Primitivity of positive matrix pairs: algebraic characterization, graph theoretic description and 2D systems interpretation

Ettore Fornasini and Maria Elena Valcher
Dip. di Elettronica ed Informatica - Univ. di Padova
via Gradenigo 6a, 35131 Padova, ITALY
phone: +39-49-827-7605, fax: +39-49-827-7699
e-mail:fornasini@paola.dei.unipd.it

Abstract

In this paper the primitivity of a positive matrix pair (A, B) is introduced as a strict positivity constraint on the asymptotic behavior of the associated two-dimensional (2D) state model.
The state evolution is first considered under the assumption of periodic initial conditions. In this case the system evolves according to a one-dimensional state updating equation, described by a block circulant matrix. Strict positivity of the asymptotic dynamics is equivalent to the primitivity of the circulant matrix, a property that can be restated as a set of conditions on the spectra of $A + e^{i\omega}B$, for suitable real values of ω.

The theory developed in this context provides a foundation whose analytical ideas may be generalized to nonperiodic initial conditions. To this purpose the spectral radius and the maximal modulus eigenvalues of the matrices $e^{i\theta}A + e^{i\omega}B$, θ and $\omega \in$, are related to the characteristic polynomial of the pair (A, B) as well as to the structure of the graphs associated with A and B, and to the factorization properties of suitable integer matrices.
A general description of primitive positive matrix pairs is finally derived, including both spectral and combinatorial conditions on the pair.

Keywords : primitive matrices, circulant matrices, directed graphs, integer matrices, multidimensional systems

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Abbreviated title: Primitivity of positive matrix pairs

1 Introduction

The notion of primitive matrix grew out of the study of the spectra and the directed graphs of positive irreducible matrices, in a purely algebraic context [2, 3, 15]. Indeed, an irreducible matrix $F \in_+^{n\times n}$ is primitive if and only if its spectral radius is the only maximal modulus eigenvalue of F, or, equivalently, if and only if in the associated directed graph the g.c.d. of the lenghts of all circuits is unitary.

An alternative definition of primitivity arises in the asymptotic analysis of the homogeneous discrete time positive system

$$\mathbf{x}(t+1) = F\mathbf{x}(t), \qquad t = 0, 1, \dots \tag{1}$$

when $\mathbf{x}(0)$, the initial state, is a nonnegative vector. Positive systems appear quite frequently in modelling real processes whose variables represent intrinsically nonnegative quantities, such as pressures, concentrations, densities, population levels, etc..., and have been the object of a long stream of research, aiming to explore basic issues of linear system theory, like controllability, reachability [4, 6, 7, 17] and realizability [1, 14], under positivity constraints.
In this context, the primitivity of F can be equivalently restated as the property that every positive initial condition $\mathbf{x}(0)$ produces a state evolution which becomes strictly positive within a finite number of steps.

When trying to introduce a notion of primitivity for a positive matrix pair (A, B), with A and B in $_+^{n\times n}$, an extension of the above algebraic characterizations is not immediately apparent, whereas it is easy to figure out a reasonable extension of the dynamical behavior we have just described. To this end, we associate with the pair (A, B) the discrete homogeneous two-dimensional (2D) system [8]

$$\mathbf{x}(h+1, k+1) = A\mathbf{x}(h, k+1) + B\mathbf{x}(h+1, k), \qquad h, k \in, \ h+k \geq 0, \tag{2}$$

where the doubly indexed *local states* $\mathbf{x}(h, k)$ are elements of the positive orthant $_+^n$ and *initial conditions* are given by assigning a sequence $\mathcal{X}_0 := \{\mathbf{x}(\ell, -\ell) : \ell \in\}$ of nonnegative local states on the *separation set* $\mathcal{C}_0 := \{(\ell, -\ell) : \ell \in\}$. A 2D system satisfying these constraints is called 2D *positive system*.

2D state models described in (1) allow to represent processes or devices whose evolutions depend upon two independent variables, according to a quarter plane causality law, and provide suitable descriptions of a large class of phaenomena. They have been introduced in the early seventies, and most of their internal and external features have been subsequently investigated. 2D positive systems, instead, have made their appearance only recently, in some contributions dealing with the discretization of the set of PDE's describing a diffusion process [9, 12], but still their relevance for modelling certain classes of physical processes has been immediately apparent.

By assuming the aforementioned dynamical viewpoint, and in analogy with the one-dimensional case, we express the primitivity of the pair (A, B) as a strict positivity constraint on the asymptotic behavior of (1). It is easy to see, however, that the structure of the sequence $\mathcal{X}_0$ has to be somehow constrained. In fact, if $\mathcal{X}_0$ includes $N+1$ consecutive zero local states

$$\mathbf{x}(h,-h) = \mathbf{x}(h+1,-h-1) = \ldots = \mathbf{x}(h+N,-h-N) = \mathbf{0},$$

then zero local states occur also on the separation sets

$$\mathcal{C}_t := \{(t+\ell,-\ell) : \ell \in\}, \qquad t = 1,2,\ldots,N,$$

irrespective of the remaining initial conditions on $\mathcal{C}_0$. So, in order to guarantee that for some finite t all local states on $\mathcal{C}_t$ are strictly positive, we must restrict our attention to *admissible* sequences of initial conditions, namely to nonnegative sequences $\mathcal{X}_0$ which satisfy the following assumption: *there is an integer* $N > 0$ *such that* $\sum_{\ell=h}^{h+N} \mathbf{x}(\ell,-\ell) > 0$ *for all* $h \in$. We are now in a position to introduce the following definition of primitivity for a nonnegative matrix pair.

Definition A pair of nonnegative matrices (A,B) is *primitive* if, for every admissible sequence $\mathcal{X}_0$ of initial conditions, all local states $\mathbf{x}(h,k)$ become strictly positive when $h+k$ is sufficiently large.

Notice that when a 2D system is described by a primitive matrix pair, eventually all its variables appear "permanently excited", independently of the particular set of admissible initial conditions that originated its evolution. This seems to be particularly relevant when the system describes, for instance, a diffusion process, and the two independent variables represent a spatial and a temporal coordinate. In that case, primitivity guarantees that, after a certain time instant, at every point all system variables represent strictly positive quantities.

To investigate the spectral and combinatorial properties of a primitive matrix pair, we consider first the dynamics of system (1) when the initial conditions sequence $\mathcal{X}_0$ has a periodic pattern, of period T. Under this assumption, the 2D system exhibits a behavior which is somewhat intermediate between those of (1) and (1), as its state evolution can be equivalently described by a model (1) with an $nT \times nT$ block circulant system matrix $F = C_T(A,B)$.
It is clear that $\mathbf{x}(h,k)$ eventually becomes strictly positive if and only if $C_T(A,B)$ is primitive, a property that easily translates into the condition that the spectral radii of the matrices $A + e^{i2\pi\ell/T}B,\ \ell = 1,\ldots,T-1$, are smaller than the spectral radius of $A+B$.

So, the primitivity of all circulant matrices $C_T(A,B)$, $T = 1,2,\ldots$, which is a necessary condition for the primitivity of (A,B), is equivalent to assume that $A + e^{i\omega}B$, has spectral radius smaller than that of $A+B$, whenever ω is a rational, but not an integer, multiple of 2π.

This remark suggests a way for obtaining equivalent descriptions of the primitivity of (A,B), based on the spectral properties of the matrix family $\{A + e^{i\omega}B : \omega \in\}$. Actually, searching for a graph theoretic interpretation of the primitivity condition of all circulant matrices $C_T(A,B)$, we can show that it corresponds to simple constraints on the structure of a certain directed graph, $\mathcal{D}^*(A,B)$, associated with the pair (A,B), and on the integer matrix $L_{A,B}$ which describes its cyclic structure.

Tying together these combinatorial characterizations with a result [10] on the Hurwitz products involved in the state updating of (1), we prove that the primitivity of all $C_T(A, B)$, $T = 1, 2, \ldots$, is also sufficient for that of the pair (A, B).

