finite memory is essentially a 1D property. In fact, the nilpotency of $A+B$ can be restated by saying that the 1D system

$$\hat{\mathbf{x}}(h + 1) = (A + B)\ \hat{\mathbf{x}}(h) \tag{2.5}$$

is finite memory, and therefore its state evolution $\hat{\mathbf{x}}(\cdot)$ dies off in a finite number of steps for any initial condition $\hat{\mathbf{x}}(0) \in \mathbf{R}^n_+$.

On the other hand, when initializing the 2D system (1.3) on $\mathcal{C}_0$ with a constant sequence of local states

$$\mathbf{x}(i, -i) = \mathbf{x}_0 \in \mathbf{R}^n_+, \qquad \forall i \in \mathbf{Z},$$

it is clear that all local states $\mathbf{x}(i+\ell, -i)$ on the separation set $\mathcal{C}_\ell = \{(h,k) : h+k = \ell\}$, $\ell \in \mathbf{Z}_+$, have the same value $(A+B)^\ell\, \mathbf{x}_0$, which is exactly the value of $\hat{\mathbf{x}}(\ell)$ when $\hat{\mathbf{x}}(0) = \mathbf{x}_0$.

If system (1.3) eventually reaches the zero sequence on some separation set, for every choice of $\mathbf{x}_0$, the 1D system (2.5) is finite memory and the same holds true for (1.3). This means that for nonnegative 2D systems the finite memory property can be checked in a very easy way, by resorting only to constant sequences of nonnegative local states.

We turn now our attention to a characterization of finite memory nonnegative pairs, which is based on their zero pattern only.

**Definition** A pair of $n \times n$ matrices $(A, B)$ is said to be *cogredient* to a pair $(\bar{A}, \bar{B})$ if there exists a permutation matrix $P$ such that $\bar{A} = P^T AP$ and $\bar{B} = P^T BP$.

The combinatorial structure of finite memory nonnegative pairs is completely explained by the following proposition. We point out that the nonnegativity assumption is an essential ingredient for proving the simultaneous triangularizability of a finite memory pair. Actually, the matrix pair in (2.4) is finite memory; yet, no similarity transformation exists which triangularizes both $A$ and $B$.

**Proposition 2.2** A pair of $n \times n$ nonnegative matrices $(A, B)$ is finite memory if and only if it is cogredient to a pair of upper triangular nonnegative nilpotent matrices.

Proof Assume first that $(A, B)$ is finite memory. By Proposition 2.1 *ii)*, $A+B$ is a nilpotent and hence a reducible matrix. Consequently, there exists a permutation matrix $P_1$ such that

$$P_1^T(A+B)P_1 = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}.$$

As $C_{11}$ and $C_{22}$ are nilpotent, we can apply the above procedure to both diagonal blocks. By iterating this reasoning, we end up with one dimensional nilpotent diagonal blocks and, therefore, with an upper triangular matrix:

$$P^T(A+B)P = P^T AP + P^T BP = \begin{bmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{bmatrix}.$$

Since $P^T AP$ and $P^T BP$ are nonnegative, they are both upper triangular with zero diagonal. The converse is obvious. ■

In analyzing nonnegative separable pairs we follow the same lines, and end up with several results that strictly parallel those obtained so far in the finite memory case. A fairly complete spectral characterization of separability is summarized in the following proposition.

**Proposition 2.3** For a pair of $n \times n$ positive matrices $A$ and $B$, the following statements are equivalent:

*i)* $\Delta_{A,B}(z_1,z_2) = r(z_1)s(z_2)$;

*ii)* $\det[I-(A+B)z] = \det[I-Az]\det[I-Bz]$;

*iii)* $A^i\sqcup\!\sqcup^j B$ is nilpotent for all $(i,j)$ with $i,j>0$;

*iv)* $w(A,B)$ is nilpotent, for all $w\in\Xi^*$ such that $|w|_i>0,\ i=1,2$;

*v)* there exists a complex valued nonsingular matrix $T$ such that $\hat{A}=T^{-1}AT$ and $\hat{B}=T^{-1}BT$ are upper triangular matrices, and $[\hat{A}]_{hh}\neq 0$ implies $[\hat{B}]_{hh}=0$.

Proof *i)* $\Rightarrow$ *ii)* Assuming either $z_1=0$ or $z_2=0$ in (2.3) we obtain $s(z_2)=\det[I-Bz_2]$ or $r(z_1)=\det[I-Az_1]$, respectively. Consequently, letting $z_1=z_2=z$, we get

$$\det[I-(A+B)z] = \det[I-Az]\det[I-Bz].$$

*ii)* $\Rightarrow$ *iii)* Introduce the matrix

$$M=\begin{bmatrix} A & 0\\ 0 & B\end{bmatrix}.$$

Assumption *ii)* implies that $M$ and $A+B$ have the same characteristic polynomial

$$\det[I-Mz]=\det[I-(A+B)z]$$

and, consequently,

$$\operatorname{tr}(M^h)=\operatorname{tr}\Big((A+B)^h\Big)\qquad \forall\ h\geq 1. \tag{2.6}$$

As $(A+B)^h=\sum_{i+j=h}A^i\sqcup\!\sqcup^j B$, by the linearity of the trace operator we get

$$\operatorname{tr}(A^h)+\operatorname{tr}(B^h)=\sum_{i+j=h}\operatorname{tr}A^i\sqcup\!\sqcup^j B,$$

which, in turn, implies

$$\sum_{\substack{i,j>0\\ i+j=h}}\operatorname{tr}A^i\sqcup\!\sqcup^j B=0,\qquad \forall\ h\geq 1. \tag{2.7}$$

Since $(A,B)$ is a nonnegative pair, (2.7) is equivalent to the assumption that all Hurwitz products $A^i\sqcup\!\sqcup^j B$ have zero trace whenever $i,j\geq 1$. Just recalling that a matrix is nilpotent if and only if all its positive powers have zero traces, *iii)* is an easy consequence of the inequality

$$\operatorname{tr}\Big((A^i\sqcup\!\sqcup^j B)^\nu\Big)\leq \operatorname{tr}A^{i\nu}\sqcup\!\sqcup^{j\nu}B=0,\quad i,j\geq 1,\ \nu=1,2,\ldots$$

*iii)* $\Rightarrow$ *iv)* Let $|w|_1=i\geq 1,\ |w|_2=j\geq 1$. As $w(A,B)\leq A^i\sqcup\!\sqcup^j B$, we have $[w(A,B)]^n\leq (A^i\sqcup\!\sqcup^j B)^n=0$.

*iv)* $\Rightarrow$ *v)* We resort to the following extension of Levitzki theorem [16]: "Let $A, B \in \mathbf{C}^{n\times n}$. All matrices of the multiplicative semigroup

$$\mathcal{S} := \{w(A,B), w \in \Xi^*, |w|_1 \geq 1, |w|_2 \geq 1\}$$

are nilpotent if and only if the pair $(A, B)$ is separable and simultaneously triangularizable via a (complex) similarity transformation".
Clearly, assumption *iv)* implies simultaneous triangularizability. Moreover, as the trace is invariant under similarity, one gets

$$\mathrm{tr}\ w(A,B) = \sum_{h=1}^{n} ([\hat{A}]_{hh})^i ([\hat{B}]_{hh})^j = 0 \qquad \forall\ i,j > 0. \tag{2.8}$$

Thus (2.8) holds true if and only if $[\hat{A}]_{hh} \neq 0 \Rightarrow [\hat{B}]_{hh} = 0$.

*v)* $\Rightarrow$ *i)* Obvious. ■

The combinatorial structure of separable matrix pairs is quite appealing, and easily determined as a consequence of the following lemma.

