REACHABILITY OF A CLASS OF DISCRETE-TIME POSITIVE SWITCHED SYSTEMS

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Abstract. In this paper, monomial reachability and reachability properties for the special class of single-input discrete-time positive systems that switch among p subsystems, sharing the same state transition matrix, are investigated. Necessary and sufficient conditions for these properties to hold, together with some examples, are provided.

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1. Introduction

Switched linear systems are dynamic systems consisting of a family of linear state-space models and a switching law, specifying when and how the switching among the subsystems takes place [14, 17, 20]. Reachability and controllability of these systems have been explored in a number of papers [10, 22, 23]. Nonetheless, there are still several open problems. Indeed, reachability has found a rather complete characterization for continuous-time switched systems and for "reversible" discrete-time switched systems (whose subsystems have nonsingular state transition matrices). In the general discrete-time case, necessary and sufficient conditions for reachability have been provided under the assumption that all subsystems share the same state transition matrix [8]. Finally, some interesting properties of the controllable sets for (both reversible and non-reversible) discrete-time switched systems have been investigated in the pioneering works of Conner and Stanford [5, 19].

Positive systems, on the other hand, are linear systems in which the variables only take nonnegative values. They naturally arise in fields like bioengineering (compartmental models), economic modelling, behavioral science, and stochastic processes (Markov chains), where the quantities involved are typically nonnegative. The theory of positive systems [9, 12] is built upon classical positive matrix theory and graph theory, essential tools in the analysis of controllability and reachability [3, 7, 21].

The interest in positive switched systems is motivated both by practical applications and by theoretical reasons. Indeed, switching among different models naturally arises as a way to formalize the fact that the system laws change under different operating conditions. If in each operating condition the system has to be positive, we naturally obtain a positive switched

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system. Discrete-time positive switched systems (DPSS) have been adopted for describing networks employing TCP and other congestion control applications [18], for modeling consensus and synchronization problems [11], for investigating distributed coloring problems [13], and wireless power control applications [1]. Reachability and controllability analysis of DPSS [15, 16] is intrinsically challenging. In particular, there is no upper bound on the number of steps required by the algorithms presently available for checking reachability.

In this paper we will focus on the reachability properties of a specific class of DPSS, namely those commuting among p single-input positive subsystems, that share the same transition matrix, and differ in the input-to-state matrix (see [8]). For these systems the internal dynamics is governed by an invariant law, while the actuator action commutes among p possible configurations. These DPSS practically arise when trying to ensure reachability to a discrete-time positive system, with state transition matrix A, under the constraint of resorting to a single actuator at each time. Reachability in this context represents the possibility of achieving arbitrary distributions of the various state components. In particular, it may be important to freely act on certain entries, while leaving the others unaffected, a situation that corresponds to the reachability of states belonging to the boundary of the positive orthant. On the other hand, the constraint of using a single actuator may arise from economical/practical reasons (a single actuator which may act on the system in different configurations) or from opportunity reasons (in pharmacokinetic, for instance, it is often not appropriate to simultaneously apply different therapies or inject different tracers). In the general case, even if the structure of A does not ensure the existence of a single positive vector b such that the pair (A, b) is reachable, reachability may be achieved by switching among a finite number of configurations b_i, each of them corresponding to a different action of the actuator on the state components.

The paper is organized as follows: in section 2 we introduce the class of DPSS under investigation and define the reachability properties. Section 3 addresses monomial reachability, and section 4 reachability along a single switching sequence. The last three sections are devoted to the general reachability analysis. Specifically, in section 5 two technical lemmas provide necessary conditions, which are exploited in section 6 and 7 to characterize reachability when all input-to-state matrices are monomial or when some of them are nonmonomial, respectively.

