Directed graphs, 2D state models and characteristic polynomials of irreducible matrix pairs

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Running title: Irreducible matrix pairs

Abstract

In the paper the definition and main properties of a 2D-digraph, namely a directed graph with two kinds of arcs, are introduced. Under the assumption of strong connectedness, the analysis of its paths and cycles is performed, basing on an integer matrix whose rows represent the compositions of all circuits, and on the corresponding row-module.

Natural constraints on the composition of the paths connecting each pair of vertices lead to the definition of a 2D-strongly connected digraph. For a 2D-digraph of this kind the set of vertices can be partitioned into disjoint 2D-imprimitivity classes, whose number and composition are strictly related to the structure of the row module.

Irreducible matrix pairs, i.e. pairs endowed with a 2D-strongly connected digraph, are subsequently discussed. Equivalent descriptions of irreducibility, naturally extending those available for a single irreducible matrix, are obtained. These refer to the free evolution of the 2D state models described by the pairs and to their characteristic polynomials.

Finally, primitivity is viewed as a special case of irreducibility, and completely characterized in terms of 2D-digraphs, characteristic polynomials and 2D system dynamics.

1 Introduction

Among the several known approaches to the theory of discrete positive dynamical systems one can distinguish, apart from the achieved generality, two basically different methods. The first one defines the updating of states via a certain set of difference equations, involving positive matrices, and hence extracts summarizing features - such as supports, recurrencies, permanent patterns, etc. - from the system trajectories. The second defines the admissible state transitions via a directed graph, and hence reduces the analysis of the above features to simple conditions on its connecting structure.

The former method is more powerful, as it provides answers to metric questions the latter is unable to deal with, like the directions of the dominating eigenvectors or the quantitative

evaluation of the asymptotic behavior. There are problems, however, whose combinatorial nature is better enlightened when formulated in graph theoretic terms. In this contribution we adopt this second viewpoint to discuss some aspects of positive homogeneous linear systems evolving on a two-dimensional discrete grid (2D systems) [2].
The notion of 2D-connectedness of a graph will constitute a natural starting point, because it induces dynamical evolutions in ^{2} whose supports are "large enough", and has the added advantage of offering an easy introduction to imprimitivity classes, congruence relations etc., with which the interested reader is likely to be familiar in the one-dimensional context. The stress in the present approach is on the extension of classical Perron-Frobenius theory to matrix pairs and to their characteristic polynomials, and on the dynamical interpretation of some consequences of this theory. Moreover, the zero patterns of 2D system trajectories, arising either from a single initial condition or from an infinite set of initial conditions, are easily understood and described in neat algebraic terms when irreducible and primitive matrix pairs are considered.

The paper is organized as follows: in sections 2 and 3 the elementary concepts and properties of a 2D-digraph $\mathcal{D}^{(2)}$, i.e. a directed graph with two kinds of arcs (named $\mathcal{A}$-arcs and $\mathcal{B}$-arcs), are introduced. Under the assumption that $\mathcal{D}^{(2)}$ is strongly connected, the structure of its paths and cycles is investigated both in a purely graph theoretic setting and from an algebraic point of view, based on certain integer matrices and the associated -modules.

In sections 4 and 5, after introducing the notion of irreducible matrix pair in terms of the associated 2D-digraph, we investigate the variety of the characteristic polynomial and the dynamical behavior of a linear homogeneous 2D system, described by an irreducible pair. Finally, primitive pairs, i.e. irreducible pairs with unitary imprimitivity index are discussed.

In order not to digress too far, the notions of directed graphs, positive matrices and a few of their elementary properties and interrelations are (though briefly explained for notational purposes) assumed as known: adequate information can be found e.g. in [1, 3, 11]. Also, in the attempt to gain the basic information on the subject as economically as possible, no detailed account is included on the basics of 2D system theory; the interested reader is referred, for instance, to [2, 6, 7].

Matrices and vectors are represented by capital italic and lower case boldface letters, respectively, while their entries by the corresponding lower case italic letters.
If $F = [f_{ij}]$ is a matrix (in particular, a vector), we write $F \gg 0$ (F *strictly positive*), if $f_{ij} > 0$ for all i, j; $F > 0$ (F *positive*), if $f_{ij} \geq 0$ for all i, j, and $f_{\bar{i}\bar{j}} > 0$ for some pair $(\bar{i}, \bar{j})$; $F \geq 0$ (F *nonnegative*), if $f_{ij} \geq 0$ for all i, j.
The *spectral radius* of a matrix F is denoted by $\rho(F)$.

2 Cyclic structure of 2D-digraphs

A *2D-digraph* $\mathcal{D}^{(2)}$ is a triple $(V, \mathcal{A}, \mathcal{B})$, where $V = \{v_1, v_2, \dots, v_n\}$ is the set of *vertices*, and $\mathcal{A}$ and $\mathcal{B}$ are subsets of $V \times V$ whose elements are called $\mathcal{A}$-*arcs* and $\mathcal{B}$-*arcs*, respectively. There is an $\mathcal{A}$-arc from v_i to v_j if (v_i, v_j) is in $\mathcal{A}$, and a $\mathcal{B}$-arc if (v_i, v_j) is in $\mathcal{B}$.
When assigning a *path* p in $\mathcal{D}^{(2)}$ one has to specify, for each pair of consecutive vertices, which kind of arc they are connected by, thus giving p a representation like $(v_{i_0}, v_{i_1})_{\mathcal{A}}$, $(v_{i_1}, v_{i_2})_{\mathcal{B}}$, $\dots, (v_{i_{k-1}}, v_{i_k})_{\mathcal{B}}$. Sometimes, however, when we are interested only in the vertices p passes through, we drop the subscripts. Also, when the emphasis is only in the initial and final vertices, we adopt the shorthand notation $v_{i_0} \xrightarrow[p]{} v_{i_k}$.

If we denote by $\alpha(p)$ and $\beta(p)$ the number of $\mathcal{A}$-arcs and $\mathcal{B}$-arcs occurring in p, then $[\alpha(p) \quad \beta(p)]$ is the *composition* of p and $|p| = \alpha(p) + \beta(p)$ its *length*. A path whose extreme vertices coincide, i.e. $v_{i_0} = v_{i_k}$, is called a *cycle*. In particular, if each vertex in a cycle appears exactly once as the first vertex of an arc, the cycle is called a *circuit*.

Definition A 2D-digraph $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ is called

i) *strongly connected* if for every pair of vertices v_i and v_j in V there is a path p connecting v_i to v_j;

ii) *2D-strongly connected* if for every pair of vertices v_i and v_j in V there are two paths $v_i \xrightarrow[p_1]{} v_j$ and $v_i \xrightarrow[p_2]{} v_j$, connecting v_i to v_j, for which

$$\det \begin{bmatrix} \alpha(p_1) & \beta(p_1) \\ \alpha(p_2) & \beta(p_2) \end{bmatrix} \neq 0. \tag{1}$$

The 2D-digraph $\mathcal{D}^{(2)}$ is naturally associated with a 1-*digraph* (i.e. a standard digraph) $\mathcal{D}^{(1)} = (V, \mathcal{E})$, having the same vertices as $\mathcal{D}^{(2)}$ and $\mathcal{E} := \mathcal{A} \cup \mathcal{B}$ as its set of arcs. So, property i) corresponds to the fact that $\mathcal{D}^{(1)}$ is strongly connected (in the ordinary sense), while property ii) requires something more, namely that for every pair of vertices, v_i and v_j, the ratio $\beta(p)/\alpha(p)$, (that is considered ∞ when $\alpha(p) = 0$), between the number of $\mathcal{B}$-arcs and $\mathcal{A}$-arcs is not invariant as p varies over the set of all paths connecting v_i to v_j.

In this paper all 2D-digraphs will be assumed strongly connected with both sets $\mathcal{A}$ and $\mathcal{B}$ nonempty.

We associate with the (finite) set $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ of all circuits in $\mathcal{D}^{(2)}$ (arbitrarily ordered) the *circuit matrix*

$$L(\mathcal{D}^{(2)}) := \begin{bmatrix} \alpha(\gamma_1) & \beta(\gamma_1) \\ \alpha(\gamma_2) & \beta(\gamma_2) \\ \vdots & \vdots \\ \alpha(\gamma_t) & \beta(\gamma_t) \end{bmatrix} \quad \in {}^{t\times 2}, \tag{2}$$

and denote by $M(\mathcal{D}^{(2)})$ the -module generated by its rows. Since every cycle γ in $\mathcal{D}^{(2)}$ decomposes into a certain number of circuits, it follows that there exist $n_1, n_2, \ldots, n_t \in$, such that

$$[\alpha(\gamma) \quad \beta(\gamma)] = [n_1 \ n_2 \ \ldots \ n_t] \quad L(\mathcal{D}^{(2)}), \tag{3}$$

i.e. $[\alpha(\gamma) \quad \beta(\gamma)]$ is an element of $M(\mathcal{D}^{(2)}) \cap {}^2$. In general, however, $M(\mathcal{D}^{(2)}) \cap {}^2$ properly includes the set of integer pairs representing the compositions of the cycles in $\mathcal{D}^{(2)}$.

As a submodule of ^{2}, $M(\mathcal{D}^{(2)})$ admits a basis consisting either of one or of two elements. In the first case $M(\mathcal{D}^{(2)})$ has only two possible bases, namely $\{[\ell \ m]\}$, for some positive integers ℓ and m, and its opposite $\{-[\ell \ m]\}$, and every circuit γ_j in $\mathcal{D}^{(2)}$ consists of $k_j \ell$ $\mathcal{A}$-arcs and $k_j m$ $\mathcal{B}$-arcs, for a suitable k_j in . On the other hand, when $L(\mathcal{D}^{(2)})$ has rank 2, we can consider its Hermite form over

$$\bar{H} := \begin{bmatrix} H \\ \hline 0 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & h_{22} \\ \hline \multicolumn{2}{c}{0} \end{bmatrix} = \bar{U} L(\mathcal{D}^{(2)}), \tag{4}$$

$\bar{U} \in^{t\times t}$ unimodular, and assume (without loss of generality) that h_{11} and h_{22} are positive integers, and $0 \leq h_{12} < h_{22}$. The rows of H provide a particular basis of $M(\mathcal{D}^{(2)})$, $\{[h_{11}\ h_{12}], [0\ h_{22}]\}$, and the rows $[w_{11}\ w_{12}]$ and $[w_{21}\ w_{22}]$ of $W = UH$, as U varies over the group of unimodular matrices in $^{2\times 2}$, give all possible bases of $M(\mathcal{D}^{(2)})$. Since the determinants of all matrices UH have the same modulus, which is the g.c.d. of the second order minors of $L(\mathcal{D}^{(2)})$, all parallelograms $\{\varepsilon[w_{11}\ w_{12}] + \delta[w_{21}\ w_{22}] : \varepsilon, \delta \in [0, 1)\}$, have the same area, that coincides with the number of integer pairs they include [4, 9].

The cyclic structure of $\mathcal{D}^{(2)}$ and the module $M(\mathcal{D}^{(2)})$ provide enough information to decide whether the 2D-digraph is 2D-strongly connected, as shown in the following proposition.