**Lemma 2.4** If $A > 0$ and $B > 0$ constitute a separable pair of $n \times n$ matrices, then $A + B$ is reducible.

Proof Consider any $w = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_m} \in \Xi^*$, with $|w|_1 > 0$ and $|w|_2 > 0$. Because of the characterization *iii)* of separability given in Proposition 2.3, each diagonal element of $w(A,B)$ is zero. Therefore, for any sequence of integers $\ell_1, \ell_2, ..., \ell_m \in \{1, 2, ..., n\}$,

$$[\psi(\xi_{i_1})]_{\ell_1\ell_2}[\psi(\xi_{i_2})]_{\ell_2\ell_3}\ \cdots\ [\psi(\xi_{i_m})]_{\ell_m\ell_1} = 0. \tag{2.9}$$

As both $A$ and $B$ are nonzero, there exist entries $[A]_{ij} > 0$ and $[B]_{hk} > 0$. If $A + B$ were irreducible, there would be integers $p$ and $q$ such that $[(A+B)^p]_{jh} > 0$ and $[(A+B)^q]_{ki} > 0$. Consequently, we would have

$$[\psi(\xi_{t_1})]_{j\ell_1}[\psi(\xi_{t_2})]_{\ell_1\ell_2}\ \cdots\ [\psi(\xi_{t_p})]_{\ell_{p-1}h} > 0$$

and

$$[\psi(\xi_{s_1})]_{kr_1}[\psi(\xi_{s_2})]_{r_1r_2}\ \cdots\ [\psi(\xi_{s_q})]_{r_{q-1}i} > 0$$

for appropriate choices of $\xi_{t_\nu}$ and $\xi_{s_\mu}$ and of the indexes $\ell_\nu$ and $r_\mu$.
This implies

$$[A]_{ij}[\psi(\xi_{t_1})]_{j\ell_1}\ \cdots\ [\psi(\xi_{t_p})]_{\ell_{p-1}h}[B]_{hk}[\psi(\xi_{s_1})]_{kr_1}...[\psi(\xi_{s_q})]_{r_{q-1}i} > 0,$$

which contradicts (2.9). ■

**Proposition 2.5** A pair of $n\times n$ nonnegative matrices $(A, B)$ is separable if and only if there exists a permutation matrix $P$ such that $P^TAP$ and $P^TBP$ are conformably partitioned into block triangular matrices

$$P^TAP = \begin{bmatrix} A_{11} & * & * & * \\ & A_{22} & * & * \\ & & \ddots & * \\ & & & A_{tt} \end{bmatrix} \qquad P^TBP = \begin{bmatrix} B_{11} & * & * & * \\ & B_{22} & * & * \\ & & \ddots & * \\ & & & B_{tt} \end{bmatrix}, \tag{2.10}$$

where $A_{ii} \neq 0$ implies $B_{ii} = 0$. It entails no loss of generality assuming that the nonzero diagonal blocks in $P^T AP$ and $P^T BP$ are irreducible.

PROOF Assume that $A$ and $B$ constitute a separable pair. If one of the matrices is zero, the proposition is trivially true. So we confine ourselves to the case of $A$ and $B$ both nonzero. By the previous lemma, there exists a permutation matrix $P_1$ s.t.

$$P_1^T AP_1 + P_1^T BP_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

where $A_{ii}$ and $B_{ii}$, $i = 1, 2$, are square submatrices. As the nonnegative matrix pairs $(A_{ii}, B_{ii})$ are separable, we can apply the same procedure as before to both of them. By iterating this method we end up with a pair of matrices with structure (2.10). The converse is obvious. ∎

In many cases the information available on the physical process we aim to model allows us to assume that no interaction exists among certain variables, and, consequently, that some entries of matrices $A$ and $B$ are exactly 0, whereas the others can be assumed nonnegative, and known with some level of uncertainty.
This is always the case of compartmental models, where nonzero entries correspond to the existence of flows between different compartments, and physical or biological reasons guarantee that some pairs of compartments have no direct interaction at all. The combinatorial characterizations given in the above propositions make it clear that the situation when all uncertain values are positive represents the "worst case" for the existence of finite memory and separability, and, therefore, if such properties are verified in the worst case, they are preserved under all perturbations of the nonnegative entries.

# 3 Pairs of matrices with property L

The examples in the previous section make it clear that there is a strong relation between the characteristic polynomial factors of a matrix pair and the properties the associated 2D state model may exhibit. The idea of connecting the factors of $\Delta_{A,B}(z_1, z_2)$ with the geometric properties of the state evolution can be applied to the more general situation, when the characteristic polynomial of the pair $(A, B)$ splits into linear factors. It turns out that such pairs are special enough to provide a basis for a rich and interesting theory, but also general enough to include models of practical importance, such as finite memory and separable systems, already discussed in the previous section, and systems described by triangular or commutative matrix pairs.

**Definition** A pair of $n \times n$ matrices $(A, B)$ is said to have *property L* if its characteristic polynomial factors into linear factors

$$\Delta_{A,B}(z_1, z_2) = \prod_{i=1}^{n} (1 - \lambda_i z_1 - \mu_i z_2), \tag{3.1}$$

over the complex field.

An equivalent definition [22] of property L is that the eigenvalues of $A$ and $B$ can be ordered into two $n$-tuples

$$\Lambda(A) = (\lambda_1, \lambda_2, ..., \lambda_n), \qquad \Lambda(B) = (\mu_1, \mu_2, ..., \mu_n) \tag{3.2}$$

such that, for all $\alpha, \beta$ in $\mathbf{C}$, the spectrum of $\alpha A + \beta B$ is given by

$$\Lambda(\alpha A + \beta B) = (\alpha\lambda_1 + \beta\mu_1, ..., \alpha\lambda_n + \beta\mu_n). \tag{3.3}$$

In other words, property L means that the spectrum of any linear combination of $A$ and $B$ is the linear combination of the spectra $\Lambda(A)$ and $\Lambda(B)$.

Propositions 2.2 and 2.5 show that, for a nonnegative matrix pair, finite memory and separability properties depend on its zero pattern only. This is no more true when property L is considered. To see that it cannot be deduced from the structure of the directed graphs, $D(A)$ and $D(B)$, of the nonnegative matrices $A$ and $B$, just consider

$$(A_1, B_1) = \left( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right)$$

and

$$(A_2, B_2) = \left( \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right).$$

The pairs $(A_1, B_1)$ and $(A_2, B_2)$, of course, have the same zero pattern, but only the first one is endowed with property L.

So, it is quite natural to ask under which conditions a nonnegative pair has property L, independently of the values of its nonzero entries. When so, property L turns out to be a feature which depends only on the combinatorial structure of the matrices, and therefore will be called *structural linearity* (property SL, for short).

**Definition** A pair of $n \times n$ nonnegative matrices $(A, B)$ has property SL if $(A * M, B * N)$ has property L for all $n \times n$ nonnegative matrices $M$ and $N$.

A matrix pair with property SL is cogredient to a particular block triangular form. To obtain this form, we need the following lemma.

**Lemma 3.1** Let $(A, B)$ be a pair of $n \times n$ nonnegative matrices, with $n > 1$. If $(A, B)$ has property SL, then either $A + B$ is reducible or one of the two matrices is zero.

Proof As a preliminary step, we prove that if one of the matrices, say $A$, is irreducible, the other is zero. Assume, by contradiction, that $B$ has an element $b_{ij} > 0$, and suppose first $i \neq j$. As $A$ is irreducible, we can find in $D(A)$ a minimal path $\gamma$, of length $m-1$, which connects vertex $j$ to vertex $i$. Possibly after relabelling the vertices, we can assume $i = 1$, $j = m$ and

$$\gamma = \{(m, m-1), (m-1, m-2), \dots, (2, 1)\}. \tag{3.4}$$

This implies the existence of nonnegative matrices $M$ and $N$ which reduce $A$ and $B$ to the following form

$$\hat{A} := A * M = \sum_{i=1}^{m-1} E_{i+1,i} \qquad \hat{B} := B * N = E_{1,m}.$$

By the SL property of $(A, B)$, the pair $(\hat{A}, \hat{B})$ should be endowed with property L, which is impossible since

$$\det \begin{bmatrix} 1 & & & & -z_2 & \\ -z_1 & 1 & & & & \\ \vdots & -z_1 & \ddots & & \vdots & \mathbf{0} \\ & & \ddots & 1 & & \\ 0 & 0 & \dots & -z_1 & 0 & \\ & & \mathbf{0} & & & \mathbf{I} \end{bmatrix} = 1 - z_1^{m-1} z_2$$

does not factor into linear factors, as $m-1$ is positive.

On the other hand, if $b_{ii}$ were nonzero for some $i$, by the irreducibility assumption on $A$, there would be in $D(A)$ a circuit $\gamma$, of length $m-1 > 1$, connecting vertex $i$ to itself and, as before, we can assume $i = 1$ and

$$\gamma = \{(1, m), (m, m-1), \dots, (2, 1)\}. \tag{3.5}$$

Hence, there exist nonnegative matrices $M$ and $N$ such that

$$\bar{A} := A * M = E_{1,m} + \sum_{i=1}^{m-1} E_{i+1,i} \qquad \bar{B} := B * N = E_{1,1}. \tag{3.6}$$

The characteristic polynomial of $(\bar{A}, \bar{B})$ is $1 - z_2 - z_1^m$, which cannot factor into linear terms as $m > 1$, thus contradicting the SL assumption on $(A, B)$.