Notation. Given two integers h and k, with $h \leq k$, we set $[h, k] := \{h, h+1, \dots, k\}$. Given a positive integer n and a set $V \subsetneq [1, n]$, we denote by $\overline{V}$ the complementary set of V, i.e., $[1, n] \setminus V$. The semiring of nonnegative real numbers is $\mathbb{R}_+$. A matrix $M \in \mathbb{R}_+^{n\times m}$ is a *nonnegative matrix.* M is *positive* $(M > 0)$ if nonnegative and nonzero, and *strictly positive* $(M \gg 0)$ if all its entries are positive. The (i, j)th entry of a matrix M is $[M]_{ij}$, the ith entry of a vector v is $[v]_i$, and $\mathbf{e}_i$ is the ith vector of the *canonical basis* in $\mathbb{R}^n$. The *nonzero pattern* of $v \in \mathbb{R}_+^n$, $\overline{\mathrm{ZP}}(v) := \{i : [v]_i \neq 0\}$, is the set of indices corresponding to its nonzero entries, and $|\overline{\mathrm{ZP}}(v)|$ is the number of nonzero entries of v. If $v_1, v_2 \in \mathbb{R}_+^n$ have the same nonzero pattern, we use the notation $v_1 \sim v_2$. If $M_1, M_2 \in \mathbb{R}_+^{n\times k}$, the symbol $M_1 \sim M_2$ means that $M_1\mathbf{e}_i \sim M_2\mathbf{e}_i$ for every index i. A vector $v \in \mathbb{R}_+^n$ is an *ℓth monomial vector* if $v \sim \mathbf{e}_\ell$. A *monomial (permutation) matrix* is a nonsingular square positive matrix whose columns are monomial (canonical) vectors. Any $n \times k$ submatrix of an $n \times n$ permutation matrix is a *selection matrix*. The matrices $C_r \in \mathbb{R}_+^{r\times r}$ and $S_r \in \mathbb{R}_+^{r\times(r-1)}$, given by

$$C_r = \begin{bmatrix} 0 & 0 & \dots & 0 & c_{1r} \\ c_{21} & 0 & \dots & 0 & 0 \\ 0 & c_{32} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & c_{r,r-1} & 0 \end{bmatrix} \qquad S_r = \begin{bmatrix} 0 & 0 & \dots & 0 \\ c_{21} & 0 & \dots & 0 \\ 0 & c_{32} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{r,r-1} \end{bmatrix}, \ c_{r,r-1}\cdots c_{21}c_{1r} \neq 0,$$

represent an $r \times r$ *cyclic monomial matrix* and the submatrix consisting of its first $r-1$ columns, respectively.

Given an n-dimensional *positive system* with p inputs

$$x(k+1) = Ax(k) + Bu(k), \qquad k \in \mathbb{Z}_+, \tag{1.1}$$

where $x(\cdot)$ and $u(\cdot)$ denote the n-dimensional nonnegative state variable and the p-dimensional nonnegative input variable, respectively, $A \in \mathbb{R}_+^{n\times n}$ and $B \in \mathbb{R}_+^{n\times p}$, we may associate with it [3, 4, 21] a *digraph* (directed graph) $\mathcal{D}(A,B)$, with n vertices, indexed by $1, 2, \dots, n$, and p sources $s_1, s_2, \dots, s_p$. There is an arc (j,i) from j to i if and only if $[A]_{ij} > 0$, and an arc (s_j, i) from s_j to i if and only if $[B]_{ij} > 0$.