Proposition 2.1 *Let $\mathcal{D}^{(2)}$ be a strongly connected 2D-digraph. The following facts are equivalent*

i) *$\mathcal{D}^{(2)}$ is 2D-strongly connected;*

ii) *there are a vertex v_i and two cycles γ and $\tilde{\gamma}$, passing through v_i, for which*

$$\det \begin{bmatrix} \alpha(\gamma) & \beta(\gamma) \\ \alpha(\tilde{\gamma}) & \beta(\tilde{\gamma}) \end{bmatrix} \neq 0;$$

iii) *there are two circuits γ_i and γ_j satisfying*

$$\det \begin{bmatrix} \alpha(\gamma_i) & \beta(\gamma_i) \\ \alpha(\gamma_j) & \beta(\gamma_j) \end{bmatrix} \neq 0;$$

iv) *rank $L(\mathcal{D}^{(2)}) = 2$;*

v) *$M(\mathcal{D}^{(2)})$ has a basis consisting of two elements.*

Proof The equivalence of statements $iii), iv)$ and $v)$ is obvious from the previous discussion. Also, $i) \Rightarrow ii)$ follows from the definition, and $ii) \Rightarrow iii)$ from the possibility of decomposing every cycle into circuits. To prove $iii) \Rightarrow i)$, pick two arbitrary vertices v_r and v_q in V, and consider a path $v_r \xrightarrow[p]{} v_q$ that intersects circuits γ_i and γ_j in two vertices, say v_{k_i} and v_{k_j}, respectively. The composite path

$$p^{(i,n)} : v_r \to v_{k_i} \xrightarrow[n \text{ times } \gamma_i]{} v_{k_i} \to v_q$$

obtained by extending p with n repetitions of γ_i, includes $\alpha(p) + n\alpha(\gamma_i)$ $\mathcal{A}$-arcs and $\beta(p) + n\beta(\gamma_i)$ $\mathcal{B}$-arcs. Similarly

$$p^{(j,n)} : v_r \to v_{k_j} \xrightarrow[n \text{ times } \gamma_j]{} v_{k_j} \to v_q$$

includes $\alpha(p) + n\alpha(\gamma_j)$ $\mathcal{A}$-arcs and $\beta(p) + n\beta(\gamma_j)$ $\mathcal{B}$-arcs. Both $p^{(i,n)}$ and $p^{(j,n)}$ connect v_r to v_q, and it is clear that

$$\det \begin{bmatrix} \alpha(p) + n\alpha(\gamma_i) & \beta(p) + n\beta(\gamma_i) \\ \alpha(p) + n\alpha(\gamma_j) & \beta(p) + n\beta(\gamma_j) \end{bmatrix}$$

is nonzero for sufficiently large values of n.

As it is well-known, the lengths of all cycles in a strongly connected 1-digraph $\mathcal{D}^{(1)}$, with imprimitivity index h, are multiples of h, and there exists a positive integer T such that, for all integers $t \in [T, +\infty) \cap (h)$, there is a cycle in $\mathcal{D}^{(1)}$ of length t [11]. A similar statement holds for a 2D-strongly connected digraph $\mathcal{D}^{(2)}$, upon considering for each cycle γ in $\mathcal{D}^{(2)}$ not just its length, but its composition $[\alpha(\gamma) \;\; \beta(\gamma)]$. In this case the module (h) and the half-line $[T, +\infty)$ have to be replaced by $M(\mathcal{D}^{(2)})$ and by a suitable convex cone in ${}^2_+$, respectively.
To prove this fact, we need the following technical lemma, which extends a well-known result on the subsets of [10].

Lemma 2.2 *Let $\mathcal{S}$ be a nonempty subset of ^{2}, closed under addition, and M the -module generated by $\mathcal{S}$. If $\mathcal{K}$ denotes the convex cone generated in ${}^2_+$ by the elements of $\mathcal{S}$, there exists $[u \;\; w] \in \mathcal{K} \cap M$ such that all elements of $\Big([u \;\; w] + \mathcal{K}\Big) \cap M$ are in $\mathcal{S}$.*

PROOF Assume that $\{[\alpha_1 \;\; \beta_1], \ldots, [\alpha_t \;\; \beta_t]\}$ is a set of elements in $\mathcal{S}$ which generates M, and set

$$r := \sum_{i=1}^{t} (\alpha_i + \beta_i) \qquad \text{and} \qquad \mathcal{T} := \{[h \;\; k] : h, k \in, h + k \leq r\}.$$

For every nonnegative pair $[h \;\; k]$ in $\mathcal{T} \cap M$ we may determine integer coefficients $c_i^{h,k}$, $i = 1, 2, \ldots, t$, such that

$$[h \;\; k] = \sum_{i=1}^{t} c_i^{h,k} [\alpha_i \;\; \beta_i].$$

Set $R := \max\, \{|c_i^{h,k}| : [h \;\; k] \in \mathcal{T} \cap M, i = 1, 2, \ldots, t\}$, and define

$$[u \;\; w] := \sum_{i=1}^{t} R\; [\alpha_i \;\; \beta_i].$$

Let $\mathcal{K}^{(2)} := [u \;\; w] + \mathcal{K}$. We will show that all integer pairs in $\mathcal{K}^{(2)} \cap M$ belong to $\mathcal{S}$. Every pair $[c \;\; d]$ in $\mathcal{K} \cap M$ can be expressed as

$$[c \;\; d] = \sum_{i=1}^{t} q_i\; [\alpha_i \;\; \beta_i],$$

for some $q_i \in_+$, and therefore as

$$[c \;\; d] = \sum_{i=1}^{t} (q_i - \lfloor q_i \rfloor)[\alpha_i \;\; \beta_i] + \sum_{i=1}^{t} \lfloor q_i \rfloor [\alpha_i \;\; \beta_i],$$

where $\lfloor q_i \rfloor$ denotes the integer part of q_i. Since $0 \leq q_i - \lfloor q_i \rfloor < 1 \leq R$, the pair $[\bar{c} \;\; \bar{d}] := \sum_{i=1}^{t} (q_i - \lfloor q_i \rfloor)[\alpha_i \;\; \beta_i]$ is an element of $\mathcal{T} \cap M$, and consequently we have

$$[c \;\; d] = [\bar{c} \;\; \bar{d}] + \sum_{i=1}^{t} \lfloor q_i \rfloor\; [\alpha_i \;\; \beta_i], \qquad \lfloor q_i \rfloor \in . \tag{5}$$

So, every integer pair $[h \;\; k]$ in $\mathcal{K}^{(2)} \cap M$ can be expressed as $[h \;\; k] = [u \;\; w] + [c \;\; d]$, $[c \;\; d] \in \mathcal{K} \cap M$, thus giving

$$[h \;\; k] \;\; = \;\; [u \;\; w] + [\bar{c} \;\; \bar{d}] + \sum_{i=1}^{t} \lfloor q_i \rfloor\; [\alpha_i \;\; \beta_i]$$

$$= \sum_{i=1}^{t} R\ [\alpha_i \quad \beta_i] + \sum_{i=1}^{t} c_i^{\bar{c},\bar{d}} [\alpha_i \quad \beta_i] + \sum_{i=1}^{t} \lfloor q_i \rfloor\ [\alpha_i \quad \beta_i]$$

$$= \sum_{i=1}^{t} (R + \lfloor q_i \rfloor + c_i^{\bar{c},\bar{d}}) [\alpha_i \quad \beta_i],$$

with $\lfloor q_i \rfloor$ and $c_i^{\bar{c},\bar{d}}$ in , $i = 1, 2, \ldots, t$. Since $R + c_i^{\bar{c},\bar{d}} + \lfloor q_i \rfloor$ is a nonnegative integer for every i, and $\mathcal{S}$ is closed under addition, $[h \quad k]$ belongs to $\mathcal{S}$.

Proposition 2.3 *Let $\mathcal{D}^{(2)}$ be a strongly connected 2D-digraph, and let*

$$\mathcal{S} := \left\{ [\alpha(\gamma) \quad \beta(\gamma)] \in^2 : \gamma \text{ a cycle in } \mathcal{D}^{(2)} \right\}$$

be the set of compositions of all cycles in $\mathcal{D}^{(2)}$.

i) If $M(\mathcal{D}^{(2)})$ has rank 1 and is generated by $[\ell \ m] \in^2$, there exists $\tau \in$ s.t.

$$\{t\ [\ell \quad m] : t \in, t \geq \tau\} \subseteq \mathcal{S} \subseteq \{t\ [\ell \quad m] : t \in\}. \tag{6}$$

ii) If $M(\mathcal{D}^{(2)})$ has rank 2 and $\mathcal{K} \subseteq_+^2$ denotes the solid (i.e., with nonempty interior) convex cone generated by the rows of $L(\mathcal{D}^{(2)})$, there exists $[u \quad w] \in M(\mathcal{D}^{(2)}) \cap \mathcal{K}$ such that

$$M(\mathcal{D}^{(2)}) \cap \Big([u \quad w] + \mathcal{K} \Big) \subseteq \mathcal{S} \subseteq M(\mathcal{D}^{(2)}) \cap \mathcal{K}. \tag{7}$$

Proof Consider a cycle $\bar{\gamma}$ passing through all vertices of $\mathcal{D}^{(2)}$ and the set $\bar{\mathcal{S}}$ of compositions of all cycles having $\bar{\gamma}$ as a subcycle. As $\bar{\mathcal{S}}$ is an additively closed subset of $\mathcal{S}$ and generates $M(\mathcal{D}^{(2)})$, the lefthand inclusions in (2) and (2) follow from the previous lemma. The righthand inclusions are consequences of (2).

Example 2.1 In the 2D-digraph $\mathcal{D}^{(2)}$ of Fig. 2.1 $\mathcal{A}$-arcs are represented by thicklines and $\mathcal{B}$-arcs by thinlines (this notation will be adopted in all subsequent pictures). The circuit matrix

$$L(\mathcal{D}^{(2)}) = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 3 & 0 \end{bmatrix}$$

is right prime and hence generates the -module ^{2}. The set of all vectors $[\alpha(\gamma) \quad \beta(\gamma)]$ that correspond to some cycle γ in $\mathcal{D}^{(2)}$ is represented in Fig. 2.2, and we see that all integer pairs inside $[7 \quad 3] + \mathcal{K}$, $\mathcal{K}$ the cone generated in $_+^2$ by $[3 \quad 0]$ and $[1 \quad 1]$, correspond to a cycle in $\mathcal{D}^{(2)}$.

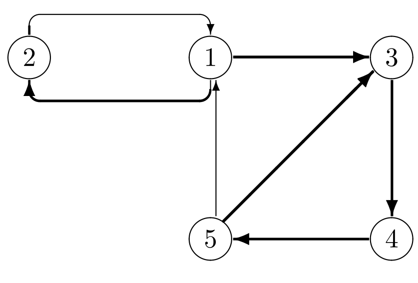

Fig. 2.1 Structure of $\mathcal{D}^{(2)}$

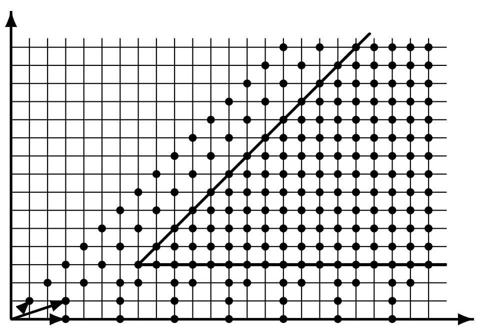
Fig. 2.2 Cycles in $\mathcal{D}^{(2)}$

Example 2.2 In the 2D-digraph of Fig. 2.3 the circuit matrix

$$L(\mathcal{D}^{(2)}) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

has rank 1 and generates the module $\cdot[1 \;\; 1]$. The compositions $[\alpha(\gamma) \;\; \beta(\gamma)]$ of all cycles are represented in Fig. 2.4.