So, every pair of positive matrices with property SL consists of reducible matrices. It remains to prove that also their sum, $A + B$, is reducible. Suppose, by contradiction, that it is not. As the irreducibility of $A + B$ only depends on its zero pattern, when both $a_{ij}$ and $b_{ij}$ are nonzero we may replace one of them with zero without destroying the irreducibility of $A + B$ and the property SL.
Since both $A$ and $B$ are still nonzero (otherwise $A + B$ would coincide either with $A$ or with $B$, and hence would be reducible), and $A * B = 0$, there exist nonzero entries $a_{ij}$ and $b_{hk}$ with $(i, j) \neq (h, k)$. By the irreducibility of $A + B$, two directed

paths can be found in $D(A+B)$ connecting $j$ with $h$ and $k$ with $i$, thus producing a closed walk $\tau$ with distinct edges, including arcs of both $D(A)$ and $D(B)$.
Among the elementary circuits of $\tau$, either there is a circuit $\gamma$ including arcs of both $D(A)$ and $D(B)$, or there are two circuits, $\gamma_A$ in $D(A)$ and $\gamma_B$ in $D(B)$, with a common vertex.
In the first case, by resorting to reasonings of the same kind as in the first part of the proof, we obtain a matrix pair $(\tilde{A},\tilde{B}) := (A*M, B*N)$ whose characteristic polynomial

$$\Delta_{\tilde{A},\tilde{B}}(z_1,z_2) = 1 - z_1^r z_2^s \qquad r,s \geq 1$$

does not split into linear factors.
In the second case, by relabelling the vertices of the digraph $D(A+B)$, we can assume that the common vertex of $\gamma_A$ and $\gamma_B$ is 1 and

$$\gamma_A = \{(1,2),(2,3),...,(m-1,m)\} \quad \gamma_B = \{(1,m+1),(m+1,m+2),...,(m+n-1,1)\}.$$

By suitably choosing $M$ and $N$, we obtain once more a pair $(A',B')$, with

$$A' := A*M = E_{m,1} + \sum_{i=1}^{m-1} E_{i,i+1} \qquad B' := B*N = E_{m+n-1,1} + E_{1,m+1} + \sum_{i=1}^{n-2} E_{m+i,m+1+i},$$

whose characteristic polynomial

$$\Delta_{A',B'}(z_1,z_2) = 1 - z_1^m - z_2^n, \qquad n+m>2$$

does not factor into linear factors. Property SL, however, would imply that $(A',B')$ has property L, a contradiction. ■

**Proposition 3.2** Let $(A,B)$ be a pair of $n \times n$ nonnegative matrices with property SL. Then there exists a permutation matrix $P$ such that $P^TAP$ and $P^TBP$ are conformably partitioned into block triangular matrices

$$P^TAP = \begin{bmatrix} A_{11} & * & * & * \\ & A_{22} & * & * \\ & & \ddots & * \\ & & & A_{tt} \end{bmatrix} \qquad P^TBP = \begin{bmatrix} B_{11} & * & * & * \\ & B_{22} & * & * \\ & & \ddots & * \\ & & & B_{tt} \end{bmatrix}, \qquad (3.8)$$

where the diagonal pairs $(A_{ii}, B_{ii})$ of dimension greater than 1 consist of an irreducible and a zero matrix.

Proof The case $n=1$ is trivial.
If $n>1$ and $A+B$ is irreducible, either $A$ or $B$ is zero, and we can reduce the other matrix to Frobenius normal form [3]. If $A+B$ is reducible, we can reduce it to Frobenius normal form by using a cogredience transformation, and apply to each irreducible diagonal block the previous arguments. ■

As a consequence of Proposition 3.2, the zero pattern of a nonnegative matrix $A$ completely characterizes the class of nonnegative matrices $B$ such that $(A,B)$ has property SL. A more difficult problem is that of obtaining, for a given (nonnegative) matrix $A$, all nonnegative pairs $(A, B)$ with property L. In the case of a diagonal matrix $A$, however, a complete solution is available, which sheds ligth on some interesting connections between properties L and SL.

**Lemma 3.3** Let $M = [m_{ij}]$ be an $n \times n$ nonnegative matrix, $n > 1$, such that

$$[M^r]_{ii} = (m_{ii})^r, \qquad i = 1, 2, \ldots, n, \quad r = 0, 1, 2, \ldots \tag{3.9}$$

Then $M$ is cogredient to a triangular matrix.

Proof We first prove that $M$ is reducible. If not, for any pair $(i, j)$ with $i \neq j$ there were integers $h$ and $k$ such that $[M^h]_{ij} > 0$, $[M^k]_{ji} > 0$. Consequently, we would have

$$[M^{h+k}]_{ii} \geq [M^h]_{ij}[M^k]_{ji} + (m_{ii})^{h+k} > (m_{ii})^{h+k},$$

which contradicts assumption (3.9).
Next we remark that, for any permutation matrix $P$ and any positive integer $r$, we have $(P^T M P)^r = P^T M^r P$. This implies that the diagonal elements in $(P^T M P)^r$ and in $M^r$ are connected by the same index permutation which connects the diagonal elements in $P^T M P$ and in $M$. So, using (3.9), we get

$$[(P^T M P)^r]_{ii} = [P^T M^r P]_{ii} = ([P^T M P]_{ii})^r, \tag{3.10}$$

for all nonnegative integers $r$ and for $i = 1, 2, \ldots n$. Now we apply a cogredience transformation which reduces $M$ to block triangular form

$$P^T M P = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}$$

and notice that, as a consequence of (3.10), both $M_{11}$ and $M_{22}$ have property (3.9). So, we can iterate the above procedure until a triangular matrix is obtained. ∎

**Proposition 3.4** Let $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$, $a_i \neq a_j$ if $i \neq j$, and $B = [b_{ij}]$ be $n \times n$ nonnegative matrices. The following statements are equivalent:

*i)* $(A, B)$ has property L;

*ii)* $\Lambda(B) = (b_{11}, b_{22}, \ldots, b_{nn})$;

*iii)* $B$ is cogredient to a triangular matrix.

Proof *i)* ⇒ *ii)* Property L implies [10] that there exists a suitable ordering of the spectrum of $B$, $\Lambda(B) = (\mu_1, \mu_2, \ldots, \mu_n)$, such that, for all $h > 0$,

$$\mathrm{tr} A^h \sqcup\!\sqcup^1 B = \binom{h+1}{h} \sum_{i=1}^{n} a_i^h \mu_i. \tag{3.11}$$

On the other hand we have

$$\mathrm{tr} A^h \sqcup^1 B = (h+1)\mathrm{tr}(A^h B) = (h+1)\sum_{i=1}^{n} a_i^h b_{ii}. \tag{3.12}$$

(3.11) and (3.12) together imply $\sum_i a_i^h(\mu_i - b_{ii}) = 0,\ h = 0, 1, \ldots, n-1$, and, taking into account that the Vandermonde matrix of the system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix} \begin{bmatrix} \mu_1 - b_{11} \\ \mu_2 - b_{22} \\ \vdots \\ \mu_n - b_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is nonsingular, we get $\mu_i = b_{ii},\ i = 1, 2, \ldots n$.

*ii)* $\Rightarrow$ *iii)* The assumption on $\Lambda(B)$ implies

$$\sum_{i=1}^{n} (b_{ii})^r = \mathrm{tr}(B^r) = \sum_{i=1}^{n} [B^r]_{ii}, \qquad r = 0, 1, \ldots \tag{3.13}$$

On the other hand, since $B$ is nonnegative, we have also

$$(b_{ii})^r \leq [B^r]_{ii} \qquad \forall\ i = 1, 2, ..., n. \tag{3.14}$$

Using (3.13) and (3.14) we get $(b_{ii})^r = [B^r]_{ii},\ r = 0, 1, \ldots,\ i = 1, 2, \ldots, n$, and therefore, by Lemma 3.3, $B$ is cogredient to a triangular matrix.

*iii)* $\Rightarrow$ *i)* Obvious. ∎

As a corollary of the above proposition, when $A$ is diagonal with distinct elements, a nonnegative pair $(A, B)$ has property L if and only if it has property SL. This is no more true, however, if two diagonal elements in $A$ coincide. Actually, the pair

$$A = I_2 \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is endowed with property L. Yet, when $A$ is modified into $\hat{A} = \mathrm{diag}\{1, 2\}$, the pair $(\hat{A}, B)$ loses property L, since $B$ is not cogredient to a triangular matrix.

The analysis of the case when the $a_i$'s in $A = \mathrm{diag}\{a_1, a_2, ..., a_n\}$ are not distinct is based on a more refined version of Lemma 3.3, given below.