The paper is organized as follows: next section investigates the spectral and combinatorial features of the pair (A, B) by means of the complex matrices $e^{i\theta}A + e^{i\omega}B$, $\theta, \omega \in$, and of the directed graph $\mathcal{D}^*(A, B)$, respectively. Section 3 analyses the periodic dynamics of system (1) and the properties of the associated circulant matrices $C_T(A, B)$, $T = 1, 2, \ldots$. Finally, in section 4, the primitivity of (A, B) is shown to be equivalent to a set of conditions involving the cyclic structure of $\mathcal{D}^*(A, B)$, the spectra of $e^{i\theta}A + e^{i\omega}B$, $\theta, \omega \in$, and the positivity of at least one Hurwitz product.

As we assume familiarity with the basic results of graph theory and positive matrix theory, they will be only touched upon in this introduction, to explain the notation in use throughout the paper. Although some elementary background on 2D systems will be provided later in this section, a couple of algebraic facts will be stated without proof. The interested reader is referred to [10], which includes further references on the subject.

Matrices and vectors will usually be represented by capital italic and lower case boldface letters, respectively, while their entries by the corresponding lower case italic letters. Sometimes, however, when a matrix F is expressed as the product or the sum of other matrices, it will be convenient to denote its (i, j)-th entry as $[F]_{ij}$.
If $F = [f_{ij}]$ is a matrix (in particular, a vector), we write $F \gg 0$ (F *strictly positive*), if $f_{ij} > 0$ for all i, j; $F > 0$ (F *positive*), if $f_{ij} \geq 0$ for all i, j, and $f_{hk} > 0$ for some pair (h, k); $F \geq 0$ (F *nonnegative*), if $f_{ij} \geq 0$ for all i, j.
The *spectral radius* of a matrix F, i.e. the modulus of its maximal eigenvalue, is denoted by $\rho(F)$.
To every $n \times n$ nonnegative matrix F we make correspond [3] a *digraph* (directed graph) $\mathcal{D}(F)$ of *order* n, with vertices indexed by $1, 2, \ldots, n$. There is an arc (i, j) from i to j if and only if $f_{ij} > 0$. Similarly, we associate with a pair of $n \times n$ nonnegative matrices (A, B) a digraph of order n, $\mathcal{D}^*(A, B)$, with arcs of two different kinds, namely A-arcs and B-arcs. There is an A-arc from vertex i to vertex j iff $a_{ij} > 0$, and a B-arc iff $b_{ij} > 0$.

Example 1 Consider the pair of positive matrices

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}.$$

The associated digraph $\mathcal{D}^*(A, B)$ is given in Fig. 1.1, where A-arcs and B-arcs have been represented (as in the sequel) by thicklines and thinlines, respectively.

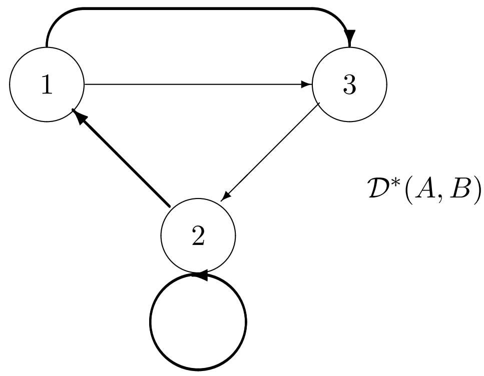

Fig. 1.1

A sequence of arcs in $\mathcal{D}(F)$ of the form $(i_0,i_1),(i_1,i_2),\dots,(i_{k-1},i_k)$ defines a *path of length* k in $\mathcal{D}(F)$, connecting i_0 to i_k. When assigning a path p in $\mathcal{D}^*(A,B)$, we have also to specify, for each pair of consecutive vertices, which kind of arc they are connected by, so that p will have a representation like $(i_0,i_1)_A,(i_1,i_2)_B,\dots,(i_{k-1},i_k)_B$. Thus, it is natural to associate p with a couple of nonnegative integers, $\alpha(p)$ and $\beta(p)$, representing the number of A-arcs and B-arcs occurring in p, respectively. A path whose extreme vertices coincide, i.e. $i_0 = i_k$, is called a *cycle*. In particular, if each vertex in a cycle appears exactly once as the first vertex of an arc, the cycle is called a *circuit*.

Given a pair of square matrices (A,B), not necessarily nonnegative, the *Hurwitz products* of A and B are inductively defined [10] as

$$A^i \sqcup\!\sqcup^0 B = A^i, \quad i \geq 0, \qquad \text{and} \qquad A^0 \sqcup\!\sqcup^j B = B^j, \quad j \geq 0, \tag{3}$$

and, when i and j are both greater than zero,

$$A^i \sqcup\!\sqcup^j B = A(A^{i-1} \sqcup\!\sqcup^j B) + B(A^i \sqcup\!\sqcup^{j-1} B). \tag{4}$$

One easily sees that $A^i \sqcup\!\sqcup^j B$ is the sum of all matrix products that include the factors A and B, i and j times, respectively. For notational convenience sometimes we allow either i or j to be negative integers, and in these cases we assume $A^i \sqcup\!\sqcup^j B = 0$.
Hurwitz products allow to express any local state $\mathbf{x}(h,k)$ of system (1) in terms of the sequence of initial conditions. Actually, if $\mathcal{X}_0 = \{\mathbf{x}(\ell,-\ell) : \ell \in\}$ is an arbitrary sequence of initial conditions on $\mathcal{C}_0$, for all $h,k \in$, $h+k \geq 0$, $\mathbf{x}(h,k)$ can be represented as

$$\mathbf{x}(h,k) = \sum_{\ell} (A^{h-\ell} \sqcup\!\sqcup^{k+\ell} B)\ \mathbf{x}(\ell,-\ell). \tag{5}$$

In particular, if the initial conditions on the separation set $\mathcal{C}_0$ are all zero, except at $(0,0)$, we have

$$\mathbf{x}(h,k) = (A^h \sqcup\!\sqcup^k B)\ \mathbf{x}(0,0), \qquad \forall\ h,k \geq 0.$$

The *characteristic polynomial* of a pair of $n \times n$ matrices (A,B) is defined as

$$\Delta_{A,B}(z_1,z_2) := \det(I_n - Az_1 - Bz_2),$$

and plays for system (1) the same role as $\det(I_n - Fz)$ for system (1). In particular, there is a bijective correspondence [10] between the characteristic polynomial of a pair (A, B) and the family of traces $\mathrm{tr}(A^i \sqcup^j B)$, $i, j \in$, a result which generalizes the well-known relation [13] between the coefficients of $\det(I_n - Fz)$ and the traces of all powers of F.

2 Spectral properties of the matrices $e^{i\theta}A + e^{i\omega}B$

Perron-Frobenius theory estabilishes, for an $n \times n$ irreducible matrix F, very tight connections among its characteristic polynomial, the invariance under rotation of its spectrum and the lengths of all cycles in the associated digraph $\mathcal{D}(F)$. These connections can be specialized to primitive matrices, thus leading to a set of characterizations of primitivity which represent suitable strengthenings of those available for irreducibility.
Trying to determine necessary and sufficient conditions for the primitivity of a positive matrix pair (A, B), it seems natural to ask to what extent the above results admit a generalization, once the spectrum of F is replaced by the variety of $\Delta_{A,B}(z_1, z_2)$ and the digraph of F by $\mathcal{D}^*(A, B)$. To this purpose, in this section and throughout the paper, we will steadily assume that the matrix pair (A, B) we are considering has the following properties:

a) A and B are both positive;
b) $A + B$ is irreducible;
c) $A + B$ has a unitary maximal eigenvalue.

The set of $n \times n$ pairs endowed with these properties will be denoted by $\mathcal{I}_n$.
Assumptions a) and b) easily prove to be necessary conditions for 2D primitivity, which is our final goal. Actually, requiring that all states on $\mathcal{C}_t$ are strictly positive for large values of t, implies that both A and B are nonzero, otherwise any sequence $\mathcal{X}_0$ including a zero local state would produce on every $\mathcal{C}_t$ a state sequence with the same property. Analogously, if $A + B$ were reducible, a positive, but not strictly positive, vector $\mathbf{c}$ could be found such that the initial state sequence $\mathcal{X}_0 = \{\mathbf{x}(\ell, -\ell) = \mathbf{c} : \forall\ \ell \in\}$ produces a constant sequence of nonstrictly positive local states on every separation set $\mathcal{C}_t$.
Assumption c) entails no loss of generality. Actually, we can divide both A and B by $\rho(A + B)$ without affecting the properties we aim to investigate, which are independent of the spectral radius of $A + B$. The case when $A + B$ is nilpotent would constitute the unique exception to this rescaling procedure, but then $A + B$ would not be irreducible.