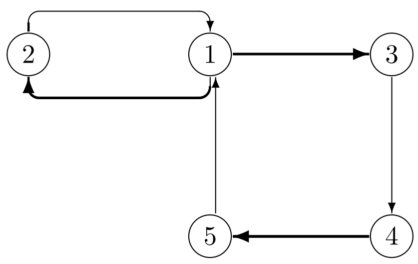

Fig. 2.3 Structure of $\mathcal{D}^{(2)}$

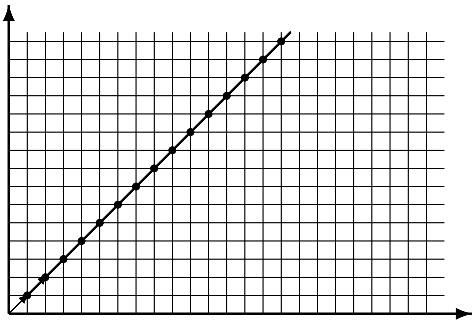

Fig. 2.4 Cycles in $\mathcal{D}^{(2)}$

3 Paths and imprimitivity classes

The vertices of a strongly connected 1-digraph $\mathcal{D} = (V, \mathcal{E})$, with imprimitivity index h, can be partitioned into h 1-*imprimitivity classes*, $C_1^{(1)}, C_2^{(1)}, \ldots, C_h^{(1)}$. Any path moves from a starting vertex through all 1-imprimitivity classes, in a definite cyclic order, and returns to the class of the starting vertex after h arcs. We may always index the classes so that any arc originated in $C_i^{(1)}$ enters $C_{i+1 \bmod h}^{(1)}$, and hence any path p of length $|p|$ originated in $C_i^{(1)}$ ends in $C_{i+|p| \bmod h}^{(1)}$. Also, if $|p|$ is sufficiently large, the terminal vertex can be arbitrarily chosen within the class $C_{i+|p| \bmod h}^{(1)}$.

The above features extend to a strongly connected 2D-digraph $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$, provided that we look at the associated 1-digraph $\mathcal{D}^{(1)}$, and consider only the lengths of the paths and not their compositions. If we take into account the numbers of $\mathcal{A}$-arcs and $\mathcal{B}$-arcs appearing in the paths, however, we generally obtain a finer partition of the set V, as every 1-imprimitivity class splits into a certain number of subclasses, called 2D-*imprimitivity classes.* Two vertices belong to the same subclass if and only if they are connected by paths p with composition $[\alpha(p) \;\; \beta(p)] \in M(\mathcal{D}^{(2)})$, and vertices of two distinct subclasses are connected by paths p whose composition satisfies $[\alpha(p) \;\; \beta(p)] \equiv [\bar{\alpha} \;\; \bar{\beta}] \bmod M(\mathcal{D}^{(2)})$, where $\bar{\alpha}$ and $\bar{\beta}$ are positive integers depending only of the two subclasses.
We introduce, now, a formal definition of the abovementioned notions.

Definition Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a strongly connected 2D-digraph. A pair of vertices v_i and $v_j \in V$ are said $\sim$-*equivalent* ($v_i \sim v_j$) if for every $v_k \in V$ there are paths $v_k \xrightarrow[p_{ik}]{} v_i$ and $v_k \xrightarrow[p_{jk}]{} v_j$ such that

$$[\alpha(p_{ik}) \;\; \beta(p_{ik})] = [\alpha(p_{jk}) \;\; \beta(p_{jk})].$$

This amounts to say that it is possible to connect each vertex of V to v_i and v_j by resorting to two paths with the same composition. It is easy to check that $\sim$ is an equivalence relation on V. As reflexivity and symmetry are obvious, it is sufficient to prove transitivity. Assume $v_i \sim v_j$ and $v_j \sim v_\ell$, and select an arbitrary vertex v_k. There are paths

$$v_k \xrightarrow[p_{ik}]{} v_i, \quad v_k \xrightarrow[p_{jk}]{} v_j, \qquad v_k \xrightarrow[\tilde{p}_{jk}]{} v_j, \quad v_k \xrightarrow[\tilde{p}_{\ell k}]{} v_\ell$$

such that

$$\begin{aligned} [\alpha(p_{ik}) \;\; \beta(p_{ik})] &= [\alpha(p_{jk}) \;\; \beta(p_{jk})] \\ [\alpha(\tilde{p}_{jk}) \;\; \beta(\tilde{p}_{jk})] &= [\alpha(\tilde{p}_{\ell k}) \;\; \beta(\tilde{p}_{\ell k})]. \end{aligned}$$

If we consider, now, any path $v_j \xrightarrow[p_{kj}]{} v_k$, we have that the two paths

$$v_k \xrightarrow[p_{jk}]{} v_j \xrightarrow[p_{kj}]{} v_k \xrightarrow[p_{ik}]{} v_i \qquad v_k \xrightarrow[\tilde{p}_{jk}]{} v_j \xrightarrow[p_{kj}]{} v_k \xrightarrow[\tilde{p}_{\ell k}]{} v_\ell$$

have the same composition, and therefore $v_i \sim v_\ell$.

The equivalence relation $\sim$ induces a partition of V into disjoint 2D-imprimitivity classes, whose number is called *2D-imprimitivity index* and denoted by $h^{(2)}$. As paths with the same composition have the same length, $v_i \sim v_j$ implies that v_i and v_j belong to the same $C_\nu^{(1)}$, thus showing that every 2D-imprimitivity class is a subset of a 1-imprimitivity class.

Lemma 3.1 *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a strongly connected 2D-digraph. Two vertices v_i and v_j belong to the same $\sim$-equivalence class if and only if, for every path $v_i \xrightarrow[p_{ji}]{} v_j$, one has*

$$[\alpha(p_{ji}) \;\; \beta(p_{ji})] \in M(\mathcal{D}^{(2)}). \tag{8}$$

PROOF As we have seen, every cycle γ passing through v_j has composition $[\alpha(\gamma) \;\; \beta(\gamma)] \in M(\mathcal{D}^{(2)})$. So, if v_i and v_j are $\sim$-equivalent and we assume $v_k = v_i$ in the definition, we can find a path $v_i \xrightarrow[\tilde{p}_{ji}]{} v_j$ which satisfies

$$[\alpha(\tilde{p}_{ji}) \;\; \beta(\tilde{p}_{ji})] \in M(\mathcal{D}^{(2)}).$$

On the other hand, if p_{ji} is any path from v_i to v_j, and p_{ij} connects v_j to v_i, the concatenated path from v_i to v_j given by

$$v_i \xrightarrow[\tilde{p}_{ji}]{} v_j \xrightarrow[p_{ij}]{} v_i \xrightarrow[p_{ji}]{} v_j$$

differs both from p_{ji} and $\tilde{p}_{ji}$ in a cycle, and we have

$$\begin{aligned} [0 \;\; 0] &\equiv [\alpha(\tilde{p}_{ji}) \;\; \beta(\tilde{p}_{ji})] \equiv [\alpha(\tilde{p}_{ji}) + \alpha(p_{ij}) + \alpha(p_{ji}) \quad \beta(\tilde{p}_{ji}) + \beta(p_{ij}) + \beta(p_{ji})] \\ &\equiv [\alpha(p_{ji}) \;\; \beta(p_{ji})] \qquad \mathrm{mod} M(\mathcal{D}^{(2)}). \end{aligned}$$

Thus p_{ji} satisfies (3).
Conversely, assume that p_{ji} is a path satisfying (3). By Proposition 2.3, there exist two cycles passing through v_i, say γ and $\bar{\gamma}$, such that

$$\alpha(p_{ji}) = \alpha(\bar{\gamma}) - \alpha(\gamma) \qquad \beta(p_{ji}) = \beta(\bar{\gamma}) - \beta(\gamma).$$

Consequently, for every $v_k \in V$ and every path p_{ik} from v_k to v_i,

$$v_k \xrightarrow[p_{ik}]{} v_i \xrightarrow[\bar{\gamma}]{} v_i$$

and

$$v_k \xrightarrow[p_{ik}]{} v_i \xrightarrow[\gamma]{} v_i \xrightarrow[p_{ji}]{} v_j$$

represent two paths from v_k to v_i and to v_j, respectively, having the same composition, and hence $v_i \sim v_j$.

Example 3.1 The 2D-digraph $\mathcal{D}^{(2)}$ of Fig. 2.3 has 2D-imprimitivity classes $\{1,4\}$ and $\{2,3,5\}$, and it is easy to check that all paths $v_1 \xrightarrow[p_{41}]{} v_4$, as well as all paths $v_2 \xrightarrow[p_{32}]{} v_3$ and $v_2 \xrightarrow[p_{52}]{} v_5$, satisfy

$$\begin{aligned} [\alpha(p_{41}) \;\; \beta(p_{41})] &\equiv [0 \;\; 0] && \bmod M(\mathcal{D}^{(2)}) \\ [\alpha(p_{32}) \;\; \beta(p_{32})] &\equiv [\alpha(p_{52}) \;\; \beta(p_{52})] \equiv [0 \;\; 0] && \bmod M(\mathcal{D}^{(2)}). \end{aligned}$$

The result of Lemma 3.1 extends to arbitrary pairs of vertices of V, as shown in the following lemma.

Lemma 3.2 *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a strongly connected 2D-digraph. For every pair of vertices v_i and v_j in V, there exist nonnegative integers $\alpha_{ji}, \beta_{ji} \in$ such that any path $v_i \xrightarrow[p]{} v_j$ satisfies*

$$[\alpha(p) \;\; \beta(p)] \equiv [\alpha_{ji} \;\; \beta_{ji}] \qquad \bmod M(\mathcal{D}^{(2)}). \tag{9}$$

Moreover, if $v_\ell \sim v_i$ and $v_m \sim v_j$, condition (3) holds for any path p connecting v_ℓ to v_m.
PROOF Select two specific paths $v_i \xrightarrow[p_{ji}]{} v_j$ and $v_j \xrightarrow[p_{ij}]{} v_i$, and set

$$[\alpha_{ji} \;\; \beta_{ji}] := [\alpha(p_{ji}) \;\; \beta(p_{ji})].$$

If $v_i \xrightarrow[p]{} v_j$ is an arbitrary path from v_i to v_j then

$$v_i \xrightarrow[p]{} v_j \xrightarrow[p_{ij}]{} v_i \xrightarrow[p_{ji}]{} v_j$$

is a path from v_i to v_j, that differs both from p and p_{ji} in a cycle passing through v_i. Consequently

$$\begin{aligned} [\alpha(p) \;\; \beta(p)] &\equiv [\alpha(p) + \alpha(p_{ij}) + \alpha(p_{ji}) \quad \beta(p) + \beta(p_{ij}) + \beta(p_{ji})] \\ &\equiv [\alpha(p_{ji}) \;\; \beta(p_{ji})] \equiv [\alpha_{ji} \;\; \beta_{ji}] \qquad \bmod M(\mathcal{D}^{(2)}). \end{aligned}$$

The second part is proved along the same lines.