**Lemma 3.5** Let $n = \nu_1 + \nu_2 + \ldots + \nu_k$, and suppose that the $n \times n$ nonnegative matrix $M$ is partitioned into blocks $M_{ij}$ of dimension $\nu_i \times \nu_j$. If

*a)* $\mathrm{tr}\Big([M^r]_{ii}\Big) = \mathrm{tr}\Big((M_{ii})^r\Big) \quad i = 1, 2, ..., k, \quad r = 0, 1, 2, ...$

*b)* $M_{ii}$ is irreducible, $i = 1, 2, ..., k$, then $M$ is cogredient to a block-triangular matrix whose diagonal blocks coincide (except, possibly, for the order) with the $M_{ii}$'s.

PROOF Let's consider the digraph $D(M)$ associated with the matrix $M$. The block-partitioning of the matrix corresponds to a partitioning of the vertices of $D(M)$ into classes, $J_1, J_2, ..., J_k$, such that, by assumption $b$), each element of a class communicates with all the others in the same class. We want to show that there exists a suitable relabelling of the classes which makes $M$ cogredient to a block-triangular matrix, with diagonal blocks $M_{ii}$. To this purpose, it is enough to prove that for every pair of distinct classes $J_i$ and $J_h$ there is no path starting from a vertex $k_i \in J_i$, reaching a vertex $k_h \in J_h$ and going back to $k_i$. If a closed path of length $\ell$ could be found with the above property, we would have

$$[M^\ell]_{k_i k_i} > [(M_{J_i J_i})^\ell]_{k_i k_i},$$

and hence $\operatorname{tr}\,[M^\ell]_{J_i J_i} > \operatorname{tr}\Big((M_{J_i J_i})^\ell\Big)$ and $\operatorname{tr}(M^\ell) > \sum_i \operatorname{tr}\Big(M_{ii})^\ell\Big)$, thus contradicting assumption $a$). ■

**Proposition 3.6** Let $A = \operatorname{diag}\{a_1 I_{\nu_1},\ a_2 I_{\nu_2}, \ldots, a_k I_{\nu_k}\}$ be a nonnegative (block) diagonal matrix, with $a_i \neq a_j$ if $i \neq j$, and let $B \geq 0$ be partitioned conformably with the partition of $A$, as follows

$$B = \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1k} \\ B_{21} & B_{22} & \ldots & B_{2k} \\ \vdots & & \ddots & \vdots \\ B_{k1} & B_{k2} & \ldots & B_{kk} \end{bmatrix}. \tag{3.16}$$

The following statements are equivalent:

*i*) $(A, B)$ has property L;

*ii*) $\det(zI_n - B) = \prod_{i=1}^k \det(zI_{\nu_i} - B_{ii})$;

*iii*) there exists a permutation matrix $P$ such that

$$P^T A P = \begin{bmatrix} \hat{A}_{11} & & & \\ & \hat{A}_{22} & & \\ & & \ddots & \\ & & & \hat{A}_{pp} \end{bmatrix} \qquad P^T B P = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \ldots & \hat{B}_{1p} \\ & \hat{B}_{22} & & \hat{B}_{2p} \\ & & \ddots & \vdots \\ & & & \hat{B}_{pp} \end{bmatrix}, \tag{3.17}$$

where the $\hat{A}_{ii}$'s are scalar matrices. Moreover, each of the $\hat{B}_{ii}$'s is a diagonal block of the Frobenius normal form of $B_{jj}$, for some $j$.

PROOF *i*) $\Rightarrow$ *ii*) If $(A, B)$ has property L and $A$ has the structure indicated above, then, according to a result of O.Taussky and T.S. Motzkin [22], the characteristic polynomial of matrix $B$ has to meet condition *ii*).

$ii) \Rightarrow iii)$ For each diagonal block $B_{ii}$ in (3.16) consider a permutation matrix $Q_i$ such that $Q_i^T B_{ii} Q_i$ is in Frobenius normal form with irreducible diagonal blocks $B_{ii}^{(j)}$, $j = 1, 2, .., r_i$. Setting $Q = \text{diag}\{Q_1, Q_2, ..., Q_k\}$, we obtain the matrix

$$\tilde{B} := Q^T B Q = \begin{bmatrix} B_{11}^{(1)} & * & * & & & & & & & \\ & \ddots & * & & * & & * & & * & \\ \mathbf{0} & & B_{11}^{(r_1)} & & & & & & & \\ & & & B_{22}^{(1)} & * & * & & & & \\ & * & & & \ddots & * & * & & * & \\ & & & \mathbf{0} & & B_{22}^{(r_2)} & & & & \\ & & & & & & & & & \\ & * & & & * & & \ddots & & * & \\ & & & & & & & & & \\ & & & & & & & B_{kk}^{(1)} & * & * \\ & * & & & * & & & & \ddots & * \\ & & & & & & & \mathbf{0} & & B_{kk}^{(r_k)} \end{bmatrix}$$

that has irreducible diagonal blocks. From the identity

$$\det(zI - \tilde{B}) = \det(zI - B) = \prod_{i=1}^{k} \det(zI - B_{ii}) = \prod_{i=1}^{k} \prod_{j=1}^{r_i} \det(zI - B_{ii}^{(j)}),$$

we get

$$\text{tr}(\tilde{B}^r) = \sum_{i=1}^{k} \sum_{j=1}^{r_i} \text{tr}\Big((B_{ii}^{(j)})^r\Big).$$

As we know that

$$\text{tr}(\tilde{B}^r) = \sum_{i=1}^{k} \sum_{j=1}^{r_i} \text{tr}\Big([B^r]_{ii}^{(j)}\Big),$$

and $\text{tr}\Big((B_{ii}^{(j)})^r\Big) \leq \text{tr}\Big([B^r]_{ii}^{(j)}\Big)$, it immediately follows that

$$\text{tr}\Big((B_{ii}^{(j)})^r\Big) = \text{tr}\Big([B^r]_{ii}^{(j)}\Big), \qquad \forall i = 1, 2, ..., k, \ j = 1, 2, ..., r_i,$$

and hence all the assumptions of the previous lemma are satisfied.
So, a block permutation matrix $\hat{Q}$ can be found, so that $\hat{Q}^T \tilde{B} \hat{Q} = (Q\hat{Q})^T B (Q\hat{Q}) =: P^T B P$ is block triangular with diagonal blocks $B_{ii}^{(j)}$. The same permutation matrix, when applied to $A$, sets a correspondence between irreducible diagonal blocks in $P^T B P$ and scalar diagonal blocks in $P^T A P$, as shown in (3.17).

$iii) \Rightarrow i)$ From (3.17) it is easy to see that

$$\det(I - A_1 z_1 - A_2 z_2) = \prod_{i=1}^{p} \det(I - \hat{A}_{ii} z_1 - \hat{B}_{ii} z_2).$$

Since the pairs $(\hat{A}_{ii}, \hat{B}_{ii})$ commute, and hence have property L, $(A, B)$ has property L, too. ■

The best known, and perhaps the most important result of the Perron-Frobenius theory concerns the existence of a positive simple maximal eigenvalue and a strictly positive maximal eigenvector for irreducible matrices. As property L induces a one to one coupling of the eigenvalues of $A$ and $B$, it seems quite natural to ask whether an irreducibility assumption on the nonnegative matrices $A$ and $B$, or on their sum, allows for a precise statement concerning the coupling of the maximal eigenvalues. The answer is affirmative and given in the following proposition.

**Proposition 3.7** Let $(A, B)$ be a pair of $n \times n$ nonnegative matrices, endowed with property L w.r.t. the orderings (3.2), and assume $A + B$ irreducible.
Then there exists a unique index $i$ such that

$$\lambda_i, \mu_i \in \mathbf{R}_+, \quad \lambda_i \geq |\lambda_j|, \quad \mu_i \geq |\mu_j|, \quad j = 1, 2, ..., n, \tag{3.17}$$

and $\alpha\lambda_i + \beta\mu_i$ is the maximal positive eigenvalue of the irreducible matrix $\alpha A + \beta B$, for all $\alpha, \beta > 0$.