The answer to the previous question is given by the following proposition, which enlightens, under different points of view, which rotations of θ and ω radians in the z_1- and z_2-planes, respectively, leave the variety of $\Delta_{A,B}(z_1, z_2)$ invariant.
The proof is based on the following remarkable result due to Wielandt [15]:

Wielandt's Theorem If an $n \times n$ complex matrix $C = [c_{ij}]$ is dominated by an irreducible matrix $F = [f_{ij}] > 0$, i.e. $|c_{ij}| \leq f_{ij}$, for all i and j, then for all eigenvalues λ_C of C

$$|\lambda_C| \leq \rho(F). \tag{6}$$

Equality holds in (2) if and only if

$$C = e^{i\phi} DFD^{-1}, \tag{7}$$

where $\lambda_C = e^{i\phi}\rho(F)$ and $D = \mathrm{diag}\{e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_n},\}$, $\omega_1, \omega_2, \ldots, \omega_n \in$.

Proposition 2.1 *Let $(A, B) \in \mathcal{I}_n$. For any θ and $\omega \in$ the following facts are equivalent:*

i) 1 is an eigenvalue of $e^{i\theta}A + e^{i\omega}B$;

ii) there exists a diagonal matrix $D = \mathrm{diag}\{e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_n},\}$, $\omega_1, \omega_2, \ldots, \omega_n \in$, such that

$$A = e^{i\theta} DAD^{-1} \qquad \text{and} \qquad B = e^{i\omega} DBD^{-1}; \tag{8}$$

iii) for every cycle γ in $\mathcal{D}^*(A, B)$, including $\alpha(\gamma)$ A-arcs and $\beta(\gamma)$ B-arcs,

$$\alpha(\gamma)\theta + \beta(\gamma)\omega \equiv 0 \qquad \mathrm{mod}\ 2\pi; \tag{9}$$

iv) the characteristic polynomial of the pair (A, B) satisfies

$$\Delta_{A,B}(z_1, z_2) = \Delta_{A,B}(z_1 e^{i\theta}, z_2 e^{i\omega}). \tag{10}$$

Proof i) $\Rightarrow$ ii) As the matrix $e^{i\theta}A + e^{i\omega}B$ is dominated by $A+B$ and condition i) holds, by Wielandt's theorem we have $\rho(e^{i\theta}A + e^{i\omega}B) = \rho(A + B) = 1$ and

$$A + B = D(e^{i\theta}A + e^{i\omega}B)D^{-1}, \tag{11}$$

for some diagonal matrix $D = \mathrm{diag}\{e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_n},\}$, $\omega_1, \omega_2, \ldots, \omega_n \in$.
If $a_{hk} \neq 0$, from (2) one gets

$$e^{i\omega_h}(e^{i\theta}a_{hk} + e^{i\omega}b_{hk})e^{-i\omega_k} = a_{hk} + b_{hk},$$

and consequently

$$(1 - e^{i(\theta+\omega_h-\omega_k)})a_{hk} = -(1 - e^{i(\omega+\omega_h-\omega_k)})b_{hk}. \tag{12}$$

As the real parts on the left and right sides of (2) are nonnegative and nonpositive, respectively, they must be zero, and hence $\omega_k \equiv \omega_h + \theta$ mod 2π. So, we have

$$[e^{i\theta}DAD^{-1}]_{hk} = e^{i\theta}e^{i\omega_h}a_{hk}e^{-i\omega_k} = a_{hk},$$

which proves the first equation in (2). The second one immediately follows from (2).
ii) $\Rightarrow$ iii) Let $\gamma = (g_1, g_2), \ldots, (g_{\ell-1}, g_\ell), (g_\ell, g_1)$ be a cycle of length ℓ in $\mathcal{D}^*(A, B)$, including $\alpha(\gamma)$ A-arcs and $\beta(\gamma)$ B-arcs. For every arc (g_i, g_j) in γ, let $c_{g_i g_j}$ denote $a_{g_i g_j}$ if (g_i, g_j) is an A-arc, and $b_{g_i g_j}$ if it is a B-arc. By (2), we have, then,

$$0 < c_{g_1 g_2} c_{g_2 g_3} \ldots c_{g_\ell g_1} = e^{i[\alpha(\gamma)\theta + \beta(\gamma)\omega]} c_{g_1 g_2} c_{g_2 g_3} \ldots c_{g_\ell g_1},$$

which implies (2).

iii) $\Rightarrow$ iv) Consider any Hurwitz product $A^h \sqcup^k B$, with $h, k \in, h+k > 0$. If $\mathrm{tr}(A^h \sqcup^k B) \neq 0$, there is a circuit γ in $\mathcal{D}^*(A,B)$, including h A-arcs and k B-arcs, and, by assumption, the congruence relation $h\theta + k\omega \equiv 0 \mod 2\pi$ is satisfied. Consequently, the identity $\mathrm{tr}(A^h \sqcup^k B)[1 - e^{i(h\theta+k\omega)}] = 0$ holds for all integers h and k, and we get

$$\mathrm{tr}(A^h \sqcup^k B) = e^{i(h\theta+k\omega)}\mathrm{tr}(A^h \sqcup^k B) = \mathrm{tr}\Big((e^{i\theta}A)^h \sqcup^k (e^{i\omega}B)\Big). \tag{13}$$

As the traces of the Hurwitz products uniquely determine the coefficients of the characteristic polynomial of a matrix pair [10], it follows that

$$\Delta_{A,B}(z_1, z_2) = \det(I - Az_1 - Bz_2) = \det\Big(I - (e^{i\theta}A)z_1 - (e^{i\omega}B)z_2\Big) = \Delta_{A,B}(z_1e^{i\theta}, z_2e^{i\omega}).$$

iv) $\Rightarrow$ i) As the pair (A,B) is in $\mathcal{I}_n$, 1 is an eigenvalue of $A+B$. Consequently,

$$0 = \det(I - A - B) = \Delta_{A,B}(1,1) = \Delta_{A,B}(e^{i\theta}, e^{i\omega}) = \det(I - e^{i\theta}A - e^{i\omega}B),$$

which implies $1 \in \Lambda(e^{i\theta}A + e^{i\omega}B)$.

Remarks a) In order to check condition iii) of Proposition 2.1, it is not necessary to consider all cycles but only the circuits in $\mathcal{D}^*(A,B)$. So, point iii) reduces to a finite number, say t, of congruence relations which can be expressed in matrix form as

$$L_{A,B}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix}\alpha(\gamma_1) & \beta(\gamma_1)\\ \alpha(\gamma_2) & \beta(\gamma_2)\\ \vdots & \vdots\\ \alpha(\gamma_t) & \beta(\gamma_t)\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix} \equiv \mathbf{0} \qquad \mod 2\pi. \tag{14}$$

b) If in $\mathcal{D}^*(A,B)$ both an A-arc and a B-arc can be found, connecting a vertex h to a vertex k, there are two cycles γ_1 and γ_2 with $\alpha(\gamma_2) = \alpha(\gamma_1) - 1$ and $\beta(\gamma_2) = \beta(\gamma_1) + 1$. As the pairs (θ, ω) which satisfy (2) for all γ in $\mathcal{D}^*(A,B)$, must, in particular, satisfy

$$\begin{aligned}\alpha(\gamma_1)\theta + \beta(\gamma_1)\omega &\equiv 0 \qquad \mod 2\pi\\ \Big(\alpha(\gamma_1) - 1\Big) + \Big(\beta(\gamma_1) + 1\Big)\omega &\equiv 0 \qquad \mod 2\pi\end{aligned}$$

we have $\theta \equiv \omega \mod 2\pi$ for all solutions of (2).

c) Finally, notice that condition $1 \in \Lambda(e^{i\theta}A + e^{i\omega}B)$ for some real pair (θ, ω) is equivalent to the fact that, for a suitable real pair (ϕ, ψ), $e^{i\phi}$ is an eigenvalue of $A + e^{i\psi}B$.