Example 3.2 The strongly connected 2D-digraph of Fig. 3.1

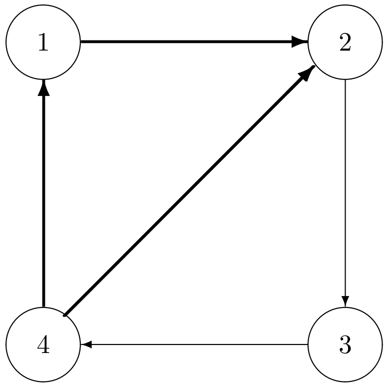

Fig. 3.1 Structure of $\mathcal{D}^{(2)}$

has circuit matrix $L(\mathcal{D}^{(2)}) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ of rank 2, and 2D-imprimitivity classes $\{1, 2, 4\}$ and $\{3\}$. For all paths $v_1 \xrightarrow[p_{21}]{} v_2$ and $v_1 \xrightarrow[p_{41}]{} v_4$ one has

$$[\alpha(p_{21}) \;\; \beta(p_{21})] \equiv [\alpha(p_{41}) \;\; \beta(p_{41})] \equiv [0 \;\; 0] \qquad \bmod M(\mathcal{D}^{(2)}),$$

and for all paths $v_2 \xrightarrow[p_{32}]{} v_3$

$$[\alpha(p_{32}) \;\; \beta(p_{32})] \equiv [0 \;\; 1] \qquad \bmod M(\mathcal{D}^{(2)}).$$

As a consequence of (3), once a particular 2D-imprimitivity class has been selected as a reference, all classes can be unambiguously indexed by the elements of the quotient module $^2/M(\mathcal{D}^{(2)})$, in the sense that each class is indexed by a coset $[\alpha(p) \;\; \beta(p)] + M(\mathcal{D}^{(2)})$, p being any path that reaches the class, starting from the reference one. We may ask under what conditions the above correspondence, mapping 2D-imprimitivity classes into cosets, is bijective, which amounts to say that for every coset $[h \;\; k] + M(\mathcal{D}^{(2)})$ there is a path p starting from the reference class and having composition $[\alpha(p) \;\; \beta(p)] \equiv [h \;\; k] \bmod M(\mathcal{D}^{(2)})$. Clearly, when $M(\mathcal{D}^{(2)})$ has rank 1, the quotient module $^2/M(\mathcal{D}^{(2)})$ includes infinitely many elements, and no bijection exists between the (finite) set of 2D-imprimitivity classes and $^2/M(\mathcal{D}^{(2)})$.
On the other hand, when the module $M(\mathcal{D}^{(2)})$ has rank 2, this correspondence always exists. The result follows from Proposition 3.3, below, which shows that every integer pair of the cone $\mathcal{K}$, generated by the rows of $L(\mathcal{D}^{(2)})$, represents the composition of some path in $\mathcal{D}^{(2)}$.

Proposition 3.3 *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a strongly connected 2D-digraph and $\mathcal{K}$ the solid convex cone generated in $^2_+$ by the rows of $L(\mathcal{D}^{(2)})$. For every integer pair $[h \;\; k]$ in $\mathcal{K}$ there exist a pair of vertices v_i and v_j and a path $v_i \xrightarrow[p]{} v_j$ such that*

$$[\alpha(p) \;\; \beta(p)] = [h \;\; k]. \tag{10}$$

PROOF Possibly after reordering the rows of $L(\mathcal{D}^{(2)})$, we can assume that the ratios $\beta(\gamma_i)/\alpha(\gamma_i) \in_{+} \cup\{+\infty\}$ satisfy

$$\frac{\beta(\gamma_i)}{\alpha(\gamma_i)} \leq \frac{\beta(\gamma_{i+1})}{\alpha(\gamma_{i+1})}, \qquad i = 1, 2, \ldots, t-1.$$

So, $[\alpha(\gamma_1) \;\; \beta(\gamma_1)]$ and $[\alpha(\gamma_t) \;\; \beta(\gamma_t)]$ determine the extremal rays of $\mathcal{K}$, and for every pair $[h \;\; k]$ in $\mathcal{K}$ we have

$$\frac{\beta(\gamma_1)}{\alpha(\gamma_1)} \leq \frac{k}{h} \leq \frac{\beta(\gamma_t)}{\alpha(\gamma_t)}.$$

Consider some $[h \;\; k] \in \mathcal{K}\cap^2$ and a path $p_0 = (v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{h+k-1}}, v_{i_{h+k}})$ of length $h+k$. If $\alpha(p_0) = h$, and hence $\beta(p_0) = k$, we are done; if not, suppose, for instance, $\alpha(p_0) > h$. As depicted in Fig. 3.2, we can first extend p_0 into a cycle $\tilde{\gamma}$, passing through some vertices v_p of γ_1 and v_q of γ_t, arbitrarily selected, and then extend $\tilde{\gamma}$ into a new cycle γ, by adding n_1 copies of circuit γ_1 and n_t copies of γ_t.

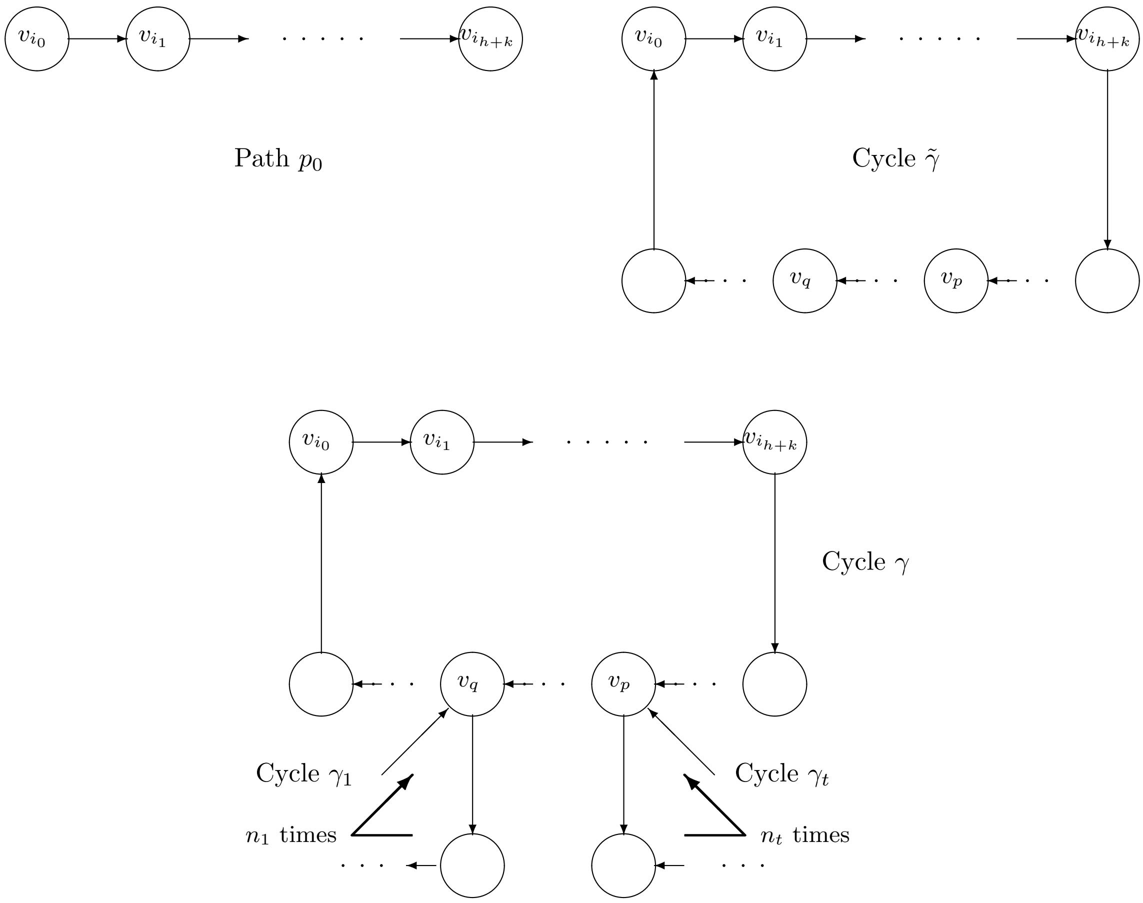

Fig. 3.2

We can select n_1 and n_t so that

$$\frac{n_1\beta(\gamma_1) + n_t\beta(\gamma_t) + \beta(\tilde{\gamma})}{n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})} \geq \frac{k}{h} \geq \frac{\beta(\gamma_1)}{\alpha(\gamma_1)}. \tag{11}$$

Set $N := n_1\Big(\alpha(\gamma_1) + \beta(\gamma_1)\Big) + n_t\Big(\alpha(\gamma_t) + \beta(\gamma_t)\Big) + \Big(\alpha(\tilde{\gamma}) + \beta(\tilde{\gamma})\Big)$ and let γ be the cycle $(v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2}), \dots, (v_{i_{h+k-1}}, v_{i_{h+k}}), \dots, (v_{i_{N-1}}, v_{i_0})$. Consider the family of all paths of length $h + k$ described as follows

$$p_r = (v_{i_r}, v_{i_{r+1}}), \dots, (v_{i_{r+h+k-1 \bmod N}}, v_{i_{r+h+k \bmod N}}), \qquad r = 0, 1, \dots, N-1.$$

As $|\alpha(p_r) - \alpha(p_{r+1})| \leq 1$ for every r, either the family includes a path with h $\mathcal{A}$-arcs, and the proof is complete, or all paths p_r have $\alpha(p_r) > h$ $\mathcal{A}$-arcs, and hence

$$\sum_{r=0}^{N-1} \alpha(p_r) > hN.$$

As in γ there are $n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})$ $\mathcal{A}$-arcs, each of them belonging to $h+k$ different paths p_r, it follows that

$$\sum_{r=0}^{N-1} \alpha(p_r) = \Big(n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})\Big)(h + k).$$

So, the assumption $\alpha(p_r) > h$ for all paths p_r implies

$$\Big(n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})\Big)(h + k) > hN,$$

and therefore

$$\Big(n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})\Big)k > \Big(n_1\beta(\gamma_1) + n_t\beta(\gamma_t) + \beta(\tilde{\gamma})\Big)h,$$

which contradicts (3).

Example 3.3 Consider the strongly connected 2D-digraph $\mathcal{D}^{(2)}$ of Fig. 3.1. The integer pairs giving the compositions of the paths in $\mathcal{D}^{(2)}$ are represented in Fig. 3.3, below, where full circles correspond to cycles and empty circles to open paths.

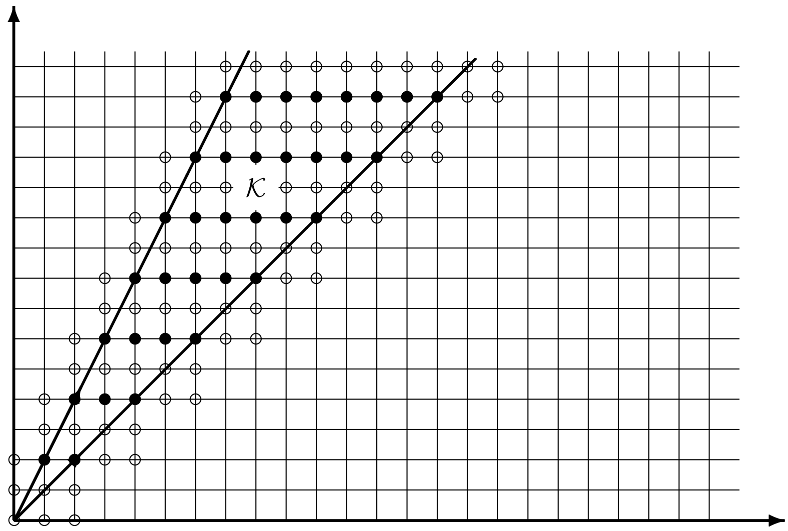

Fig. 3.3 Cycles and paths in $\mathcal{D}^{(2)}$

Example 3.4 Consider the 2D-digraph of Fig. 3.4. It is immediate to verify that $M(\mathcal{D}^{(2)})$ has rank 1 and basis $\{[2\ \ 2]\}$. The set of all pairs corresponding to paths/cycles in $\mathcal{D}^{(2)}$ is represented in Fig. 3.5.