Proof To prove the result it is sufficient to consider the convex combinations $\alpha A + (1-\alpha)B$, for $\alpha \in (0, 1)$. Note that such matrices, having the same zero pattern as $A + B$, are irreducible and hence have a simple maximal eigenvalue $\nu_{\max}(\alpha)$.
Denote by $r_i(\alpha)$, $i = 1, 2, ..., n$, the straight line in the complex plane $\mathbf{C}$, passing through $\lambda_i$ and $\mu_i$

$$r_i(\alpha) := \{\alpha\lambda_i + (1 - \alpha)\mu_i, \alpha \in \mathbf{R}\}. \tag{3.18}$$

For each $\alpha$ in $(0, 1)$, $\nu_{\max}(\alpha)$ lies on the straight lines $r_i(\alpha)$ and cannot belong to any line intersection, as irreducible matrices have simple maximal eigenvalues. So, as $\alpha$ varies from 0 to 1, $\nu_{\max}(\alpha)$ continuously moves along the same line, say $r_k(\alpha)$. It remains to show that $\lambda_k$ and $\mu_k$ are maximal eigenvalues of $A$ and $B$, respectively. Suppose, for instance, that $A$ possesses a positive maximal eigenvalue $\lambda_h$ distinct from $\lambda_k$. As the eigenvalues of $\alpha A + (1-\alpha)B$ are continuous functions of $\alpha$, $|\alpha\lambda_h + (1-\alpha)\mu_h|$ would be greater than $\nu_{\max}(\alpha)$ for all values of $\alpha$ in a suitable neighbourhood of 1, a contradiction. ■

It is worthwhile to underline that the irreducibility of $A + B$ is an essential ingredient of the proof. Once we drop this assumption, as, for instance, with the pair

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.19}$$

the maximal eigenvalues of $A$ and $B$ are not necessarily coupled in the ordered spectra, and hence do not appear in the same linear factor of the characteristic polynomial $\Delta_{A,B}(z_1, z_2)$.

## 4 Common maximal eigenvectors

When trying to extend the Perron-Frobenius theorem on positive maximal eigenvectors to a matrix pair $(A, B)$ with property L and irreducible sum, we are naturally faced with the following question: "what is the structure of the maximal eigenvector of $\alpha A + \beta B$ when both $\alpha$ and $\beta$ are positive?"
Based on the coupling of maximal eigenvalues, a first guess could be that $A$ and $B$ have parallel maximal eigenvectors. Unfortunately this is not generally true, as shown by the following counterexample.

**Example** Consider the matrix pair

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 4 \\ \delta & 1 \end{bmatrix} \tag{4.1}$$

and select for the parameter $\delta$ the nonnegative real values that make $\Delta_{A,B}(z_1, z_2)$ split into linear factors, or, equivalently, that annihilate the discriminant

$$\det \begin{bmatrix} -1 & -\frac{9}{2} - \frac{\delta}{2} & -2 \\ -\frac{9}{2} - \frac{\delta}{2} & 2 - 4\delta & -\frac{3}{2} \\ -2 & -\frac{3}{2} & 1 \end{bmatrix}$$

of the quadratic equation $\Delta_{A,B}(z_1, z_2) = 0$. Clearly the discriminant is zero when $\delta$ is a solution of the equation

$$\delta^2 - 50\,\delta + 220 = 0. \tag{4.2}$$

Both solutions $\delta_1 = 25 + \sqrt{405}$ and $\delta_2 = 25 - \sqrt{405}$ of (4.2) are positive. So, the corresponding pairs $(A, B_1)$ and $(A, B_2)$, with

$$B_1 = \begin{bmatrix} 2 & 4 \\ \delta_1 & 1 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 2 & 4 \\ \delta_2 & 1 \end{bmatrix},$$

are strictly positive (which obviously implies $A + B$ irreducible) and endowed with property L. It is easy to check that $A$ and $B_1$ have a strictly positive common eigenvector, which is the maximal eigenvector of both matrices, whereas the maximal eigenvectors of $A$ and $B_2$ are nonparallel.

The above example shows that property L and the irreducibility of $A + B$ (or, even more, the strict positiveness of $A$ and $B$) do not allow to draw any conclusion about the existence of a common maximal eigenvector. As a matter of fact, the same can be said for pairs with property P, which is generally stronger than property L, but coincides with it when $2 \times 2$ matrices are considered [22].
This negative conclusion raises some interesting questions which we summarize as follows: "assuming that $A + B$ is an irreducible matrix,

*a*) how can we strenghthen property L so as to guarantee that $A$ and $B$ have a common maximal eigenvector?

*b*) when no further assumptions, except property L, are introduced, what can be said about the structure of the maximal eigenvector of $\alpha A + \beta B$, as $\alpha$ and $\beta$ vary over the positive real numbers?

*c*) finally, what kind of necessary and sufficient conditions do guarantee that $A$ and $B$ have a common maximal eigenvector?"

Given an arbitrary pair $(A, B)$ of $n \times n$ matrices, we define the matrix sets $\mathcal{C}^{(k)}, k = 1, 2, \ldots$, as follows:

$$\mathcal{C}^{(1)} = \{[A, B]\},$$

and, for $k > 1$,

$$\mathcal{C}^{(k)} = \{[A, C^{(k-1)}]\} \cup \{[B, C^{(k-1)}]\},$$

where $C^{(k-1)}$ runs over the elements of the set $\mathcal{C}^{(k-1)}$, and $[M, N]$ denotes the *commutator* $MN - NM$.
If $\mathcal{C}^{(k)} = \{0\}$, we shall say that the pair $(A, B)$ has the property of (generalized) quasi commutativity of the $k$-th order or, briefly, $(A, B)$ is a *$k$-commuting pair*. For all $k \in \mathbf{N}$, $k$-commutativity implies [4] property P, and hence property L.

**Proposition 4.1** Let $A > 0$ and $B > 0$ be $k$-commutative $n \times n$ matrices, whose sum, $A+B$, is irreducible. Then $A$ and $B$ have a strictly positive common eigenvector $\mathbf{v}$

$$A\mathbf{v} = r_A \mathbf{v}, \qquad B\mathbf{v} = r_B \mathbf{v}, \tag{4.3}$$

and $r_A, r_B$ are maximal eigenvalues of $A$ and $B$, respectively.

Proof Denote by $\mathbf{v} \gg 0$ a maximal eigenvector of the irreducible matrix $A + B$, corresponding to its maximal eigenvalue $r_{A+B}$. Since $k$-commutativity implies property L, from Proposition 3.7 we get $r_{A+B} = r_A + r_B$, and hence

$$(A + B)\mathbf{v} = (r_A + r_B)\mathbf{v}.$$

It is easy to show that when $(A, B)$ is a $k$-commuting pair, $(A + B, B)$ has the same property. So, we can consider this new pair and get

$$\overbrace{[A + B, [A + B, [\ldots [A + B}^{k \text{ times}}, B] \ldots]]]\mathbf{v} = 0. \tag{4.4}$$

On the other hand, we have also

$$[A+B,\ B]\mathbf{v} = \Big(A+B-(r_A+r_B)I\Big)B\mathbf{v}$$

and, inductively, we see that

$$\overbrace{[A+B,[A+B,[\dots[A+B,}^{h-1\ \text{times}}B]\dots]]]\mathbf{v} = \Big(A+B-(r_A+r_B)I\Big)^{h-1}B\mathbf{v}$$

implies

$$\begin{aligned}
&\overbrace{[A+B,[A+B,[\dots[A+B,}^{h\ \text{times}}B]\dots]]]\mathbf{v} \\
&= (A+B\,\overbrace{[A+B,[A+B,[\dots[A+B}^{h-1\ \text{times}}B]\dots]]] - \overbrace{[A+B,[A+B,[\dots[A+B,}^{h-1\ \text{times}}B]\dots]]]A+B)\mathbf{v} \\
&= (A+B)(A+B-(r_A+r_B)I)^{h-1}B\mathbf{v} - \overbrace{[A+B,[A+B,[\dots[A+B,}^{h-1\ \text{times}}B]\dots]]](r_A+r_B)\mathbf{v} \\
&= \Big(A+B-(r_A+r_B)I\Big)^{h}B\mathbf{v}.
\end{aligned}$$

Thus (4.4) can be rewritten as

$$(A+B-(r_A+r_B)I)^k B\mathbf{v} = 0,$$

which shows that $B\mathbf{v}$ is a generalized eigenvector of $A+B$, corresponding to the maximal eigenvalue $r_A+r_B$. However, since the algebraic multiplicity of $r_A+r_B$ is 1, we have also

$$(A+B-(r_A+r_B)I)B\mathbf{v} = 0,$$

and $B\mathbf{v} > 0$ has to be a maximal eigenvector of $A+B$. Since an irreducible matrix has exactly one (maximal) eigenvector [21] in $E^n := \{x \in \mathbf{R}_+^n : \sum_{i=1}^n x_i = 1\}$, and both $\mathbf{v}$ and $B\mathbf{v}$ are positive maximal eigenvectors of $A+B$, we get

$$B\mathbf{v} = \mu\mathbf{v}, \quad \mu > 0. \tag{4.5}$$

We claim that $\mu = r_B$. If not, we would have

$$A\mathbf{v} = (A+B)\mathbf{v} - B\mathbf{v} = (r_A+r_B)\mathbf{v} - \mu\mathbf{v} = (r_A+r_B-\mu)\mathbf{v} =: \lambda\mathbf{v},$$

where $\lambda := r_A+r_B-\mu \neq r_A$. So, $(r_A, r_B)$ and $(\lambda, \mu)$ would be pairs of corresponding eigenvalues in the coupling determined by Property L and

$$\lambda + \mu = r_A + r_B = r_{A+B},$$

would imply that the maximal eigenvalue of $A+B$ is not simple, a contradiction. Therefore $\mu$ has to coincide with $r_B$. ■

We consider now briefly the second problem, namely what is the structure of the maximal eigenvector of $\alpha A + \beta B,\ \alpha, \beta \in \mathbf{R}_+$, when $A$ and $B$ are positive matrices with property L and $A+B$ is irreducible.