A sequence $i_0 \to i_1 \to \dots \to i_k$, starting from the vertex i_0, and passing through the vertices $i_1, \dots, i_k$, is a *path* of length k from i_0 to i_k provided that $(i_0, i_1), \dots, (i_{k-2}, i_{k-1})$ are all arcs of $\mathcal{D}(A,B)$. The matrix A is *irreducible* if, given any pair of vertices i and j in $[1,n]$, there is a path from i to j. When so, for each vertex $i \in [1,n]$ there exists at least one closed path (a *circuit*) including it. The greatest common divisor (g.c.d.) of the lengths of all circuits appearing in $\mathcal{D}(A,B)$ represents the *imprimitivity index* of A (equivalently, of the digraph). If A is irreducible with imprimitivity index D, then there exists $k_0 \in \mathbb{Z}_+$ such that (s.t.) $A^k + A^{k+1} + \dots + A^{k+D-1}$ is strictly positive for every $k \geq k_0$. When the imprimitivity index is unitary, A is said to be *primitive* and there exists $k_0 \in \mathbb{Z}_+$ such that A^k is strictly positive for every $k \geq k_0$. If D is the imprimitivity index of A, all vertices in $\mathcal{D}(A,B)$ can be partitioned into D *imprimitivity classes*, $\mathcal{C}_p, p \in [1,D]$. Two vertices belong to the same class if and only if the length of any path connecting one vertex to the other is a multiple of D.

On the other hand, a sequence $s_j \to i_0 \to i_1 \to \dots \to i_{k-1}$, starting from the source s_j, and passing through the vertices $i_0, \dots, i_{k-1}$, is an *s-path* of length k from s_j to i_{k-1} provided that $(s_j, i_0), (i_0, i_1), \dots, (i_{k-2}, i_{k-1})$ are all arcs of $\mathcal{D}(A,B)$. An *s-path* of length k from s_j *deterministically reaches* some vertex i, if no other vertex of the digraph can be reached in k steps starting from s_j. It is easily seen that there is an s-path of length k from s_j to i if and only if $[A^{k-1}B]_{ij} > 0$. In general, leaving from s_j, after k steps one can reach several distinct vertices: this corresponds to saying that the jth column of $A^{k-1}B$ can have more than one nonzero entry. So, a vertex i can be deterministically reached from the source s_j by means of a path of length k if and only if the jth column of $A^{k-1}B$ is an ith monomial vector.

Basic definitions and results about cones may be found in [2]. A set $\mathcal{C} \subset \mathbb{R}^n$ is a *cone* if $\alpha\mathcal{C} \subseteq \mathcal{C}$ for all $\alpha \geq 0$. A cone $\mathcal{C}$ is *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of *generating vectors*. This amounts to saying that a positive integer k and an $n \times k$ matrix G can be found, such that $\mathcal{C}$ coincides with the set of nonnegative combinations of the columns of G. In this case, we adopt the notation $\mathcal{C} := \text{Cone}(G)$.

2. Reachability of a class of discrete-time positive switched systems

Generally speaking, by a *(single-input) discrete-time positive switched system* we mean a system described, at each time $t \in \mathbb{Z}_+$, by the first-order difference equation:

$$x(t+1) = A_{\sigma(t)}x(t) + b_{\sigma(t)}u(t), \tag{2.1}$$

where $x(t)$ and $u(t)$ denote the n-dimensional state variable and the scalar input, respectively, at time t, while σ is a switching sequence, defined on $\mathbb{Z}_+$ and taking values in $[1,p]$. For each $i \in [1,p]$, the pair (A_i, b_i) represents a discrete-time positive system, which means that $A_i \in \mathbb{R}_+^{n\times n}$ and $b_i \in \mathbb{R}_+^n$. In this paper we focus on DPSSs described by

$$x(t+1) = Ax(t) + b_{\sigma(t)}u(t). \tag{2.2}$$

This amounts to saying that the system switches among p subsystems (A, b_i), sharing the same positive transition matrix A, and differing only in the positive input-to-state matrices b_i.