If (A,B) is an element of $\mathcal{I}_n$ and $A+B$ is primitive, the situation when only the trivial rotations, i.e. $\theta \equiv \omega \equiv 0 \mod 2\pi$, leave invariant the variety of $\Delta_{A,B}(z_1, z_2)$, corresponds to the special case when the congruence (2) is devoid of nonzero solutions. This happens if and only if $L_{A,B}$ is a right prime integer matrix.

Proposition 2.2 *Let $(A,B) \in \mathcal{I}_n$ and assume that $A+B$ is primitive. The following facts are equivalent:*

i) $1 \in \Lambda(e^{i\theta}A + e^{i\omega}B)$ *implies* $\theta \equiv \omega \equiv 0$ *mod* 2π*;*

ii) the integer matrix $L_{A,B}$ *is right prime;*

iii) $\Delta_{A,B}(z_1, z_2) = \Delta_{A,B}(e^{i\theta}z_1, e^{i\omega}z_2)$ *implies* $\theta \equiv \omega \equiv 0$ *mod* 2π*.*

PROOF i) $\Rightarrow$ ii) We show, first, that $L_{A,B}$ has full column rank. Consider the integer matrix

$$\bar{L}_{A,B} := \begin{bmatrix} \alpha(\gamma_1) & \alpha(\gamma_1) + \beta(\gamma_1) \\ \alpha(\gamma_2) & \alpha(\gamma_2) + \beta(\gamma_2) \\ \vdots & \vdots \\ \alpha(\gamma_t) & \alpha(\gamma_t) + \beta(\gamma_t) \end{bmatrix} = L_{A,B} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

whose second column consists of the lengths of all circuits in $\mathcal{D}^*(A, B)$. The primitivity assumption on $A + B$ implies that the g.c.d. of these lengths is 1, and hence integer coefficients x_h can be found such that $\sum_h x_h \, [\alpha(\gamma_h) + \beta(\gamma_h)] = 1$. If $\bar{L}_{A,B}$ were not full column rank, its first column, which is nonzero as (A, B) is in $\mathcal{I}_n$, would be a scalar multiple of the second one, namely

$$\bar{L}_{A,B} \begin{bmatrix} 1 \\ -q \end{bmatrix} = \mathbf{0},$$

for some rational number q, $0 < q < 1$. Consequently, we would have $0 < \sum_h x_h \alpha(\gamma_h) < 1$, which is impossible, as all addenda $x_h \alpha(\gamma_h)$ are integer numbers. So, $\bar{L}_{A,B}$, and hence $L_{A,B}$, have rank 2.

We prove, now, that $L_{A,B}$ is right prime. If not, it would factor over the ring as $L_{A,B} = L\Delta$, where L is a $t \times 2$ right prime matrix and Δ a square matrix with $\det \Delta \neq \pm 1$ [16]. As Δ^{-1} is not an integer matrix, the pair

$$\begin{bmatrix} \theta \\ \omega \end{bmatrix} := \Delta^{-1} \begin{bmatrix} 2\pi \\ 2\pi \end{bmatrix} \not\equiv \mathbf{0} \mod 2\pi$$

satisfies $L_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv \mathbf{0} \bmod 2\pi$. By Proposition 2.1, this implies $1 \in \Lambda(e^{i\theta}A + e^{i\omega}B)$.

ii) $\Rightarrow$ i) If $L_{A,B}$ is right prime, it admits a $2 \times t$ integer left inverse S, so that $SL_{A,B} = I_2$. Consequently, $L_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv \mathbf{0} \bmod 2\pi$ implies $\begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv \mathbf{0} \bmod 2\pi$. By Proposition 2.1, this proves the result.

i) $\Leftrightarrow$ iii) Obvious from Proposition 2.1.

The situation when in Proposition 2.1 θ is zero is particularly interesting for the subsequent analysis of circulant matrices. Clearly, the problem of determining for which ω's the matrix $A + e^{i\omega}B$ has eigenvalue 1 can be solved by resorting to the above propositions and, in particular, by analysing the cyclic structure of $\mathcal{D}^*(A, B)$. It seems more convenient, however, to associate with the pair (A, B) a simpler (strongly connected) digraph, $\mathcal{D}_A(B)$, obtained as follows: for all vertices $h \in \{1, 2, \ldots, n\}$ shrink into a single vertex $[h]$ all vertices of the communicating class of h in $\mathcal{D}(A + A^T)$, and then connect $[h]$ and $[k]$ with the arc $([h], [k])$ if there is an arc (ℓ, m) in $\mathcal{D}(B)$ for some $\ell \in [h]$ and $m \in [k]$.

The structure of the shrinked digraph $\mathcal{D}_A(B)$ of a pair (A, B) is better clarified by means of an example.

Example 2 The positive matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are associated with the following digraphs

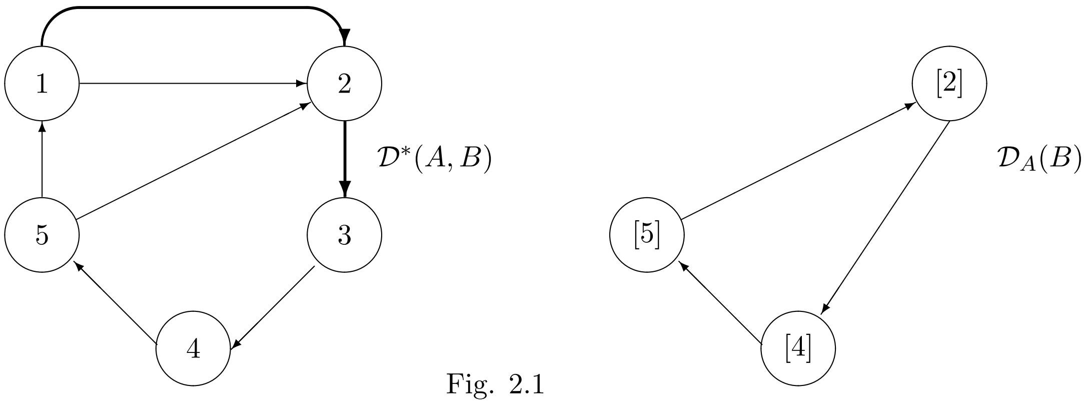

Fig. 2.1

Proposition 2.3 *Let $(A, B) \in \mathcal{I}_n$. 1 is an eigenvalue of $A + e^{i\omega}B$ if and only if the imprimitivity index $h_A(B)$ of the digraph $\mathcal{D}_A(B)$ satisfies*

$$h_A(B)\ \omega \equiv 0 \qquad \text{mod } 2\pi. \tag{15}$$

Proof By Proposition 2.1, the statement $1 \in \Lambda(A + e^{i\omega}B)$ can be replaced by the equivalent condition $\beta(\gamma)\ \omega \equiv 0$ mod 2π, where γ ranges over all cycles in $\mathcal{D}^*(A, B)$ and $\beta(\gamma)$ denotes the number of B-arcs in γ.
Assume, first, that $h_A(B)\ \omega \equiv 0$ mod 2π. Every cycle γ in $\mathcal{D}^*(A, B)$, including say $\beta(\gamma)$ B-arcs, obviously determines a cycle γ' of length $\beta(\gamma)$ in $\mathcal{D}_A(B)$. As $h_A(B)$ is the g.c.d. of the lengths of all cycles in $\mathcal{D}_A(B)$, the length $\beta(\gamma)$ of γ' satisfies $\beta(\gamma)\ \omega \equiv 0$ mod 2π.

To prove the converse, consider any cycle $\hat{\gamma}$ in $\mathcal{D}_A(B)$, of length say ℓ. By definition of $\mathcal{D}_A(B)$, there is a cycle $\bar{\gamma}$ in $\mathcal{D}^*(A + A^T, B)$ such that $\hat{\gamma}$ is obtained by identifying every pair of consecutives vertices connected in $\bar{\gamma}$ by an $(A + A^T)$-arc. As $A + B$ is irreducible, every A^T-arc (h, k) in $\bar{\gamma}$ can be replaced in $\mathcal{D}^*(A, B)$ by a suitable path p_{hk}, from h to k, thus producing a new cycle γ^*. Clearly, as $a_{kh} > 0$, p_{hk} can be completed into a cycle, γ_{hk}, of $\mathcal{D}^*(A, B)$ by means of the A-arc corresponding to a_{kh}. Since all cycles γ_{hk} as well as γ^* satisfy

$$\begin{aligned} \beta(\gamma_{hk})\ \omega &\equiv 0 \mod 2\pi \\ \beta(\gamma^*)\ \omega \equiv \Big(\ell + \sum \beta(\gamma_{hk})\Big)\ \omega &\equiv 0 \mod 2\pi, \end{aligned}$$

it follows that the length ℓ of any cycle in $\mathcal{D}_A(B)$ satisfies $\ell\omega \equiv 0 \mod 2\pi$, and hence $h_A(B)\omega \equiv 0 \mod 2\pi$.