First of all, both matrices have a strictly positive maximal eigenvalue. Otherwise, one of them would be nilpotent, which implies that $(A, B)$ is separable and hence, by Lemma 2.4, that $A+B$ is reducible. Suppose, for the moment, that both $A$ and $B$ have a unitary maximal eigenvalue, and consider any convex combination of $A$ and $B$

$$\gamma A+(1-\gamma)B, \quad \gamma \in [0,1]. \tag{4.6}$$

This combination has a simple maximal eigenvalue $r_{A+B}=1$ for all $\gamma \in (0,1)$, and, consequently, the rank of the polynomial matrix $(B-I)+s(A-B)$ over the field $\mathbf{R}(s)$ is $n-1$.

**Lemma 4.2** [13, vol.II, pp.30] Let

$$\mathbf{v}(s)=\mathbf{v}_0+\mathbf{v}_1 s+\ldots+\mathbf{v}_t s^t, \quad \mathbf{v}_t \neq 0 \tag{4.7}$$

be a minimum degree nonzero polynomial vector which satisfies the equation

$$\Big[(B-I)+s(A-B)\Big]\mathbf{v}(s)=0. \tag{4.8}$$

Then vectors $\mathbf{v}_i \in \mathbf{R}^n,\ i=0,1,\ldots,t$, are linearly independent. ■

Clearly $\mathbf{v}(s)$ is uniquely determined, up to a multiplicative constant, which can be chosen so as to guarantee that $\mathbf{v}(\bar{\gamma})$ is positive for some $\bar{\gamma} \in (0,1)$. As a consequence of Lemma 4.2, the vector $\mathbf{v}(\gamma)$ is positive for all $\gamma \in [0,1]$, and provides the structure of the maximal eigenvector.

The general case, when the maximal eigenvalues of $A$ and $B$ are not necessarily 1 and the combination $\alpha A+\beta B$ is not necessarily convex, easily reduces to the previous one. Actually, once we set $\bar{A} := A/r_A$, $\bar{B} := B/r_B$, $r_A$ and $r_B$ the maximal eigenvalues of $A$ and $B$ respectively, we can consider the minimal degree solution $\mathbf{v}(s)$ of $\Big[(\bar{B}-I)+s(\bar{A}-\bar{B})\Big]\mathbf{v}(s)=0$. Clearly $\mathbf{v}(\dfrac{\alpha r_A}{\alpha r_A+\beta r_B})$ is a maximal eigenvector of $\alpha A+\beta B$.

If we drop the assumption that $A$ and $B$ have property L and look for general statements on nonnegative pairs with a (strictly) positive common eigenvector, we get a characterization in terms of stochastic matrices.

**Proposition 4.3** Assume that $A$ and $B$ are positive matrices, with $A+B$ irreducible. $A$ and $B$ have a positive common eigenvector if and only if their maximal eigenvalues $r_A$ and $r_B$ are positive and there exists a nonsingular positive diagonal matrix $D$ such that $r_A^{-1}D^{-1}AD$ and $r_B^{-1}D^{-1}BD$ are row stochastic matrices.

Proof Assume that $r_A$ and $r_B$ are positive and, for some positive matrix

$$D=\operatorname{diag}\{d_1, d_2, \ldots, d_n\}, \quad d_i>0,$$

$r_A^{-1}D^{-1}AD$ and $r_B^{-1}D^{-1}BD$ are row stochastic.

Clearly $[1\ 1\ ...\ 1]^T \gg 0$ is a common eigenvector of $D^{-1}AD$ and $D^{-1}BD$, relative to $r_A$ and $r_B$. Thus

$$\mathbf{d} := [d_1\ d_2\ \dots d_n]^T \gg 0$$

is a common eigenvector of $A$ and $B$, associated with their maximal eigenvalues.

Conversely, suppose that $A$ and $B$ have a common eigenvector $\mathbf{d} = [d_1\ d_2\ ...\ d_n]^T > 0$. As $A + B$ is irreducible, $(A + B)\mathbf{d} = r_{A+B}\mathbf{d}$ and $\mathbf{d} > 0$ imply $\mathbf{d} \gg 0$. Moreover $A, B \neq 0$ together with $r_A\mathbf{d} = A\mathbf{d} \neq 0$ and $r_B\mathbf{d} = B\mathbf{d} \neq 0$ imply $r_A, r_B > 0$. Then $D := \text{diag}\{d_1, d_2, \dots, d_n\}$ provides [3] the similarity transformation we are looking for.

■

# 5 Inverse spectral problem

The inverse spectral problem for nonnegative matrix pairs can be stated in a very simple way as follows: "what are the necessary and sufficient conditions for a polynomial in two variables

$$p(z_1, z_2) = 1 - \sum_{i+j>0} p_{ij} z_1^i z_2^j$$

to be the characteristic polynomial of a nonnegative matrix pair $(A, B)$ ?"
The above question can be appropriately framed into the more general setting of realization theory of dynamical systems [6,15]. In the 2D case, the transfer function of some filter is given as the ratio of two coprime polynomials

$$w(z_1, z_2) = m(z_1, z_2)/p(z_1, z_2),$$

and one looks for a 2D system in state space form whose input response is $w(z_1, z_2)$. For every state space model that solves the problem, the characteristic polynomial $\Delta_{A,B}(z_1, z_2)$ of matrices $A$ and $B$ that provide the "free state updating" has to be a multiple of $p(z_1, z_2)$. Consequently, the inverse spectral problem reduces to verify whether some positive system can be found whose transfer function has $p(z_1, z_2)$ as denominator.

Although the inverse spectral problem, as set above, is still unsolved, interesting results can be obtained by introducing some restrictions on $p(z_1, z_2)$ and/or $(A, B)$. In this section we aim to present a sufficient condition for solvability, which allows to explicitely construct a matrix pair $(A, B)$ satisfying $\Delta_{A,B}(z_1, z_2) = p(z_1, z_2)$ and the extra requirement that $A + B$ is irreducible.
When we consider a polynomial which splits into linear factors

$$p(z_1, z_2) = \prod_{i=1}^{n} (1 - \lambda_i z_1 - \mu_i z_2),$$

and hence we look for matrix pairs with property L, the aforementioned condition specializes into a constraint on $\lambda_i$ and $\mu_i$, which is reminiscent of an important 1D result of Suleimanova [27].

**Lemma 5.1** Let

$$p(z_1, z_2) = 1 - \sum_{i+j>0} p_{ij} z_1^i z_2^j \in \mathbf{R}[z_1, z_2], \tag{5.1}$$

and suppose that the integers $r$ and $s$ satisfy

$$\deg_{z_1}(p) \leq r, \quad \deg_{z_2}(p) \leq s, \quad \deg(p) \leq r + s - 1. \tag{5.2}$$

Then there exists a pair $(A, B)$ of $(r + s - 1) \times (r + s - 1)$ matrices which satisfies

$$\Delta_{A,B}(z_1, z_2) = p(z_1, z_2). \tag{5.3}$$

Moreover, when all coefficients $p_{ij}$ are nonnegative, all entries of $(A, B)$ can be chosen nonnegative.

PROOF There is no restriction in assuming $r \leq s$. Thus $p(z_1, z_2)$ can be rewritten as

$$\begin{aligned} p(z_1, z_2) \;=\; & 1 - h_{0,0} - (z_1 h_{1,0} + z_2 h_{0,1}) - (z_1^2 h_{2,0} + z_1 z_2 h_{1,1} + z_2^2 h_{0,2}) - \ldots \\ & -(z_1^{r-1} z_2^{s-r} h_{r-1,s-r} + z_1^{r-2} z_2^{s-r+1} h_{r-2,s-r+1} + \ldots + z_2^{s-1} h_{0,s-1}) \\ & - \ldots - z_1^{r-1} z_2^{s-1} h_{r-1,s-1}, \end{aligned} \tag{5.4}$$

where $h_{i,j} = \alpha_{i,j} z_1 + \beta_{i,j} z_2$ are suitable linear forms. In general, $p(z_1, z_2)$ does not uniquely determine the forms $h_{i,j}(z_1, z_2)$. In any case, when the $p_{ij}$'s are nonnegative, it is always possible to assume that all linear forms in (5.4) have nonnegative coefficients.