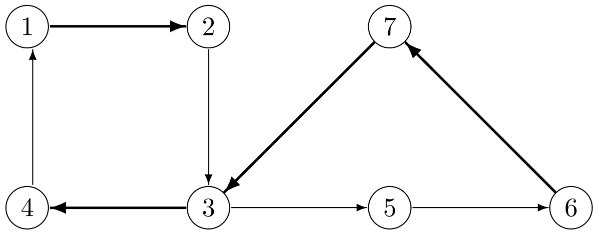

Fig. 3.4 Structure of $\mathcal{D}^{(2)}$

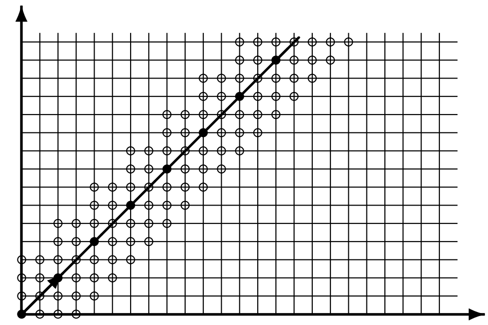

Fig. 3.5 Cycles and paths in $\mathcal{D}^{(2)}$

Corollary 3.4 *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a strongly connected 2D-digraph. If $M(\mathcal{D}^{(2)})$ has rank 2, for every pair $[h\ \ k] \in^2$ there is a pair of vertices v_i and v_j and a path $v_i \xrightarrow[p]{} v_j$ such that*

$$[\alpha(p)\ \ \beta(p)] \equiv [h\ \ k] \qquad \mathrm{mod}\ M(\mathcal{D}^{(2)}). \tag{12}$$

Proof If $[h\ \ k]$ belongs to the convex cone $\mathcal{K}$, generated in $^2_+$ by the rows of $L(\mathcal{D}^{(2)})$, then (3) holds by Proposition 3.3. If not, by exploiting the fact that $\mathcal{K}$ is a solid cone, we can find positive integers $n_1, n_2, \dots, n_t$, such that

$$[\tilde{h}\ \ \tilde{k}] := [h\ \ k] + [n_1\ \ n_2\ \ \dots\ \ n_t]\ L(\mathcal{D}^{(2)})$$

is an element of $\mathcal{K}$. By the above proposition, there is a pair of vertices v_i and v_j and a path $v_i \xrightarrow[p]{} v_j$ such that $[\alpha(p)\ \ \beta(p)] = [\tilde{h}\ \ \tilde{k}]$, and hence (3) holds true.

Remark In the above corollary the rank assumption cannot be relaxed. Actually, if $M(\mathcal{D}^{(2)})$ is generated by a single vector $[\ell\ \ m] \in^2$ and we denote by $\mathcal{P} \subset^2$ the set of compositions of all simple paths in $\mathcal{D}^{(2)}$, the composition of any path p is an integer pair of the set

$$\mathcal{H} := \bigcup_{[u\ \ w] \in \mathcal{P}} \Big([u\ \ w] + \mathcal{K}\Big),$$

where $\mathcal{K}$ is the convex cone generated by $L(\mathcal{D}^{(2)})$, namely the half-line $\{t\ [\ell\ \ m] : t \in_+\}$. As the pair $[h\ \ k] \in^2$ is in the coset $\{[h\ \ k] + t\ [\ell\ \ m], t \in\}$, all pairs which are not in $\mathcal{H}$ cannot fulfill condition (3).
Consequently, the 2D-imprimitivity classes of $\mathcal{D}^{(2)}$ are exactly the cosets of $^2/M(\mathcal{D}^{(2)})$ that include some points of $\mathcal{P}$. This situation is depicted, for instance, in Example 3.4.

As a result of Corollary 3.4, when $M(\mathcal{D}^{(2)})$ has rank 2 and a reference class has been selected, there is a bijection between 2D-imprimitivity classes and cosets of $^2/M(\mathcal{D}^{(2)})$. This correspondence maps each coset $[h\ \ k] + M(\mathcal{D}^{(2)})$ into the set of all vertices reachable from the reference via some path p with $[\alpha(p)\ \ \beta(p)] \in [h\ \ k] + M(\mathcal{D}^{(2)})$. Moreover, as

$h^{(2)}$ coincides with the cardinality of $^2/M(\mathcal{D}^{(2)})$, i.e. with the g.c.d. of the second order minors of $L(\mathcal{D}^{(2)})$, the 2D-imprimitivity classes biuniquely correspond to the elements of any set of representatives for $^2/M(\mathcal{D}^{(2)})$, in particular, to the integer pairs included in the parallelogram $\{\varepsilon\ [w_{11}\ w_{12}] + \delta\ [w_{21}\ w_{22}] : \varepsilon, \delta \in [0, 1)\}$, where $\{[w_{11}\ w_{12}], [w_{21}\ w_{22}]\}$ is an arbitrary basis of $M(\mathcal{D}^{(2)})$.
For instance, if we refer to the basis of $M(\mathcal{D}^{(2)})$ obtained from the Hermite form of $L(\mathcal{D}^{(2)})$, we can index the 2D-imprimitivity classes on the set

$$\mathcal{I} := \{[i\ \ j] \in^2 : [i\ \ j] = \varepsilon\ [h_{11}\ h_{12}] + \delta\ [0\ h_{22}],\ \exists\ \varepsilon, \delta \in [0, 1)\}$$

and denote each of them as $C^{(2)}_{ij}$, $[i\ \ j] \in \mathcal{I}$. If the reference class is denoted by $C^{(2)}_{00}$, all paths from vertices in $C^{(2)}_{00}$ to vertices in $C^{(2)}_{ij}$ have compositions congruent to $[i\ \ j]$ mod $M(\mathcal{D}^{(2)})$, and hence lengths congruent to $i + j$ mod h (the imprimitivity index of $\mathcal{D}^{(1)}$). This implies that $C^{(2)}_{ij}$ and $C^{(2)}_{00}$ are included in the same 1-imprimitivity class if and only if $i + j \equiv 0$ mod h.

Example 3.5 The circuit matrix of the 2D-digraph of Fig. 3.1 has Hermite form

$$H = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix},$$

and hence $h^{(2)} = \det H = 2$. The parallelogram generated by the basis $\{[1\ \ 2], [0\ \ 2]\}$ includes the integer pairs $[0\ \ 0]$ and $[0\ \ 1]$, and, upon indexing the classes on these pairs, we get $C^{(2)}_{00} = \{1, 2, 4\}$ and $C^{(2)}_{01} = \{3\}$.

Proposition 3.5 *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a strongly connected 2D-digraph, with rank $M(\mathcal{D}^{(2)}) = 2$, and denote by h and $h^{(2)}$ its 1- and 2D-imprimitivity indices, respectively. All 1-imprimitivity classes of $\mathcal{D}^{(2)}$ include the same number q of 2D-imprimitivity classes, so that $h^{(2)} = qh$.*

Proof Let $C^{(1)}_\nu$ be an arbitrary 1-imprimitivity class. Select as a reference any 2D-imprimitivity class included in $C^{(1)}_\nu$, and denote it by $C^{(2)}_{00}$. As $C^{(2)}_{ij}$ is included in $C^{(1)}_\nu$ if and only if $i + j \equiv 0$ mod h, it is clear that the number of 2D-imprimitivity classes included in $C^{(1)}_\nu$ coincides with that of the integer pairs $[i\ \ j]$ in $\mathcal{I}$ satisfying $h \mid (i + j)$. Since this number is independent of both the particular $C^{(1)}_\nu$ and the 2D-imprimitivity class selected in it, the result holds true.

Remarks The above result is in general false if rank $M(\mathcal{D}^{(2)})$ is 1. Consider, for instance, the 2D-digraph of Fig. 3.6, below, whose module $M(\mathcal{D}^{(2)})$ has rank 1.

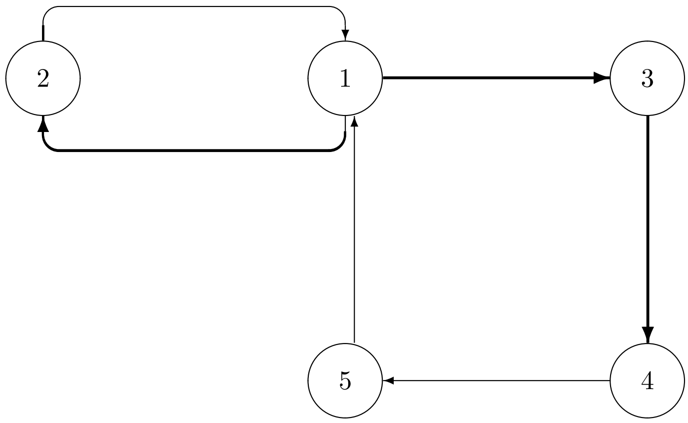

Fig. 3.6 Structure of $\mathcal{D}^{(2)}$

The 1-imprimitivity index h is 2, but $h^{(2)}$ is 3, as the 2D-imprimitivity classes are $\{1\}$, $\{4\}$ and $\{2, 3, 5\}$.

Differently from the case of 1-digraphs, in general it is not possible to cyclically visit all 2D-imprimitivity classes of $\mathcal{D}^{(2)}$, namely to find a path that meets every class exactly once, except for the first and last ones, which coincide. In fact, the strongly connected 2D-digraph of Fig. 3.7

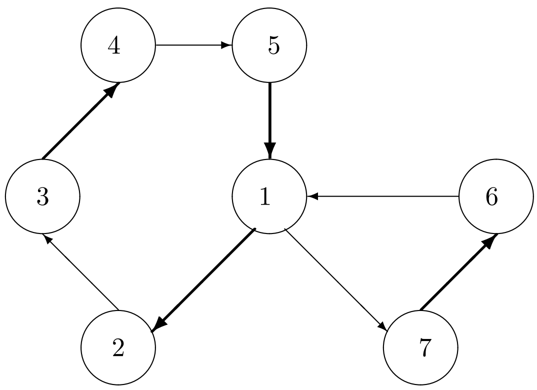

Fig. 3.7 2D-digraph $\mathcal{D}^{(2)}$

has circuit matrix

$$L(\mathcal{D}^{(2)}) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

with determinant 4 and hence there are four 2D-imprimitivity classes, $C_{00}^{(2)} = \{1\}, C_{01}^{(2)} =$

$\{4,7\}, C_{02}^{(2)} = \{2,5\}$ and $C_{03}^{(2)} = \{3,6\}$. It is not possible to find a path that ordinately visits each class once, and ends in the starting class.

4 Dynamical characterizations of irreducible matrix pairs

Given a nonnegative matrix $F = [f_{ij}] \in_+^{n\times n}$, it is possible to associate it with an essentially unique 1-digraph, $\mathcal{D}^{(1)}(F)$, with n vertices, $v_1, v_2, \dots, v_n$. There is an arc from v_i to v_j if and only if $f_{ji} > 0$.
This correspondence is highly noninjective, yet several properties of the multiplicative semigroup generated by F and of the asymptotic behavior of F^ν, as ν tends to $+\infty$, only depend on $\mathcal{D}(F)$. More precisely, paths and cycles in $\mathcal{D}(F)$ are strictly related to the nonzero patterns of the powers of F, since the (i,j)th entry of F^ν is positive if and only if there exists a path of length ν from v_j to v_i. On the other hand, the structure of a 1-digraph D can obviously be investigated in terms of the algebraic properties of any nonnegative matrix F for which $\mathcal{D}^{(1)}(F) = D$.