The results obtained in Proposition 2.2 for the linear combinations $e^{i\theta}A + e^{i\omega}B$ of the matrices A and B particularize to the case $\theta = 0$, by resorting once again to the shrinked digraph $\mathcal{D}_A(B)$.

Proposition 2.4 *Let $(A, B) \in \mathcal{I}_n$. The following facts are equivalent:*

i) $1 \in \Lambda(A + e^{i\omega}B)$ for some real number ω implies $\omega \equiv 0 \mod 2\pi$;

ii) g.c.d.$\{\beta(\gamma) : \gamma$ a cycle in $\mathcal{D}^*(A, B)\} = 1$;

iii) the imprimitivity index $h_A(B)$ of $\mathcal{D}_A(B)$ is 1.

Proof i) $\Rightarrow$ ii) If $b :=$ g.c.d.$\{\beta(\gamma) : \gamma$ a cycle in $\mathcal{D}^*(A, B)\}$ is greater than 1, then $\bar\omega := 2\pi/b$ is not an integer multiple of 2π. However, condition $\beta(\gamma)\ \bar\omega \equiv 0 \mod 2\pi$ holds true for every cycle γ in $\mathcal{D}^*(A, B)$, thus implying, by Proposition 2.1, that 1 is an eigenvalue of $A + e^{i\bar\omega}B$. This contradicts i).
ii) $\Rightarrow$ iii) Given any cycle γ in $\mathcal{D}^*(A, B)$, with say $\beta(\gamma)$ B-arcs, we can identify pairs of consecutive vertices which are connected by A-arcs, thus obtaining a cycle in $\mathcal{D}_A(B)$ of length $\beta(\gamma)$. So, as g.c.d.$\{\beta(\gamma) : \gamma$ a cycle in $\mathcal{D}^*(A, B)\} = 1$, there is a family of cycles in $\mathcal{D}_A(B)$ whose lengths are coprime, and hence $h_A(B)$ is 1.
iii) $\Rightarrow$ i) follows from Proposition 2.3.

Remark Analogous results can be obtained for the family of matrices $e^{i\theta}A + B$, $\theta \in$, by simply referring to the shrinked digraph $\mathcal{D}_B(A)$ and to the occurrencies of the A-arcs in the cycles of $\mathcal{D}^*(A, B)$. It is worthwhile to notice, however, that the digraphs $\mathcal{D}_A(B)$ and $\mathcal{D}_B(A)$ can be endowed with different structural properties and, in particular, their imprimitivity indices $h_A(B)$ and $h_B(A)$ need not coincide.

To conclude this section we investigate the set of solutions of the congruence relation (2). As the pair (A, B) is in $\mathcal{I}_n$, both columns of $L_{A,B}$ are nonzero, and therefore we can distinguish two cases, depending on the rank of $L_{A,B}$.

• *$L_{A,B}$ has rank 1* if and only if there is a pair of positive coprime integers, m and ℓ, such that

$$L_{A,B} \begin{bmatrix} m \\ -\ell \end{bmatrix} = \mathbf{0}.$$

By the same reasoning adopted to prove Proposition 2.1, we see that the traces of the Hurwitz products $A^h{\sqcup\!\sqcup}^k B$ are possibly nonzero only for $(h, k) = (t\ell, tm)$, $t \in$. This

situation corresponds [10] to have a characteristic polynomial of the form $\Delta_{A,B}(z_1,z_2) = p(z_1^\ell z_2^m)$, i.e. with support included in a straight line through the origin.
In this case the set of all distinct solutions of (2), i.e. corresponding to different pairs $(e^{i\theta}, e^{i\omega})$, includes infinitely many elements.

Example 3 The pair of matrices (A,B) in $\mathcal{I}_5$, with

$$A = \frac{1}{\sqrt[4]{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad B = \frac{1}{\sqrt[4]{2}} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

has characteristic polynomial $\Delta_{A,B}(z_1,z_2) = 1 - \ z_1 z_2^3 \in [z_1 z_2^3]$ and the associated digraph $\mathcal{D}^*(A,B)$ is

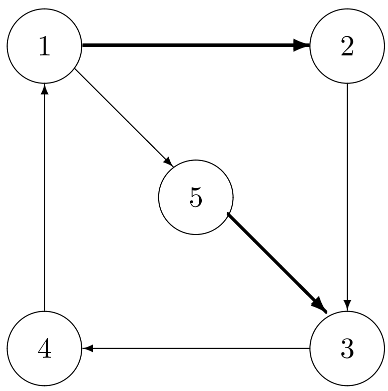

Fig. 2.2

Clearly,

$$L_{A,B} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

has rank 1.

• When the support of $\Delta_{A,B}(z_1,z_2)$ is not included in a straight line, $L_{A,B}$ *has rank 2* and there is only a finite set of distinct solutions of (2). To study this set, it is convenient to consider the Smith form of $L_{A,B}$ over , namely

$$S_{AB} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} = U L_{A,B} V,$$

where U and V are unimodular integer matrices, and the positive integers s_1 and $s_1 s_2$ represent the g.c.d.'s of the elements and of the second order minors of $L_{A,B}$, respectively.

Equation (2) can be rewritten as $\begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} V^{-1} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv \mathbf{0} \mod 2\pi,$ (15)and hence as

$$\begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \equiv \mathbf{0} \mod , \tag{16}$$

where $[\, r_1 \quad r_2 \,]^T := V^{-1} [\, \theta/2\pi \quad \omega/2\pi \,]^T$. A set of distinct representatives of all solutions of (2) is given by

$$\left\{ \begin{bmatrix} \frac{n_1}{s_1} \\ \frac{n_2}{s_2} \end{bmatrix} : n_1 = 0, 1, \ldots, s_1 - 1;\ n_2 = 0, 1, \ldots, s_2 - 1 \right\}.$$

So, letting $\mathbf{g}_1 := 2\pi V [\, 1/s_1 \quad 0 \,]^T$ and $\mathbf{g}_2 := 2\pi V [\, 0 \quad 1/s_2 \,]^T$, the set

$$\{n_1\mathbf{g}_1 + n_2\mathbf{g}_2 : n_1 = 0, 1, \ldots, s_1 - 1;\ n_2 = 0, 1, \ldots, s_2 - 1\}, \tag{17}$$

is the abelian group of the solutions (mod 2π) of (2), represented as the direct sum of two cyclic groups.
The case when both cyclic groups are nontrivial is quite special, because it occurs only when all elements of $L_{A,B}$ have a nontrivial common divisor s_1. In terms of Hurwitz products, this amounts to require that tr $(A^h \sqcup\!\sqcup^k B)$ is possibly nonzero only when both h and k are multiples of s_1, or equivalently [10] $\Delta_{A,B}(z_1, z_2)$ is in $[z_1^{s_1}, z_2^{s_1}]$.

Example 4 The pair of positive matrices $(A, B) \in \mathcal{I}_5$, with

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

has characteristic polynomial $\Delta_{A,B}(z_1, z_2) = 1 - \frac{1}{4}z_1^2 - \frac{3}{4}z_1^2 z_2^2 \in [z_1^2, z_2^2]$. The associated digraph $\mathcal{D}^*(A, B)$ is

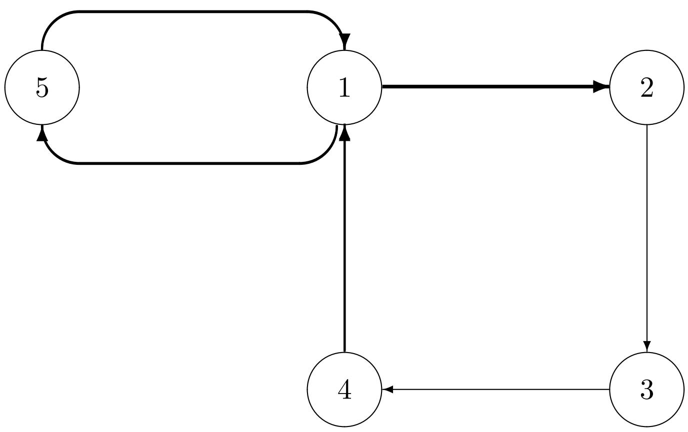

Fig. 2.3

and all entries of the matrix

$$L_{A,B} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

are multiples of 2.