Applying the Laplace theorem for the expansion of a determinant, one sees that the $(r + s - 1) \times (r + s - 1)$ polynomial matrix

$$L(z_1, z_2) = \begin{bmatrix} 1 & & & & & -h_{r-1,0} & -h_{r-1,1} & \cdots & \cdots & -h_{r-1,s-1} \\ -z_1 & 1 & & & & -h_{r-2,0} & -h_{r-2,1} & \cdots & \cdots & -h_{r-2,s-1} \\ & -z_1 & 1 & & & -h_{r-3,0} & -h_{r-3,1} & \cdots & \cdots & -h_{r-3,s-1} \\ & & \ddots & \ddots & & \vdots & \vdots & \cdots & \cdots & \vdots \\ & & & -z_1 & 1 & -h_{1,0} & -h_{1,1} & \cdots & \cdots & -h_{1,s-1} \\ & & & & -z_1 & 1 - h_{0,0} & -h_{0,1} & \cdots & \cdots & -h_{0,s-1} \\ & & & & & -z_2 & 1 & & & \\ & & & & & & -z_2 & \ddots & & \\ & & & & & & & \ddots & & \\ & & & & & & & & -z_2 & 1 \end{bmatrix} \tag{5.5}$$

satisfies

$$\det L(z_1, z_2) = p(z_1, z_2).$$

Consequently, the following matrices, whose elements are the opposite of the coefficients of $z_1$ and $z_2$ in $L(z_1, z_2)$

$$A = \begin{bmatrix}
0 & 0 & \dots & 0 & 0 & \alpha_{r-1,0} & \alpha_{r-1,1} & \cdots & \cdots & \alpha_{r-1,s-1} \\
1 & 0 & & & & \alpha_{r-2,0} & \alpha_{r-2,1} & \cdots & \cdots & \alpha_{r-2,s-1} \\
& 1 & & & & \alpha_{r-3,0} & \alpha_{r-3,1} & \cdots & \cdots & \alpha_{r-3,s-1} \\
& & \ddots & \ddots & & \vdots & \vdots & \dots & \dots & \vdots \\
& & & 1 & & \alpha_{1,0} & \alpha_{1,1} & \cdots & \cdots & \alpha_{1,s-1} \\
& & & & 1 & \alpha_{0,0} & \alpha_{0,1} & \cdots & \cdots & \alpha_{0,s-1} \\
& & & & & & & & & \\
& & \mathbf{0} & & & & & \mathbf{0} & & \\
& & & & & & & & &
\end{bmatrix}$$

and

$$B = \begin{bmatrix}
& \beta_{r-1,0} & \beta_{r-1,1} & \cdots & \cdots & \beta_{r-1,s-1} \\
& \beta_{r-2,0} & \beta_{r-2,1} & \cdots & \cdots & \beta_{r-2,s-1} \\
& \beta_{r-3,0} & \beta_{r-3,1} & \cdots & \cdots & \beta_{r-3,s-1} \\
\mathbf{0} & \vdots & \vdots & \dots & \dots & \vdots \\
& \beta_{1,0} & \beta_{1,1} & \cdots & \cdots & \beta_{1,s-1} \\
& \beta_{0,0} & \beta_{0,1} & \cdots & \cdots & \beta_{0,s-1} \\
& 1 & & & & 0 \\
\mathbf{0} & & 1 & & & 0 \\
& & & \ddots & & \vdots \\
& & & & 1 & 0
\end{bmatrix}$$

satisfy equation (5.3). ∎

**Proposition 5.2** If in (5.1) all coefficients $p_{ij}$ of the polynomial $p(z_1, z_2)$ are nonnegative, there exists a pair of nonnegative matrices $(A, B)$, with $A + B$ irreducible, such that (5.3) is satisfied.

Proof Let $r = \deg_{z_1}(p)$, $s = \deg_{z_2}(p)$, and suppose first $r + s > \deg(p)$. The previous lemma allows to construct two nonnegative matrices $A$ and $B$, of dimension $(r + s - 1) \times (r + s - 1)$, which satisfy equation (5.3). Moreover the assumption on the degree implies that in $M := A + B$ there exist at least a nonzero element $m_{1,\kappa}$, $\kappa \geq r$, in the first row, and at least a nonzero element $m_{\rho,r+s-1}$, $\rho \leq r$, in the last column. As for every $i = 1, 2, .., r+s-1$, the elements $m_{i+1,i}$ are 1, we see that, given two arbitrary positive integers $i, j \leq r + s - 1$, the digraph $D(M)$ includes a directed path from vertex $i$ to vertex $j$. This is trivial if $i > j$. If not, just consider the sequence of arcs:

$$(i, i-1)(i-1, i-2) \dots (1, \kappa)(\kappa, \kappa-1) \dots (\rho+1, \rho)(\rho, r+s-1)(r+s-1, r+s-2) \dots (j+1, j).$$

Therefore $M$ is an irreducible matrix.
If $\deg(p) = r + s$, assume that $p(z_1, z_2)$ has "formal" degree $r + 1$ in the variable $z_1$ (e.g. by introducing in the expression of $p$ the monomial $0z_1^{r+1}$), and repeat the

construction of Lemma 5.1. In this case we end up with two nonnegative matrices $A$ and $B$ of dimension $r+s$, and $M$ exhibits a nonzero element in position $(1, r+s)$. This again proves that $M$ is irreducible. ■

An obvious necessary condition for the solvability of the inverse spectral problem is that the 1D inverse spectral problems corresponding to the polynomials

$$p(z_1, 0) = 1 - \sum_i p_{i0} z_1^i \quad \text{and} \quad p(0, z_2) = 1 - \sum_i p_{0j} z_2^j$$

have a solution, which amounts to say that nonnegative matrices $A$ and $B$ can be found, such that

$$p(z_1, 0) = \det(I - Az_1) \quad \text{and} \quad p(0, z_2) = \det(I - Bz_2). \tag{5.6}$$

In general it is not possible to reduce a 2D inverse spectral problem to a pair of 1D problems, as the solvability of (5.6) is far from implying that equation (5.3) is solvable by resorting to a nonnegative matrix pair. Moreover, as no general solution to the 1D spectral problem is available [21], this kind of approach seems to be even more questionable.
A special case, however, deserves some attention, namely when

1. in $p(z_1, 0) = \prod_{i=1}^n (1 - \lambda_i z_1)$ and $p(0, z_2) = \prod_{i=1}^n (1 - \mu_i z_2)$, $\lambda_i$ and $\mu_i$ are real, for every $i$, and satisfy the Suleimanova conditions for the solvability of the 1D inverse spectral problem:

$$\begin{array}{ll} \lambda_1 > 0 \geq \lambda_i, \quad \forall i \geq 2 \;\text{ and }\; \sum_{i=1}^n \lambda_i > 0 \\ \mu_1 > 0 \geq \mu_i, \quad \forall i \geq 2 \;\text{ and }\; \sum_{i=1}^n \mu_i > 0; \end{array} \tag{5.7}$$

2. $p(z_1, z_2)$ factors into a product of linear factors, as follows

$$p(z_1, z_2) = \prod_{i=1}^n (1 - \lambda_i z_1 - \mu_i z_2). \tag{5.8}$$

When (5.7) and (5.8) are fulfilled, the 2D inverse spectral problem is solvable and a solution $(A, B)$ can be found with $A + B$ irreducible.
Taking into account Lemma 5.1 and Proposition 5.2, we are reduced to prove that the coefficients $p_{ij}$ of $p(z_1, z_2)$ are nonnegative, which is the content of the next proposition.

**Proposition 5.3** Suppose that $\lambda_i$ and $\mu_i$, $i = 1, 2 \ldots, n$, are real numbers satisfying (5.7). Then in

$$p(z_1, z_2) = \prod_{i=1}^n (1 - \lambda_i z_1 - \mu_i z_2) = 1 - \sum_{i+j=1}^n p_{ij} z_1^i z_2^j$$

all coefficients $p_{ij}$ are nonnegative.