In this section we aim to extend the above correspondence to matrix pairs, by associating with every pair (A,B) of $n\times n$ nonnegative matrices a 2D-digraph $\mathcal{D}^{(2)}(A,B)$ with n vertices, $v_1, v_2, \dots, v_n$. There is an $\mathcal{A}$-arc (a $\mathcal{B}$-arc) from v_j to v_i if and only if the (i,j)th entry of A (of B) is nonzero [1]. For instance, the pair of positive matrices

$$(A,B) = \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right)$$

corresponds to the 2D-digraph of Fig. 4.1.

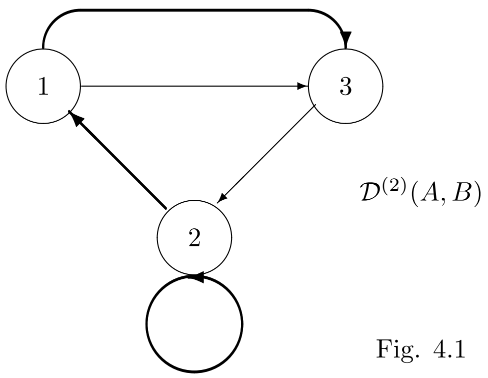

Fig. 4.1

We will show that the combinatorial properties of a pair (A,B) with a 2D-strongly connected digraph can be viewed as natural generalizations of those of an irreducible matrix, i.e. a matrix with a strongly connected digraph. Moreover, the dynamical behavior of the 2D state model described by (A,B) eventually exhibits a two-dimensional periodic pattern, and the "extremal" zeros of its characteristic polynomial are periodically distributed on a torus. This motivates the following definition.

[1] As in the sequel we always refer to the 2D-digraph $\mathcal{D}^{(2)}(A,B)$, associated with a specific matrix pair (A,B), we denote the circuit matrix $L(\mathcal{D}^{(2)}(A,B))$ by $L_{A,B}$ and the corresponding module by $M_{A,B}$.

Definition A pair (A, B) of $n\times n$ positive matrices is *irreducible* if $\mathcal{D}^{(2)}(A, B)$ is 2D-strongly connected.

It is worth noticing that this amounts to require that $A+B$ is irreducible and $L_{A,B}$ has rank 2. So, in particular, all pairs (A, B) with $A + B$ primitive are irreducible, but the converse is not true.

In positive matrix theory, the irreducibility of a single matrix F has received several equivalent descriptions. Among the others, there are algebraic and system theoretic characterizations, which connect this property to the zero-patterns of the powers of F and to the behavior of the associated state model. More precisely, a positive matrix $F \in^{n\times n}$ is irreducible if and only if positive integers h and T can be found, such that for every $t \geq T$

$$\sum_{i=t+1}^{t+h} F^i \gg 0, \tag{13}$$

or equivalently, if for every positive initial condition $\mathbf{x}(0) > 0$ the dynamical model

$$\mathbf{x}(t+1) = F\mathbf{x}(t), \qquad t = 0, 1, \ldots \tag{14}$$

produces state vectors satisfying

$$\sum_{i=t+1}^{t+h} \mathbf{x}(i) \gg \mathbf{0},$$

for sufficiently large values of t.
Similar results hold true for irreducible matrix pairs, if we refer to the *2D system* [5]

$$\mathbf{x}(h+1, k+1) = A\mathbf{x}(h, k+1) + B\mathbf{x}(h+1, k), \qquad h, k \in,\ h+k \geq 0, \tag{15}$$

where the doubly indexed *local states* $\mathbf{x}(h, k)$ are elements of $^n_+$ and *initial conditions* are given by assigning a sequence $\mathcal{X}_0 := \{\mathbf{x}(\ell, -\ell) : \ell \in\}$ of nonnegative local states on the *separation set* $\mathcal{S}_0 := \{(\ell, -\ell) : \ell \in\}$.
If the initial conditions on $\mathcal{S}_0$ are all zero, except at $(0, 0)$, we have

$$\mathbf{x}(h, k) = (AhkB)\ \mathbf{x}(0, 0), \qquad \forall\ h, k \in,$$

where the *Hurwitz products* $AhkB$ of A and B are inductively defined [7] as

$$Ah0B = A^h, \quad h \geq 0, \qquad \text{and} \qquad A0kB = B^k, \quad k \geq 0, \tag{16}$$

and, when h and k are both positive,

$$AhkB = A(Ah-1kB) + B(Ahk-1B). \tag{17}$$

One easily sees that $AhkB$ is the sum of all matrix products that include the factors A and B, h and k times, respectively.
For an arbitrary set of initial conditions $\mathcal{X}_0$, each local state in an arbitrary point $(h, k) \in^2$, $h + k \geq 0$, can be obtained by linearity as

$$\mathbf{x}(h, k) = \sum_{\ell}(Ah-\ell k+\ell B)\ \mathbf{x}(\ell, -\ell), \tag{18}$$

where the Hurwitz product $Ah-\ell k+\ell B$ is assumed zero when either $h - \ell$ or $k + \ell$ is negative.

An alternative description of irreducible matrix pairs, that is reminiscent of that in (4), can be obtained by replacing the power matrices with the Hurwitz products, and the half-line $[T, +\infty)$ with a suitable solid convex cone. In fact, it turns out that a positive matrix pair (A, B) is irreducible if and only if there are a finite "window" $\mathcal{F}$ and a solid convex cone such that, independently of how the window has been positioned within the cone, the sum of all Hurwitz products $ArsB$ corresponding to integer pairs in the window, is strictly positive.

Proposition 4.1 *Let (A, B) be a pair of $n \times n$ positive matrices. (A, B) is irreducible if and only if there are a solid convex cone $\mathcal{K}^*$ and a finite set $\mathcal{F} \subset^2$ such that*

$$\sum_{[r \;\; s] \in [h \;\; k] + \mathcal{F}} ArsB \gg 0, \qquad \forall\ [h \;\; k] \in^2 \text{ s.t. } [h \;\; k] + \mathcal{F} \subset \mathcal{K}^*. \tag{19}$$

Proof Suppose that (A, B) is an irreducible pair and denote by $\mathcal{K}$ the solid convex cone generated in $^2_+$ by the rows of $L_{A,B}$. Let $\{[w_{11}\ w_{12}], [w_{21}\ w_{22}]\} \subset^2$ be a basis of $M_{A,B}$, and set

$$\mathcal{F} := \{\varepsilon\ [w_{11}\ w_{12}] + \delta[w_{21}\ w_{22}] :\ \varepsilon, \delta \in [0, 1)\} \cap {}^2.$$

For every pair of vertices v_i and v_j, consider a path $\tilde{p}_{ji}$ connecting v_i to v_j and passing through all vertices of $\mathcal{D}^{(2)}(A, B)$. It is clear that we can extend $\tilde{p}_{ji}$ by means of n_1 copies of circuit γ_1, n_2 of γ_2,..., n_t of γ_t, thus obtaining a new path p_{ji} from v_i to v_j, with

$$[\alpha(p_{ji}) \;\; \beta(p_{ji})] = [\alpha(\tilde{p}_{ji}) \;\; \beta(\tilde{p}_{ji})] + [n_1 \;\; n_2 \;\; \ldots \;\; n_t]\ L_{A,B}.$$

As $n_1, \ldots, n_t$ are arbitrary positive integers, by Proposition 2.3 there exists $[u \;\; w] \in M_{A,B} \cap \mathcal{K}$, such that every pair in $M_{A,B} \cap \Big([u \;\; w] + \mathcal{K}\Big)$ can be represented as $[n_1 \;\; n_2 \;\; \ldots \;\; n_t]\ L_{A,B}$, for suitable $n_\nu \in$. Consequently, if we define

$$\mathcal{K}_{ji} := [\alpha(\tilde{p}_{ji}) \;\; \beta(\tilde{p}_{ji})] + [u \;\; w] + \mathcal{K},$$

all pairs in $\mathcal{K}_{ji} \cap \Big([\alpha(\tilde{p}_{ji}) \;\; \beta(\tilde{p}_{ji})] + M_{A,B}\Big)$ correspond to some path from v_i to v_j.
Set $\mathcal{K}^* := \cap_{i,j=1}^n \mathcal{K}_{ji}$. It is clear that for every $[h \;\; k] \in^2$ s.t. $[h \;\; k] + \mathcal{F} \subset \mathcal{K}^*$, we have $[h \;\; k] + \mathcal{F} \subset \mathcal{K}_{ji}$ for all j and i, and hence there is a path $v_i \xrightarrow[p_{ji}]{} v_j$ that fulfills

$$[\alpha(p_{ji}) \;\; \beta(p_{ji})] \in [h \;\; k] + \mathcal{F}, \tag{20}$$

This amounts to say that the (j, i)th entry of the Hurwitz product $ArsB$ is nonzero, for some $[r \;\; s] \in [h \;\; k] + \mathcal{F}$. As (4) holds for every j and i, (4) is verified.
Conversely, suppose that (4) holds true, for a suitable choice of $\mathcal{K}^*$ and $\mathcal{F} \subset^2$. Clearly, $A + B$ is irreducible. If not, every power $(A + B)^\nu$, and hence every Hurwitz product of A and B, would be reducible and (4) would not hold.
On the other hand, if the module $M_{A,B}$ is generated by a single vector $[\ell \;\; m] \in^2$, all pairs $[r \;\; s] \in^2$ which correspond to some path in $\mathcal{D}^{(2)}(A, B)$, and hence to a nonzero Hurwitz product $ArsB$, should belong to the "strip"

$$\mathcal{H} := \bigcup_{[u \;\; w] \in \mathcal{P}} \{[u \;\; w] + t\ [\ell \;\; m], t \in\},$$

where $\mathcal{P}$ is the set of compositions $[\alpha(p) \;\; \beta(p)]$ of all simple paths p in $\mathcal{D}^{(2)}$. Consequently, no solid convex cone $\mathcal{K}^*$ could be found, for which (4) is satisfied.

The above proof makes it clear that as in the case of a single positive matrix, where condition (4) involves h consecutive powers of F (h the imprimitivity index), (4) involves the Hurwitz products corresponding to $h^{(2)}$ integer pairs within any shifted version of the window $\mathcal{F}$.

If $\mathcal{X}_0$ consists of a single nonzero local state at $(0,0)$, condition (4) can be restated as

$$\sum_{[i \quad j] \in [h \quad k] + \mathcal{F}} \mathbf{x}(i,j) \gg 0, \tag{21}$$

for every pair $[h \quad k] \in^2$ s.t. $[h \quad k] + \mathcal{F} \subset \mathcal{K}^*$. When there is an infinite number of nonzero local states on $\mathcal{S}_0$, the state evolution possibly affects the whole half-plane $\{(h,k) \in^2 : h+k \geq 0\}$. We may ask whether there is a separation set $\mathcal{S}_\nu = \{(h,k) \in^2 : h+k = \nu\}$ such that condition (4) is fulfilled by all pairs $[h \quad k]$ beyond $\mathcal{S}_\nu$, i.e. satisfying $h+k \geq \nu$.
This is clearly impossible if no upper bound exists on the distance between consecutive nonzero local states on $\mathcal{S}_0$. If we confine ourselves to *admissible* sets of initial conditions, namely to nonnegative sequences $\mathcal{X}_0$ which satisfy the following constraint: *there is an integer $N > 0$ such that $\sum_{\ell=h}^{h+N} \mathbf{x}(\ell, -\ell) > 0$ for all $h \in$,* irreducibility can be characterized as follows.