In the remaining cases, and, in particular, when $(A, B) \in \mathcal{I}_n$ has primitive sum $A + B$, the set of distinct solutions of (2) is a cyclic group generated by $\mathbf{g}_2$ and including s_2 elements.
Finally, when $L_{A,B}$ *is right prime*, both cyclic groups collapse and we have only the trivial solution $\theta \equiv \omega \equiv 0 \bmod 2\pi$.

As a final remark, if our interest is in the pairs $(0, \omega)$ which satisfy $L_{A,B} \begin{bmatrix} 0 \\ \omega \end{bmatrix} \equiv \mathbf{0} \bmod 2\pi$, or, equivalently, in the values $\omega \in [0, 2\pi[$ for which

$$1 \in \Lambda(A + e^{i\omega} B), \tag{18}$$

it is more convenient to exploit condition $h_A(B)\ \omega \equiv 0 \bmod 2\pi$, given in Proposition 2.3. This way it is immediately apparent that the solutions (mod 2π) constitute a cyclic group of order $h_A(B) \leq n$.

3 Periodic initial conditions and circulant matrices

In this section we turn our attention to some conditions on the pair (A, B) which ensure a strictly positive asymptotic dynamics for the associated 2D system (1), under the assumption that the initial conditions $\mathcal{X}_0$ have a periodic pattern.

Although this situation is admittedly restrictive, it deserves a thorough discussion for at least two reasons. First, it develops intuitive insights into the combinatorial and spectral properties of a positive matrix pair, meanwhile enlightening some interesting features of block-circulant positive matrices. Second, this analysis leads the way to the solution of the general problem, we shall afford in the subsequent section.

If $\mathcal{X}_0$ is nonzero and periodic with period T, i.e.

$$\mathbf{x}(\ell, -\ell) = \mathbf{x}(\ell + T, -\ell - T) \geq 0, \qquad \forall\ \ell \in . \tag{19}$$

it is clear that the local states $\mathbf{x}(t + \ell, -\ell)$ on each subsequent separation set $\mathcal{C}_t$ still constitute a periodic sequence of period T. It is a matter of simple computation to check that the nT-dimensional vector

$$\mathbf{p}_T(t) := \begin{bmatrix} \mathbf{x}(t, 0) \\ \mathbf{x}(t + 1, -1) \\ \vdots \\ \mathbf{x}(t + T - 1, -T + 1) \end{bmatrix}, \tag{20}$$

obtained by stacking T consecutive local states on $\mathcal{C}_t$, updates according to the following equation

$$\mathbf{p}_T(t + 1) = C_T(A, B)\ \mathbf{p}_T(t), \tag{21}$$

where $C_T(A,B)$ denotes the $nT \times nT$ block circulant matrix

$$C_T(A,B) = \begin{bmatrix} A & B & & & \\ & A & B & & \\ & & \ddots & \ddots & \\ & & & & B \\ B & & & & A \end{bmatrix}, \tag{22}$$

if $T > 1$, and the $n \times n$ matrix $A + B$ if $T = 1$, i.e. if all initial local states on $\mathcal{C}_0$ coincide.

It is worth noticing that $\mathbf{p}_T(t)$ is completely determined by the initial condition, $\mathbf{p}_T(0) > 0$, and by the structure of $C_T(A,B)$. In particular

• if $C_T(A,B)$ is irreducibile, no component of $\mathbf{p}_T(t)$ remains permanently unexcited. Vice versa, if $C_T(A,B)$ is reducibile, a positive vector $\mathbf{p}_T(0)$ can be found such that for some $j \in \{1, 2, \ldots, nT\}$, the j-th entry of $\mathbf{p}_T(t)$ is zero for all $t \in$.

• If $C_T(A,B)$ is irreducibile, $\mathbf{p}_T(t)$ eventually becomes strictly positive if and only if the set of the indices corresponding to nonzero entries in $\mathbf{p}_T(0)$ includes at least one element of each communicating class in $\mathcal{D}(C_T(A,B))$.

• The matrix $C_T(A,B)$ is primitive if and only if for every $\mathbf{p}_T(0) > 0$ the vector $\mathbf{p}_T(t)$ eventually becomes strictly positive.

So, under the assumption of periodic initial conditions with period T, the asymptotic strict positivity of every state evolution of (1) is equivalent to the primitivity of $C_T(A,B)$, which describes the system dynamics according to (3). Consequently, our primary goal in this section is to investigate how the properties of a positive pair (A,B) affect those of $C_T(A,B)$, and, in particular, under what conditions $C_T(A,B)$ is irreducible or primitive. The solution of this problem relies on the results obtained in the previous section and on a couple of technical lemmas, available in the literature. The first lemma introduces a general result on the spectra of block circulant matrices, which allows to express the spectrum of $C_T(A,B)$ in terms of the spectra $\Lambda(A + e^{i2\pi\ell/T}B)$, $\ell = 0, 1, \ldots, T-1$. The second lemma provides a useful criterion for recognizing irreducible matrices.

Lemma on circulant matrices [5] The spectrum of the block circulant matrix

$$C = \begin{bmatrix} A_1 & A_2 & \ldots & A_T \\ A_T & A_1 & & A_{T-1} \\ & & \ddots & \\ A_2 & A_3 & \ldots & A_1 \end{bmatrix}, \qquad A_i \in^{n\times n},$$

is the nT-tuple given by

$$\begin{aligned} \Lambda(C) &= \Lambda(A_1 + A_2 + \ldots + A_T) \uplus \Lambda(A_1 + e^{i\omega}A_2 + \ldots + e^{i\omega(T-1)}A_T) \\ &\uplus \cdots \uplus \Lambda(A_1 + e^{i\omega(T-1)}A_2 + \ldots + e^{i\omega(T-1)(T-1)}A_T) \end{aligned}$$

where $\omega = 2\pi/T$. In particular, the spectrum of (3) is

$$\Lambda(C_T(A,B)) = \Lambda(A+B) \uplus \Lambda(A + e^{i\omega}B) \uplus \cdots \uplus \Lambda(A + e^{i\omega(T-1)}B). \tag{23}$$

Irreducibility criterion [15] An $n \times n$ matrix $F > 0$ with a simple maximal eigenvalue, $\lambda_{\max}$, is irreducible if and only if both F and F^T have strictly positive eigenvectors corresponding to $\lambda_{\max}$.

Lemma 3.1 *Let $(A, B) \in \mathcal{I}_n$ and $T \in$. The circulant matrix $C_T(A, B)$ is*

i) irreducible if and only if 1 is not an eigenvalue of anyone of the following matrices

$$A + e^{i\omega}B, A + e^{2i\omega}B, \dots, A + e^{(T-1)i\omega}B, \qquad \omega = 2\pi/T. \tag{24}$$

ii) primitive if and only if $A + B$ is primitive and none of the above matrices has an eigenvalue of unitary modulus.

Proof i) By the above lemma on circulant matrices, if none of the matrices in (3) has 1 as an eigenvalue, 1 is the simple maximal eigenvalue of $A + B$ and hence of $C_T(A, B)$. On the other hand, if $\mathbf{v}$ and $\mathbf{w}$ denote two strictly positive eigenvectors of $A + B$ and $(A + B)^T$, respectively, corresponding to the eigenvalue 1, we have

$$C_T(A, B) \begin{bmatrix} \mathbf{v} \\ \vdots \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \vdots \\ \mathbf{v} \end{bmatrix},$$

and

$$C_T(A, B)^T \begin{bmatrix} \mathbf{w} \\ \vdots \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{w} \\ \vdots \\ \mathbf{w} \end{bmatrix}.$$

Consequently, both $C_T(A, B)$ and $C_T(A, B)^T$ have a strictly positive eigenvector corresponding to the eigenvalue 1, and hence are irreducible.