Proof Let $\nu < n$ and assume that in

$$\prod_{i=1}^{\nu}(1-\lambda_i z_1-\mu_i z_2) = 1-\sum_{h+k=1}^{\nu} p_{hk}^{(\nu)} z_1^h z_2^k,$$

$\sum_{i=1}^{\nu}\lambda_i$, $\sum_{i=1}^{\nu}\mu_i$ and $p_{hk}^{(\nu)}$, for every $h,k$, are nonnegative.
Keeping in mind (5.7), it is easy to check that in

$$\prod_{i=1}^{\nu+1}(1-\lambda_i z_1-\mu_i z_2) = \Big(1-\sum_{i=1}^{\nu}\lambda_i z_1-\sum_{i=1}^{\nu}\mu_i z_2-\sum_{h+k=2}^{\nu} p_{hk}^{(\nu)} z_1^h z_2^k\Big)(1-\lambda_{\nu+1}z_1-\mu_{\nu+1}z_2)$$

all coefficients of nonconstant monomials are also nonnegative. Thus the result follows by induction on $\nu$. ■

As a consequence of the above propositions we have an algorithm for producing nontrivial examples of positive pairs with property L and arbitrarily high dimension.

**Example** Consider the polynomial

$$p(z_1,z_2) = (1-z_1-z_2)\left(1+\frac{z_1}{2}+\frac{z_2}{2}\right)\left(1+\frac{z_1}{4}\right). \tag{5.9}$$

We aim to obtain a pair of $4\times 4$ positive matrices $(A,B)$, with irreducible sum, satisfying $\Delta_{A,B}(z_1,z_2)=p(z_1,z_2)$.
First of all, note that the pair $(A,B)$ has property L w.r.t. the following orderings of the spectra

$$\Lambda(A)=(1,-1/2,-1/4,0) \quad\text{and}\quad \Lambda(B)=(1,-1/2,0,0). \tag{5.10}$$

Next rewrite $p(z_1,z_2)$ as follows

$$p(z_1,z_2) = 1-(\frac{1}{4}z_1+\frac{1}{2}z_2)-\Big(z_1(\frac{5}{8}z_1+\frac{5}{8}z_2)+z_2(\frac{1}{2}z_1+\frac{1}{2}z_2)\Big)-\Big(z_1^2(\frac{1}{8}z_1+\frac{1}{8}z_2)+z_1z_2(\frac{1}{8}z_1+\frac{1}{8}z_2)\Big),$$

and use the coefficients of the linear forms to construct $A$ and $B$, according to Lemma 5.1.
We obtain

$$A=\begin{bmatrix}0&0&1/8&0\\1&0&5/8&1/8\\0&1&1/4&1/2\\0&0&0&0\end{bmatrix} \qquad\text{and}\qquad B=\begin{bmatrix}0&0&1/8&0\\0&0&5/8&1/8\\0&0&1/2&1/2\\0&0&1&0\end{bmatrix},$$

which fulfill all the requirements.

The above example is interesting from different points of view. First of all, the traces of the matrix products $A^2B^2$ and $ABAB$ do not coincide. This rules out the possibility [10,24] that $A$ and $B$ have property P, which, therefore, is stronger than property L even in the class of nonnegative matrices.

Also, as the maximal eigenvectors of $A$ and $B$, computed by numerical methods, are

$$[.0995\ .5970\ .7690\ \ 0]^T \quad \text{and} \quad [.0778\ .4671\ .6228\ .6228]^T,$$

we see once again that the maximal eigenvectors in a nonnegative pair with property L may be nonparallel, even though the maximal eigenvalues are coupled in the characteristic polynomial.

# 6 Extensions to 1-linearity

A weaker property, that nevertheless entails some interesting consequences on the structure of the matrix pairs, especially nonnegative ones, is 1-linearity. A matrix pair $(A, B)$ is 1-*linear* [22] w.r.t. the pair of eigenvalues $(\lambda_i, \mu_i)$, $\lambda_i \in \Lambda(A), \mu_i \in \Lambda(B)$, if for every $\alpha, \beta \in \mathbf{C}$, $\alpha\lambda_i + \beta\mu_i \in \Lambda(\alpha A + \beta B)$.
It is clear that property L can be viewed as a stronger version of 1-linearity, and, as a matter of fact, some results on property L could be obtained by strengthening the corresponding statements on the other property.

Here we confine ourselves to nonnegative pairs $(A, B)$, where $A$ is diagonal with distinct elements. The result we are going to present depends on a preliminary lemma, whose proof follows the same lines as the proof of Theorem 1 in [22].

**Lemma 6.1** Let $A = \text{diag}\{a_1, a_2, \ldots, a_n\}$, with $a_i \neq a_j$ if $i \neq j$, and $B = [b_{ij}]$ be $n \times n$ matrices. If $(A, B)$ is 1-linear w.r.t. $(a_i, \mu_i)$, then $\mu_i$ coincides with $b_{ii}$.

Prooof There is no loss of generality in assuming $i = 1$. To prove the result, assume also $a_1 = 0$, as 1-linearity is not affected by translations. Write then

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & & & \\ 0 & & A_{22} & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & & & \\ b_{31} & & B_{22} & \\ \vdots & & & \\ b_{n1} & & & \end{bmatrix},$$

where $A_{22}$ is an $(n-1) \times (n-1)$ nonsingular matrix (as all $a_j$'s are different from $a_1 = 0$). By 1-linearity, it follows that

$$\det(zI - \alpha A - B) = (z - \mu_1)\ p(z; \alpha), \qquad \forall \alpha \in \mathbf{C}, \tag{6.1}$$

$p(z; \alpha)$ a suitable polynomial in $z$ and $\alpha$. Equating the coefficients of the $(n-1)$-th power of $\alpha$ on the two sides of equation (6.1) gives

$$(z - b_{11}) \det A_{22} = (z - \mu_1) \cdot \text{coeff. } \alpha^{n-1} \text{in } p(z; \alpha).$$

Since $A_{22}$ is not singular, it is clear that $z - \mu_1 = z - b_{11}$, which proves the result. ∎

**Proposition 6.2** Let $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ be a non negative matrix with $a_i \neq a_j$ if $i \neq j$, and let $B = [b_{ij}]$ be an $n \times n$ positive matrix. Suppose that $\bar{\mu} \in \Lambda(B)$ is a maximal eigenvalue of B. The following statements are equivalent:

*i)* $(A, B)$ is 1-linear w.r.t. a pair of eigenvalues $(a_i, \bar{\mu})$;

*ii)* $\bar{\mu} = b_{ii}$ for some $i$;

*iii)* there exists a permutation matrix $P$ s.t. $P^T BP$ has the block-structure

$$P^T BP = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & \bar{\mu} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix}; \tag{6.2}$$

*iv)* $(A^h, B^k)$ is 1-linear w.r.t. a pair $(a_i^h, \bar{\mu}^k)$ for every $h, k \geq 1$.

Proof *i)* $\Rightarrow$ *ii)* True by Lemma 6.1.

*ii)* $\Rightarrow$ *iii)* If $\bar{\mu} = b_{ii}$, then $B$ has to be reducible. If not, there would be an eigenvector $\mathbf{v} := [v_1\ v_2\ ...\ v_n]^T \gg 0$, corresponding to $b_{ii}$, such that $B\mathbf{v} = b_{ii}\mathbf{v}$. If we consider the $i$-th entry of the vectors on both sides, we get the equality

$$\sum_{j=1}^{n} b_{ij} v_j = b_{ii} v_i. \tag{6.3}$$

As $v_j > 0$ for every $j$, (6.3) implies $b_{ij} = 0$ for every $j \neq i$, and hence $B$ would be reducible, a contradiction. So, a permutation matrix $P_1$ can be found, such that

$$P_1^T BP_1 = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

It is obvious now that property *ii)* is inherited either by $B_{11}$ or by $B_{22}$, so we can apply to one of these submatrices the same reasoning as before, thus getting (6.2).

*iii)* $\Rightarrow$ *iv)* When the same permutation matrix $P$ that brings $B$ to the form shown in (6.2) is applied to $A$, we get a matrix pair $(P^T AP, P^T BP)$ that is clearly 1-linear w.r.t. some pair $(a_i, \bar{\mu})$. This immediately implies *iv)*.

*iv)* $\Rightarrow$ *i)* Obvious. ■

**Remark** Once we drop the assumption that $\bar{\mu}$ is a maximal eigenvalue of $B$, the above proposition is no more true. Actually, while the implications *iii)* $\Rightarrow$ *iv)* $\Rightarrow$ *i)* $\Rightarrow$ *ii)* still hold, condition *ii)* no longer implies *iii)*. This can be seen, for instance, when considering the matrix pair

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

that is 1-linear w.r.t. $(a_3, \bar{\mu}) = (2,1)$, $\bar{\mu}$ is not maximal, and satisfies *ii*) with $\bar{\mu} = b_{33} = 1$, but it clearly does not fulfill *iii*) and *iv*).

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