Definition 2.1. *A state $x_f \in \mathbb{R}_+^n$ is* reachable at time $k \in \mathbb{N}$ *(in k steps) if there exist a switching sequence $\sigma : \mathbb{Z}_+ \to [1,p]$ and an input sequence $u : \mathbb{Z}_+ \to \mathbb{R}_+$ that lead the state trajectory from $x(0) = 0$ to $x(k) = x_f$. System (2.1) (in particular, (2.2)) is* monomially reachable *if every monomial vector $x_f \in \mathbb{R}_+^n$ is reachable at some time k, and* reachable *if every state $x_f \in \mathbb{R}_+^n$ is reachable at some time k.*

Since reachability of a given vector always refers to a finite time interval, focusing on the state at the final instant k, only the values of the switching sequence σ within $[0, k-1]$ are relevant. So, we refer to the cardinality of the discrete time interval $[0, k-1]$ as to the *length* of σ and denote it by $|\sigma| = k$. While for standard positive systems monomial reachability and reachability are equivalent properties, this is not the case for DPSS (2.1) [16].

When (monomial) reachability is ensured, a natural goal is that of determining the minimum number of steps required to reach every nonnegative (monomial) state.

Definition 2.2. *Given a (monomially) reachable DPSS (2.1) (or (2.2)), we define its* monomial reachability index *as $\mathcal{I}_{MR} := \max_{i\in[1,n]} \min\{k : \mathbf{e}_i$ is reachable at time $k\}$, and its* reachability index *as $\mathcal{I}_R := \sup_{x\in\mathbb{R}_+^n} \min\{k : x$ is reachable at time $k\}$.*

In [15, 16] necessary and sufficient conditions for the reachability of DPSS (2.1) have been given. Such conditions, however, are not computationally efficient, as the algorithms for testing them may require an arbitrarily large number of steps. This, however, seems to be an intrinsic problem, since reachable DPSS can be found [16] endowed with an infinite $\mathcal{I}_R$. For DPSS described as in (2.2) these situations cannot arise, and indeed we will derive canonical forms to which reachable DPSS may be reduced, thus leading to finite checking conditions.

To explore reachability properties of system (2.2), we provide the expression of the state at time $k \in \mathbb{N}$, starting from $x(0) = 0$, under the action of a nonnegative input $u(\cdot)$ and of a switching sequence $\sigma(\cdot)$. If we define the *reachability matrix associated with the switching sequence σ of length k* as

$$\mathcal{R}_k(\sigma) := [\, A^{k-1}b_{\sigma(0)} \quad A^{k-2}b_{\sigma(1)} \quad \dots \quad Ab_{\sigma(k-2)} \quad b_{\sigma(k-1)} \,],$$

it is easy to see that

$$(2.3)\quad x(k) = A^{k-1}b_{\sigma(0)}u(0) + A^{k-2}b_{\sigma(1)}u(1) + \ldots + b_{\sigma(k-1)}u(k-1) = \mathcal{R}_k(\sigma)\begin{bmatrix} u(0) \\ \vdots \\ u(k-1) \end{bmatrix},$$

and hence $x(k) \in \mathrm{Cone}(\mathcal{R}_k(\sigma))$. Therefore, a positive state x_f is reachable if and only if there exists a switching sequence σ such that $x_f \in \mathrm{Cone}(\mathcal{R}_{|\sigma|}(\sigma))$, or, equivalently, there exist $k \in \mathbb{N}$ and $i_0, \ldots, i_{k-2}, i_{k-1} \in [1, p]$ such that $x_f \in \mathrm{Cone}\left([\, A^{k-1}b_{i_0} \quad \ldots \quad Ab_{i_{k-2}} \quad b_{i_{k-1}}\,]\right)$.

3. Monomial reachability analysis

Monomial reachability of the DPSS (2.2) proves to be equivalent to the monomial reachability (and hence the reachability) of the (non-switched) positive system with p inputs (1.1) having the same state transition matrix A, and input-to-state matrix B obtained by juxtaposing the p columns b_i's. As a consequence, monomial reachability of (2.2) can be easily checked by resorting to either the algebraic or the graph-based algorithms available for positive systems [3, 7, 21].