Conversely, if 1 is an eigenvalue of some matrix in (3), the multiplicity of 1 as maximal eigenvalue of $C_T(A, B)$ is greater than one, and $C_T(A, B)$ is reducible.

ii) Assume that $C_T(A, B)$ is primitive. As its spectral radius $\rho(C_T(A, B)) = 1$ is an eigenvalue of $A + B$, none of the matrices $A + e^{i\omega\ell}B$, $\ell = 1, 2, \dots, T - 1$, has an eigenvalue of unitary modulus. In particular, the irreducible matrix $A + B$, having no eigenvalue of unitary modulus except for 1, is primitive.

Vice versa, if $A + B$ is primitive and none of the matrices in (3) has an eigenvalue of unitary modulus, by the first part of the proof $C_T(A, B)$ is an irreducible matrix with 1 as simple maximal eigenvalue. As any other eigenvalue of $C_T(A, B)$ has modulus strictly less than 1, $C_T(A, B)$ must be primitive.

It is easy to obtain dual statements for the block circulant matrices $C_T(B, A)$, $T = 1, 2, \dots$, thus relating the irreducibility and primitivity of these matrices to the spectra $\Lambda(e^{i\theta\ell}A + B)$, $\theta = 2\pi/T$, $\ell = 1, 2, \dots, T - 1$. In general, however, the irreducibility of $C_T(B,A)$ needs not imply that of $C_T(A,B)$, as a consequence of the fact that the imprimitivity indices $h_A(B)$ and $h_B(A)$ need not coincide.

Example 5 The pair of matrices (A,B), with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

is an element of $\mathcal{I}_3$. It is immediate to see from the digraphs $\mathcal{D}_A(B)$ and $\mathcal{D}_B(A)$ that the block circulant matrix $C_2(A,B) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is reducible, whereas $C_2(B,A) = \begin{bmatrix} B & A \\ A & B \end{bmatrix}$ is irreducible.

Notice that, differently from the case of irreducibility, A and B play a symmetric role in determining the primitivity of $C_T(A,B)$. Actually, if $C_T(A,B)$ is primitive, none of the matrices $A + e^{i\omega\ell}B$, $\omega = 2\pi/T$ and $\ell = 1,2,\ldots,T-1$, has an eigenvalue of unitary modulus, and this happens if and only if the same holds true for the family $B + e^{i\omega\ell}A$, $\ell = 1,2,\ldots,T-1$. Thus, $C_T(B,A)$ is primitive, too.

Proposition 3.2 *Let $(A,B) \in \mathcal{I}_n$. The following facts are equivalent:*

i) all circulant matrices $C_T(A,B)$, $T = 1,2,\ldots$, are irreducible;

ii) $1 \in \Lambda(A + e^{i\omega}B)$ for some real number ω implies $\omega \equiv 0$ mod 2π.

Proof i) $\Rightarrow$ ii) Assume, by contradiction, that 1 is an eigenvalue of $A + e^{i\omega}B$, for some $\omega \not\equiv 0$ mod 2π. By Proposition 2.3, ω must be a rational multiple of 2π, i.e. $\omega = 2\pi(\nu/\bar{T})$, for some nonzero integers ν and $\bar{T}$, $\nu \not\equiv 0$ mod $\bar{T}$. But in this case, by Lemma 3.1, $C_{\bar{T}}(A,B)$ is a reducible matrix, which contradicts assumption i).
ii) $\Rightarrow$ i) follows from Lemma 3.1, too.

Proposition 3.3 *Let $(A,B) \in \mathcal{I}_n$, with $A+B$ primitive. The following facts are equivalent:*

i) all circulant matrices $C_T(A,B)$, $T = 1,2,\ldots$, are primitive;

ii) $1 \in \Lambda(e^{i\theta}A + e^{i\omega}B)$ implies $\theta \equiv \omega \equiv 0$ mod 2π.

Proof i) $\Rightarrow$ ii) As remarked at the end of the previous section, $L_{A,B}$ has rank 2. So, all solutions of

$$L_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv \mathbf{0} \mod 2\pi,$$

and, consequently, all the pairs (θ,ω) for which 1 belongs to $\Lambda(e^{i\theta}A + e^{i\omega}B)$, must be rational multiples of 2π, namely $(\theta,\omega) = 2\pi(q_1,q_2)$, $q_1,q_2 \in$. However, if 1 would be in $\Lambda(e^{i\theta}A + e^{i\omega}B)$, for certain θ and ω rational multiples of 2π, we would have $e^{-i\theta} \in \Lambda(A + e^{i(\omega-\theta)}B)$, thus contradicting Lemma 3.1.

ii) $\Rightarrow$ i) Immediate from Lemma 3.1.

Tying together Propositions 2.2 and 2.4 with the above results, several alternative characterizations of the irreducibility and the primitivity of all circulant matrices $C_T(A,B)$, $T \in$, can be obtained, based on the digraphs $\mathcal{D}_A(B)$ and $\mathcal{D}^*(A,B)$, respectively. In particular, graph-theoretic criteria are available for checking the above properties and hence the strict positivity of the asymptotic dynamics of (1), starting from periodic initial conditions.

4 Arbitrary initial conditions and 2D primitivity

In this section we drop the periodicity assumption, and turn our attention to general (admissible) initial conditions. As every periodic $\mathcal{X}_0$ is admissible, it is clear that the primitivity of all $C_T(A,B)$, $T \in$, is necessary for 2D primitivity. We aim to prove that it is also sufficient.

In fact, we will show that when all $C_T(A,B)$, $T \in$, are primitive, and hence $L_{A,B}$ is right prime, there exists a solid convex cone $\mathcal{K}$ in ${}^2_+$ such that for all (h,k) in $\mathcal{K}\cap^2$ the Hurwitz products $A^h \sqcup\!\sqcup^k B$ are strictly positive. Consequently, every nonzero local state $\mathbf{x}(\ell,-\ell) > 0$ produces a strictly positive state evolution inside the cone $(\ell,-\ell)+\mathcal{K}$, as we have

$$\mathbf{x}(h+\ell,k-\ell) \geq (A^h \sqcup\!\sqcup^k B)\mathbf{x}(\ell,-\ell) \gg 0, \qquad \forall\ (h+\ell,k-\ell) \in (\ell,-\ell)+\mathcal{K}.$$

The admissibility assumption on $\mathcal{X}_0$ guarantees that the union of all cones $(\ell,-\ell)+\mathcal{K}$, which correspond to positive initial states $\mathbf{x}(\ell,-\ell)$, includes all separation sets $\mathcal{C}_t$, for t greater than a suitable $t_{\min}$. Consequently, for $t > t_{\min}$ all local states on the separation set $\mathcal{C}_t$ are strictly positive.

The subsequent discussion is based on the following number theoretic result, which extends a well-know lemma attributed to Schur [3].

Lemma 4.1 *Let $\mathcal{S}$ be a nonempty subset of ^{2}, closed under addition, such that the -module generated by $\mathcal{S}$ is ^{2}. Then there exists a solid convex cone $\mathcal{K}^*$ in ${}^2_+$ such that all elements in $\mathcal{K}^*\cap^2$ are in $\mathcal{S}$.*

Proof Let (α_1,β_1), $\ldots$, (α_t,β_t) be a set of elements of $\mathcal{S}$ which generate ^{2}, and let $r := \sum_{i=1}^t(\alpha_i+\beta_i)$. For every nonnegative pair (h,k) in $\mathcal{T} := \{(h,k) : h,k \in, h+k \leq r\}$ we may determine integer coefficients $c_i^{h,k}$ such that

$$(h,k) = \sum_{i=1}^t c_i^{h,k}(\alpha_i,\beta_i).$$

Let M be the maximum of the integers $|c_i^{h,k}|$, $(h,k) \in \mathcal{T}$ and $i = 1,2,\ldots,t$, and define

$$(v,w) := \sum_{i=1}^t M\ (\alpha_i,\beta_i).$$

As the -module generated by $\mathcal{S}$ is ^{2}, the cone $\mathcal{K}$ generated in $^2_+$ by the positive pairs (α_1, β_1), ..., (α_t, β_t) is convex and solid. We aim to show that all integer pairs in $\mathcal{K}^* := (v, w) + \mathcal{K}$ belong to $\mathcal{S}$.