Proposition 4.2 *A pair (A,B) of $n \times n$ positive matrices is irreducible if and only if there is a finite set $\mathcal{F} \subset^2$ such that for every admissible set of initial conditions $\mathcal{X}_0$ a positive integer T can be found such that*

$$\sum_{[i \quad j] \in [h \quad k] + \mathcal{F}} \mathbf{x}(i,j) \gg 0, \qquad \forall\ [h \quad k] \in^2 \text{ s.t. } h+k \geq T. \tag{22}$$

Proof Assume that (A,B) is irreducible and let $\mathcal{K}^*$ and $\mathcal{F}$ be the solid convex cone and the finite set, respectively, considered in Proposition 4.1. If $\mathbf{x}(\ell,-\ell)$ is a nonzero state belonging to an admissible set $\mathcal{X}_0$, (4) holds for every integer pair $[h \quad k]$ such that $[h \quad k] + \mathcal{F}$ is included in $[\ell \quad -\ell] + \mathcal{K}^*$. As the distance between consecutive nonzero states on $\mathcal{S}_0$ is bounded above, the union of all solid convex cones $[\ell \quad -\ell] + \mathcal{K}^*$, for $[\ell \quad -\ell]$ varying over the support of $\mathcal{X}_0$, includes the half-plane $\{[h \quad k] \in^2 : h+k \geq T\}$ for T large enough, which proves the statement.
Conversely, suppose that (A,B) is not irreducible. If $A+B$ is reducible, we can assume, possibly by simultaneously permuting its rows and columns, that it has block triangular form

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ 0 & A_{22} + B_{22} \end{bmatrix}, \qquad A_{11}, B_{11} \in_+^{r \times r},\ 0 < r < n.$$

Then, for any admissible set of initial conditions $\mathcal{X}_0$ whose local state vectors have the last $n-r$ entries identically zero, condition (4) is not fulfilled.
On the other hand, if $A+B$ is irreducible and rank $L_{A,B} = 1$, there is a strip $\mathcal{H} \subset^2$, including all pairs $[h \quad k]$ corresponding to some paths in $\mathcal{D}^{(2)}(A,B)$, and hence to nonzero Hurwitz products. Consequently, admissible sets of initial conditions can be found such that two strips $[\ell \quad -\ell] + \mathcal{H}$, corresponding to consecutive nonzero local states $\mathbf{x}(\ell,-\ell)$, do not intersect. For this kind of admissible sets (4) is not verified.

5 Characteristic polynomials of irreducible matrix pairs

Up to this point, we have considered nonnegative matrix pairs only from the point of view of the corresponding graph. Important tools for analysing the properties of a pair (A,B)

are its *characteristic polynomial*, defined as

$$\Delta_{A,B}(z_1, z_2) := \det(I_n - Az_1 - Bz_2) = \sum_{h,k\in} d_{hk} z_1^h z_2^k, \qquad d_{00} = 1,$$

and the associated *variety* $\mathcal{V}(\Delta_{A,B})$, namely the set of points $(\lambda, \mu) \in^2$ such that $\det(I_n - A\lambda - B\mu) = 0$. Indeed, many features of the pair and of the corresponding 2D system, like internal stability, finite memory, separability, etc. [7], can be expressed directly in terms of $\Delta_{A,B}(z_1, z_2)$. The positivity constraint on the matrix entries makes these tools even more powerful, as there exists a strict relation between the cyclic structure of the 2D-digraph $\mathcal{D}^{(2)}(A, B)$ and the *support of* $\Delta_{A,B}(z_1, z_2)$, defined as

$$\mathrm{supp}(\Delta_{A,B}) := \{(h_i, k_i) \in^2 : d_{h_i k_i} \neq 0\}.$$

In this section we aim to enlighten certain connections between $\mathrm{supp}(\Delta_{A,B})$ and the circuit matrix $L_{A,B}$, and to show that the *support matrix*

$$S_{A,B} := \begin{bmatrix} h_1 & k_1 \\ h_2 & k_2 \\ \vdots & \vdots \\ h_r & k_r \end{bmatrix} \tag{23}$$

and $L_{A,B}$ provide the same information about the irreducibility of (A, B). This approach is intimately connected with the classical Perron-Frobenius theory for a single positive matrix, and suggests the possibility of obtaining a description of irreducible pairs in terms of the associated characteristic polynomials.

Proposition 5.1 *Let A and B be positive matrices with $A+B$ irreducible and $\rho(A+B) = r$. For any θ and $\omega \in$ the following facts are equivalent:*

i) $(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$ *belongs to* $\mathcal{V}(\Delta_{A,B})$*;*

ii) for every cycle γ in $\mathcal{D}^{(2)}(A, B)$, including $\alpha(\gamma)$ $\mathcal{A}$-arcs and $\beta(\gamma)$ $\mathcal{B}$-arcs,

$$\alpha(\gamma)\theta + \beta(\gamma)\omega \equiv 0 \qquad \mathrm{mod}\ 2\pi; \tag{24}$$

iii) the characteristic polynomial of the pair (A, B) satisfies

$$\Delta_{A,B}(z_1, z_2) = \Delta_{A,B}(z_1 e^{i\theta}, z_2 e^{i\omega}); \tag{25}$$

iv) for every pair $(h, k) \in \mathrm{supp}(\Delta_{A,B})$

$$h\theta + k\omega \equiv 0 \qquad \mathrm{mod}\ 2\pi. \tag{26}$$

Proof i) $\Rightarrow$ ii) As $e^{i\theta}A + e^{i\omega}B$ is dominated by $A+B$ and condition i) holds, by Wielandt's theorem [11] we have $\rho(e^{i\theta}A + e^{i\omega}B) = \rho(A + B) = r$ and

$$A + B = D(e^{i\theta}A + e^{i\omega}B)D^{-1}, \tag{27}$$

for some diagonal matrix $D = \mathrm{diag}\{e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_n}, \}$, $\omega_1, \omega_2, \ldots, \omega_n \in$.

If $a_{hk} \neq 0$, from (5) one gets

$$e^{i\omega_h}(e^{i\theta}a_{hk} + e^{i\omega}b_{hk})e^{-i\omega_k} = a_{hk} + b_{hk},$$

and consequently

$$(1 - e^{i(\theta+\omega_h-\omega_k)})a_{hk} = -(1 - e^{i(\omega+\omega_h-\omega_k)})b_{hk}. \tag{28}$$

As the real parts on the left and right sides of (5) are nonnegative and nonpositive, respectively, they must be zero, and hence $\omega_k \equiv \omega_h + \theta \bmod 2\pi$. So, we have

$$[e^{i\theta}DAD^{-1}]_{hk} = e^{i\theta}e^{i\omega_h}a_{hk}e^{-i\omega_k} = a_{hk},$$

and therefore $A = De^{i\theta}AD^{-1}$. From (5) it immediately follows $B = De^{i\omega}BD^{-1}$.
Let $\gamma = (v_{j_1}, v_{j_2}), \dots, (v_{j_{\ell-1}}, v_{j_\ell}), (v_{j_\ell}, v_{j_1})$ be a cycle of length ℓ in $\mathcal{D}^{(2)}(A, B)$, including $\alpha(\gamma)$ $\mathcal{A}$-arcs and $\beta(\gamma)$ $\mathcal{B}$-arcs. For every arc (v_{j_h}, v_{j_k}) in γ, let $c_{j_h j_k}$ denote $a_{j_h j_k}$ if (v_{j_h}, v_{j_k}) is an $\mathcal{A}$-arc, and $b_{j_h j_k}$ if it is a $\mathcal{B}$-arc. We have, then,

$$0 < c_{j_1 j_2}c_{j_2 j_3}\dots c_{j_\ell j_1} = e^{i[\alpha(\gamma)\theta+\beta(\gamma)\omega]}c_{j_1 j_2}c_{j_2 j_3}\dots c_{j_\ell j_1},$$

which implies (5).
ii) $\Rightarrow$ iii) Consider any Hurwitz product $AhkB$, with $h, k \in$, $h + k > 0$. If $\operatorname{tr}(AhkB) \neq 0$, there is a circuit γ in $\mathcal{D}^{(2)}(A, B)$, including h A-arcs and k B-arcs, and, by assumption, the congruence relation $h\theta + k\omega \equiv 0 \bmod 2\pi$ is satisfied. Consequently, identity $\operatorname{tr}(AhkB)[1 - e^{i(h\theta+k\omega)}] = 0$ holds for all integers h and k, and we get

$$\operatorname{tr}(AhkB) = e^{i(h\theta+k\omega)}\operatorname{tr}(AhkB) = \operatorname{tr}\Big((e^{i\theta}A)hk(e^{i\omega}B)\Big). \tag{29}$$

As the traces of the Hurwitz products uniquely determine the coefficients of the characteristic polynomial of a matrix pair [7], it follows that

$$\Delta_{A,B}(z_1, z_2) = \det(I - Az_1 - Bz_2) = \det\Big(I - (e^{i\theta}A)z_1 - (e^{i\omega}B)z_2\Big) = \Delta_{A,B}(z_1e^{i\theta}, z_2e^{i\omega}).$$

iii) $\Rightarrow$ i) As r is the spectral radius of $A + B$,

$$0 = \Delta_{A,B}(r^{-1}, r^{-1}) = \Delta_{A,B}(r^{-1}e^{i\theta}, r^{-1}e^{i\omega}),$$

and thus $(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$ belongs to $\mathcal{V}(\Delta_{A,B})$.
iii) $\Leftrightarrow$ iv) Condition (5) holds for every pair $(h, k) \in \operatorname{supp}(\Delta_{A,B})$ if and only if $d_{hk} = d_{hk}e^{i(\theta h+\omega k)}$ for every $(h, k) \in^2$, and therefore if and only if

$$\Delta_{A,B}(z_1, z_2) = \sum_{h,k\in} d_{hk}z_1^h z_2^k = \sum_{h,k\in} d_{hk}e^{i\theta h}e^{i\omega k}z_1^h z_2^k = \Delta_{A,B}(z_1e^{i\theta}, z_2e^{i\omega}).$$

We aim to show that the circuit matrix and the support matrix of any pair (A, B) generate the same -module, thus extending an analogous result [11] connecting the support of $\det(I - zF)$ with the lengths of all circuits in $\mathcal{D}^{(1)}(F)$. The proof depends upon the following technical lemma.

Lemma 5.2 *Let L and S be arbitrary integer matrices with the same number n of columns. The two congruences*

$$\begin{array}{ll} L\Theta \equiv 0 & \bmod \\ S\Theta \equiv 0 & \bmod\ , \end{array} \qquad \Theta \in^n, \tag{30}$$

have the same set of solutions if and only if the -modules M_L and M_S, generated by the rows of L and by the rows of S, respectively, coincide.

PROOF It is clear that if M_S and M_L coincide, the two congruences in (5) have the same solutions.
Conversely, assume that the solutions of (5) coincide, and suppose, by contradiction, $M_L \neq M_S$. If the rows of L and S generate different -vector spaces, there is a noninteger vector $\Theta \in^n$ in the kernel of one of the matrices, say L, such that $S\Theta$ is noninteger. As a consequence, Θ solves only the first congruence in (5).
Suppose, now, that L and S generate the same vector subspace of ^{n}, having dimension p, and let H_S and H_L be $p \times n$ integer matrices of rank p generating M_S and M_L, respectively. Then there exists a nonsingular matrix $Q \in^{p\times p}$ such that

$$H_S = QH_L.$$

We aim to show that Q is a unimodular integer matrix. If not, either Q or Q^{-1} is noninteger. In the first case, select a canonical vector $\mathbf{e}_i$ such that $Q\mathbf{e}_i$ is noninteger. As $\mathbf{e}_i$ can be expressed as $\mathbf{e}_i = H_L\Theta$ for some rational vector Θ, then Θ satisfies only the first congruence in (5). The second case is proved along the same lines, by referring to $H_L = Q^{-1}H_S$.

Proposition 5.3 *Let (A, B) be a pair of $n \times n$ positive matrices, with $A + B$ irreducible. The -modules generated by the rows of $L_{A,B}$ and by the rows of $S_{A,B}$ coincide.*
PROOF As $A + B$ is irreducible, condition (5) holds for every cycle γ if and only if

$$L_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv 0 \mod 2\pi. \tag{31}$$

So, by Proposition 5.1, the congruence

$$S_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv 0 \mod 2\pi \tag{32}$$

and that in (5) have the same sets of solutions, and the result is a direct consequence of the previous lemma, upon replacing $[\theta \;\; \omega]^T$ with $\Theta := [\theta/2\pi \;\; \omega/2\pi]^T$ in (5) and (5).

The Perron-Frobenius theorem undoubtely constitutes the most significative result about irreducible matrices, as it clarifies their spectral structure and provides useful information on the asymptotic behavior of the associated state models. Interestingly enough, the varieties of irreducible matrix pairs exhibit features that appear as natural extensions of the properties enlightened by Perron-Frobenius theorem, a result that further corroborates the definition of irreducibility introduced in section 4.

Proposition 5.4 [2D Perron-Frobenius theorem] *Let (A, B) be an irreducible pair of $n \times n$ positive matrices, with $\rho(A + B) = r$. The variety $(\Delta_{A,B})$ intersects the polydisc*

$$\mathcal{P}_{r^{-1}} := \{(z_1, z_2) \in^2 : |z_1| \leq r^{-1}, |z_2| \leq r^{-1}\} \tag{33}$$

only in (r^{-1}, r^{-1}), and in a finite number of points of its distinguished boundary $\mathcal{T}_{r^{-1}} := \{(z_1, z_2) \in^2 : |z_1| = r^{-1}, |z_2| = r^{-1}\}$. Moreover, (r^{-1}, r^{-1}) is a regular point of the variety and there exists a strictly positive vector $\mathbf{w}$ such that

$$\left(I_n - r^{-1}A - r^{-1}B\right) \mathbf{w} = \mathbf{0}. \tag{34}$$

Proof Clearly, (r^{-1}, r^{-1}) belongs to the variety $(\Delta_{A,B})$, as $I - r^{-1}A - r^{-1}B = r^{-1}[rI - (A+B)]$ is a singular matrix. If $(\rho_1 e^{j\theta}, \rho_2 e^{j\omega})$ is an arbitrary point of $(\Delta_{A,B})$, there exists a nonzero vector $\mathbf{v} = [v_1\ v_2\ \dots\ v_n]^T \in^n$ such that

$$(\rho_1 e^{j\theta} A + \rho_2 e^{j\omega} B)\mathbf{v} = \mathbf{v}. \tag{35}$$

It is clear that

$$(\rho_1 A + \rho_2 B)|\mathbf{v}| \geq |\mathbf{v}|, \tag{36}$$

where $|\mathbf{v}|$ denotes the real vector whose i-th entry is $|v_i|$, $i = 1, 2, ..., n$.
In particular, if $(\rho_1 e^{j\theta}, \rho_2 e^{j\omega})$ belongs to $\mathcal{P}_{r^{-1}}$,

$$\rho_1 \leq r^{-1} \qquad \text{and} \qquad \rho_2 \leq r^{-1}. \tag{37}$$

simultaneously hold, and (5) and (5) imply $(A+B)|\mathbf{v}| \geq r\ |\mathbf{v}|$. Since $|\mathbf{v}|$ is a positive vector and $A+B$ is an irreducible matrix with maximal eigenvalue r, then $|\mathbf{v}|$ is a strictly positive eigenvector of $A+B$ corresponding to r [11]. Notice that this proves (5) with $\mathbf{w} = |\mathbf{v}|$. If in (5) ρ_1 or ρ_2 were strictly less than r^{-1}, then $|\mathbf{v}| \gg 0$ and the assumption $A, B > 0$ would imply

$$|\mathbf{v}| = r^{-1}(A+B)|\mathbf{v}| > (\rho_1 A + \rho_2 B)|\mathbf{v}| \geq |\mathbf{v}|,$$

a contradiction. So, we must have $\rho_1 = \rho_2 = r^{-1}$ and all intersections between $\mathcal{P}_{r^{-1}}$ and $(\Delta_{A,B})$ are elements of $\mathcal{T}_{r^{-1}}$, i.e. are expressed as $(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$, for suitable θ's and ω's in .
On the other hand, by Proposition 5.1, $(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$ belongs to $(\Delta_{A,B})$ if and only if the pair (θ, ω) satisfies

$$L_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv 0 \qquad \mod 2\pi,$$

and the assumption rank $L_{A,B} = 2$ implies that the above congruence has a finite number of solutions. Thus $(\Delta_{A,B})$ intersects $\mathcal{T}_{r^{-1}}$ in a finite number of points.

To prove the regularity of (r^{-1}, r^{-1}), we show that the partial derivatives $\partial\Delta_{A,B}/\partial z_1$ and $\partial\Delta_{A,B}/\partial z_2$ evaluated at (r^{-1}, r^{-1}) are nonzero. In fact, we have

$$\begin{aligned} \frac{\partial}{\partial z_1} \det(I - Az_1 - Bz_2) &= -\sum_{i,j=1}^{n} (-1)^{i+j} \det\Big(I - Az_1 - Bz_2)(i|j)\Big) a_{ij} \\ &= -\operatorname{tr}\Big(A^T \operatorname{adj}(I - Az_1 - Bz_2)\Big), \end{aligned} \tag{38}$$

where $(I - Az_1 - Bz_2)(i|j)$ denotes the $(n-1)\times(n-1)$ matrix obtained from $(I - Az_1 - Bz_2)$ by deleting the i-th row and the j-th column.
Since r is a nonzero eigenvalue of $A+B$, we have also

$$\Big(I - (A+B)r^{-1}\Big)\operatorname{adj}\Big(I - (A+B)r^{-1}\Big) = r^n I_n \det(rI - (A+B)) = 0,$$

and therefore each nonzero column of $\operatorname{adj}(I - (A+B)r^{-1})$ is a maximal eigenvector of $A+B$ and hence is either strictly positive or strictly negative.
Applying the same reasonings to A^T and B^T, the above conclusions hold also for the columns of $\operatorname{adj}(I - (A+B)^T r^{-1})$, that is for the rows of $\operatorname{adj}(I - (A+B)r^{-1})$.

Consequently, each row and column of adj $(I - (A+B)r^{-1})$ is either strictly positive or negative or zero, and at least one of the rows and of the columns is nonzero. It follows that adj $(I - (A+B)r^{-1})$ is either strictly positive or strictly negative and, as A is positive, at least one row of $A^T \mathrm{adj}(I-(A+B)r^{-1})$ is strictly positive or strictly negative. So (2), evaluated at $(z_1,z_2)=(r^{-1},r^{-1})$, is nonzero. The same reasoning proves also the inequality $\partial\Delta_{A,B}/\partial z_2 \mid_{(z_1,z_2)=(r^{-1},r^{-1})} \neq 0$.

Corollary 5.5 *Let (A,B) be an irreducible pair of $n\times n$ positive matrices with $\rho(A+B)=r$, and let*

$$\bar{H} := \begin{bmatrix} H \\ \text{—} \\ 0 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & h_{22} \\ \hline \multicolumn{2}{c}{0} \end{bmatrix} \in^{t\times 2}$$

be the Hermite form of $L_{A,B}$. The variety of $\Delta_{A,B}(z_1,z_2)$ intersects the polydisc $\mathcal{P}_{r^{-1}}$ exactly in the points $(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$, one obtains by varying (θ,ω) in the set

$$\left\{\left(\alpha\frac{2\pi}{h_{11}}+\beta\frac{2\pi h_{12}}{h_{11}h_{22}},\ \beta\frac{2\pi}{h_{22}}\right);\alpha,\beta\in\right\}. \tag{39}$$

Consequently, the cardinality of $\mathcal{V}(\Delta_{A,B})\ \cap\ \mathcal{P}_{r^{-1}}$ coincides with the imprimitivity index of the pair.

Proof As $\bar{H}=UL_{A,B}$ for a suitable unimodular matrix $U\in^{t\times t}$, the congruences $H\begin{bmatrix}\theta\\ \omega\end{bmatrix}\equiv \mathbf{0}$ and $L_{A,B}\begin{bmatrix}\theta\\ \omega\end{bmatrix}\equiv \mathbf{0}$ modulo 2π, have the same sets of solutions which, by Proposition 5.1, coincide with $\mathcal{V}(\Delta_{A,B})\ \cap\ \mathcal{P}_{r^{-1}}$.
It is easy to verify that the distinct pairs (modulo 2π) in (5) amounts to $h_{11}h_{22}=\det H = h^{(2)}$.

6 Concluding remarks

The analysis of irreducibility just carried on allows to derive in a different way, and to partially extend, some results on primitive matrix pairs presented in a previous contribution. In [8] the notion of primitivity for a positive pair (A,B) was introduced as a strict positivity constraint on the asymptotic dynamics of the associated 2D state model.
Introducing irreducibility has required a careful analysis of the equivalent characterizations available for a single irreducible matrix, and has highlighted the graph-theoretic approach as particularly suitable to this purpose. So, irreducible pairs have been first defined in terms of the associated 2D-digraph and later endowed with alternative descriptions.

In this framework primitivity is introduced as a special case of irreducibility, as an irreducible pair is called primitive if its imprimitivity index $h^{(2)}$ is 1, and it is characterized in several alternative ways by resorting to the results derived in the previous sections. Indeed, the proof of the following proposition is an immediate corollary of the previous propositions.

Proposition 6.1 *Let (A,B) be an irreducible pair of $n\times n$ positive matrices, with $\rho(A+B)=r$. The following facts are equivalent:*

i) *(A,B) is primitive;*

ii) *$L_{A,B}$ is a right prime integer matrix;*

iii) *$S_{A,B}$ is a right prime integer matrix;*

iv) $M_{A,B}$ *coincides with* ^{2}*;*

v) *there exists a strictly positive Hurwitz product;*

vi) *there is a solid convex cone* $\mathcal{K}^*$ *in* ${}^2_+$ *such that for all* $(h,k) \in^2 \cap \mathcal{K}^*$ *the Hurwitz product* $AhkB$ *is strictly positive;*

vii) *for every admissible set of initial conditions there is a positive integer* T *such that* $\mathbf{x}(h,k) \gg 0$ *for all* $(h,k) \in^2$*,* $h + k \geq T$*;*

viii) *the variety* $(\Delta_{A,B})$ *intersects the polydisk* $\mathcal{P}_{r^{-1}}$ *only in* (r^{-1}, r^{-1})*.*

It is worthwhile to remark that the results of the paper easily extend to kD-digraphs, i.e. digraphs with k kinds of arcs, and to kD systems evolving on ^{k}, for any $k \in$. We preferred, however, to discuss only the case $k = 2$ and to avoid the notational and graphical burden connected with the general case, since, in our opinion, it tends to obscure the main features of the theory, without providing any conceptual advantage.

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