Every integer pair (c, d) in $\mathcal{K}$ can be expressed as

$$(c, d) = \sum_{i=1}^{t} q_i \ (\alpha_i, \beta_i), \qquad q_i \in_+,$$

and therefore as

$$(c, d) = \sum_{i=1}^{t} \lfloor q_i \rfloor (\alpha_i, \beta_i) + \sum_{i=1}^{t} (q_i - \lfloor q_i \rfloor)(\alpha_i, \beta_i),$$

where $\lfloor q_i \rfloor$ denotes the integer part of q_i. Since $0 \leq q_i - \lfloor q_i \rfloor < 1$, the pair $(\bar{c}, \bar{d}) := \sum_{i=1}^{t} (q_i - \lfloor q_i \rfloor)(\alpha_i, \beta_i)$ is an element of $\mathcal{T}$, and (c, d) decomposes into

$$(c, d) = (\bar{c}, \bar{d}) + \sum_{i=1}^{t} n_i \ (\alpha_i, \beta_i), \qquad n_i \in . \tag{25}$$

So, every integer pair (h, k) in $\mathcal{K}^*$ can be written as $(h, k) = (v, w) + (c, d)$, $(c, d) \in \mathcal{K}$, and hence as

$$\begin{aligned} (h, k) &= (v, w) + (\bar{c}, \bar{d}) + \sum_{i=1}^{t} n_i \ (\alpha_i, \beta_i) \\ &= \sum_{i=1}^{t} M \ (\alpha_i, \beta_i) + \sum_{i=1}^{t} c_i^{\bar{c}, \bar{d}} (\alpha_i, \beta_i) + \sum_{i=1}^{t} n_i \ (\alpha_i, \beta_i) \\ &= \sum_{i=1}^{t} (M + n_i + c_i^{\bar{c}, \bar{d}})(\alpha_i, \beta_i), \end{aligned}$$

with n_i and $c_i^{\bar{c}, \bar{d}}$ in , $i = 1, 2, \ldots, t$. Since $M + c_i^{\bar{c}, \bar{d}} + n_i$ is a nonnegative integer for every i, and $\mathcal{S}$ is closed under addition, (h, k) belongs to $\mathcal{S}$.

Proposition 4.2 *Let (A, B) be in $\mathcal{I}_n$. The following facts are equivalent*

i) *the integer matrix $L_{A,B}$ is right prime;*

ii) *there is a solid convex cone $\mathcal{K}$ in $^2_+$ such that for every pair of integers (h, k) in $\mathcal{K}$ and every couple of vertices i and j, there is a path p in $\mathcal{D}^*(A, B)$, from i to j, including h A-arcs and k B-arcs;*

iii) *there is a solid convex cone $\mathcal{K}_H$ in $^2_+$ such that for every pair of integers (h, k) in $\mathcal{K}_H$ the Hurwitz product $A^h \sqcup^k B$ is strictly positive;*

iv) *the pair (A, B) is primitive.*

Proof i) $\Rightarrow$ ii) Let $\mathcal{S}_\ell$ be the set of integer vectors $[\alpha(\gamma)\ \beta(\gamma)]$ corresponding to all cycles γ in $\mathcal{D}^*(A,B)$ passing through vertex ℓ. Clearly, $\mathcal{S}_\ell$ is nonempty and closed under addition. Moreover, the -module generated by $\mathcal{S}_\ell$ coincides with the -module generated by the rows of $L_{A,B}$, namely with ^{2}. Actually, consider a positive vector $[\alpha(\gamma)\ \beta(\gamma)]$, γ a circuit in $\mathcal{D}^*(A,B)$, which is not included in $\mathcal{S}_\ell$, and let j be any vertex γ passes through. As $\mathcal{D}^*(A,B)$ is strongly connected, it includes a cycle γ' passing through ℓ and j, and another cycle, γ", obtained by connecting γ and γ'. So, both $[\alpha(\gamma')\ \beta(\gamma')]$ and $[\alpha(\gamma")\ \beta(\gamma")]$ are in $\mathcal{S}_\ell$, and

$$[\alpha(\gamma)\ \beta(\gamma)] = [\alpha(\gamma")\ \beta(\gamma")] - [\alpha(\gamma')\ \beta(\gamma')]$$

is in the -module generated by $\mathcal{S}_\ell$. By the above lemma, then, there exists a solid convex cone $\mathcal{K}^*_\ell$ in ${}^2_+$ such that all integer vectors in $\mathcal{K}^*_\ell$ are in $\mathcal{S}_\ell$.

If i and j are arbitrary vertices in $\mathcal{D}^*(A,B)$ and $p_{i\ell}$ and $p_{\ell j}$ are two fixed paths connecting i to ℓ and ℓ to j, respectively, all integer vectors in the cone

$$\mathcal{K}^*_{ij} := [\alpha(p_{i\ell}) + \alpha(p_{\ell j}) \quad \beta(p_{i\ell}) + \beta(p_{\ell j})] + \mathcal{K}^*_\ell$$

correspond to paths connecting i to j. Clearly, $\mathcal{K} := \cap_{ij}\mathcal{K}^*_{ij}$ is a solid convex cone which satisfies ii).

ii) $\Rightarrow$ iii) Obvious, once assuming $\mathcal{K}_H = \mathcal{K}$.

iii) $\Rightarrow$ iv) Under assumption iii), it is easy to see that every admissible $\mathcal{X}_0$ eventually produces a strictly positive state evolution, and hence the pair (A,B) is primitive, by definition.

iv) $\Rightarrow$ i) When (A,B) is primitive, all nonzero periodic initial conditions eventually produce strictly positive dynamics. This implies that all $C_T(A,B)$, $T \in$, are primitive matrices and hence, by Propositions 2.2 and 3.3, $L_{A,B}$ is right prime.

To conclude, observe that the above proposition reduces the primitivity of the pair (A,B) to the existence of a solid cone $\mathcal{K}_H$ in ${}^2_+$, whose integer coordinates points correspond to strictly positive Hurwitz products. Indeed, this condition can be considerably simplified, as the existence of a primitive, and hence of a strictly positive, Hurwitz product ensures that of a whole cone $\mathcal{K}_H$ of strictly positive Hurwitz products. This property nicely extends to matrix pairs the well-known fact that a positive matrix F is primitive if and only if it has a strictly positive power.

Proposition 4.3 *Let (A,B) be in $\mathcal{I}_n$. The following facts are equivalent:*

i) there is a solid convex cone $\mathcal{K}_H$ in ${}^2_+$ such that for every pair of integers (h,k) in $\mathcal{K}_H$ the Hurwitz product $A^h \sqcup^k B$ is strictly positive ;

ii) there exists a positive pair $(\ell,m) \in \times$ such that $A^\ell \sqcup^m B$ is primitive.

Proof i) $\Rightarrow$ ii) Obvious.

ii) $\Rightarrow$ i) Assume that $A^\ell \sqcup^m B$ is primitive. Then there exists a positive integer r such that $A^{r\ell} \sqcup^{rm} B \geq (A^\ell \sqcup^m B)^r \gg 0$. So, it is not restrictive to assume that $A^\ell \sqcup^m B$ is strictly positive.

As A and B are both positive and $\mathcal{D}^*(A, B)$ is strongly connected, there exists a vertex j with an outgoing A-arc, (j, u), and an ingoing B-arc (e, j). By the assumption on $A^{\ell} \sqcup\!\sqcup^{m} B$, in $\mathcal{D}^*(A, B)$ one can find a cycle γ passing through j, a path p_{uj} from u to j and a path p_{je} from j to e, each of them including ℓ A-arcs and m B-arcs. So, the path p_{ju} can be completed into a cycle with $\ell+1$ A-arcs and m B-arcs, and similarly p_{ej} can be completed into a cycle including ℓ A-arcs and $m+1$ B-arcs.
Clearly, the -module $\mathcal{M}$ generated by the pairs (ℓ, m), $(\ell+1, m)$ and $(\ell, m+1)$ is ^{2}, as the integer matrix

$$\begin{bmatrix} \ell & m \\ \ell+1 & m \\ \ell & m+1 \end{bmatrix}$$

is right prime. Moreover, as the -module generated by the rows of $L_{A,B}$ includes $\mathcal{M}$, $L_{A,B}$ is a right prime matrix, too, and the conclusion follows from the above proposition.

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