Proposition 3.1. *The following facts are equivalent:*

i) the DPSS (2.2) is monomially reachable;

ii) the (n-dimensional) positive system with p inputs (1.1), with $B := [\, b_1 \quad b_2 \quad \ldots \quad b_p\,]$, *is reachable, i.e.* $\mathcal{R}_n(A, B) := [\, A^{n-1}B \quad \ldots \quad AB \quad B\,]$ *includes an* $n \times n$ *monomial submatrix [6, 21].*

So, if system (2.2) is monomially reachable, then $\mathcal{I}_{MR} \leq n$.

Proof. The DPSS (2.2) is monomially reachable if and only if for every $i \in [1, n]$ there exists a switching sequence σ_i of length say k_i such that $\mathbf{e}_i \in \mathrm{Cone}(\mathcal{R}_{k_i}(\sigma_i))$. This amounts to saying that for every $i \in [1, n]$ there exists $h_i \in \mathbb{Z}_+$ and $j_i \in [1, p]$ such that $A^{h_i}b_{j_i} \sim \mathbf{e}_i$. As proved in [6], if such an index h_i exists, it can always be chosen not greater than $n - 1$. But this ensures that all monomial vectors are reachable if and only if $\mathcal{R}_n(A, B)$ contains n linearly independent monomial vectors. The final statement about $\mathcal{I}_{MR}$ is an obvious consequence. □

We remark that the assumption $A_i = A$, $\forall i \in [1, p]$, ensures that the monomial reachability index of (2.2) never exceeds n, while for general DPSS (2.1) $\mathcal{I}_{MR}$ can even reach the value $2^n - 1$ (see [16], Proposition 3 and Example 3). However, as for system (2.1), the monomial reachability of (2.2) is not equivalent to reachability. In fact, if all monomial vectors can be reached by means of the same switching sequence σ, of length say k, the reachability matrix $\mathcal{R}_k(\sigma)$ includes an $n \times n$ monomial submatrix, and hence all positive vectors can be reached along σ. On the other hand, if monomial reachability cannot be achieved by means of a single switching sequence[1], different situations may arise, as illustrated by the following example.

[1]It is worthwhile to remark, however, that a discrete-time (non-positive) switched system is reachable if and only if [10] there exists a switching sequence σ (of length say k) such that $\mathrm{Im}\,(\mathcal{R}_k(\sigma)) = \mathbb{R}^n$.

EXAMPLE 1. Consider the DPSS Σ_1 and Σ_2, each of them described as in (2.2) with $p = 2$:

$$\Sigma_1: \ A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\Sigma_2: \ A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}, b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Both Σ_1 and Σ_2 are monomially reachable, but (for both of them) the canonical vectors cannot be reached along a single switching sequence. For Σ_1 this is obvious. For Σ_2 this is due to the fact that $\forall\, k > 0$ and $i \in [1, 2]$, $|\overline{\mathrm{ZP}}(A^k b_i)| > 1$. So, there is no switching sequence σ, of length say k, such that the reachability matrix $\mathcal{R}_k(\sigma)$ of Σ_2 includes a 2×2 monomial submatrix.

Σ_1 is not reachable, as no strictly positive vector can be reached. On the contrary, Σ_2 is reachable. In fact, if σ_1 and σ_2 are the switching sequences of length 2 taking only value 1 and 2 respectively, for Σ_2 we get

$$\mathrm{Cone}(\mathcal{R}_2(\sigma_1)) \cup \mathrm{Cone}(\mathcal{R}_2(\sigma_2)) = \mathrm{Cone}([\, Ab_1 \quad b_1\,]) \cup \mathrm{Cone}([\, Ab_2 \quad b_2\,])$$
$$= \mathrm{Cone}\left(\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\right) \cup \mathrm{Cone}\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\right) = \mathbb{R}^2_+. \qquad \diamondsuit$$

The following lemma, which extends Lemma 2 in [21], shows how the increasingly stronger assumptions of monomial reachability, reachability, and reachability along a single switching sequence impose strong constraints on the patterns of the matrices A and b_i, that will be very useful in the sequel.

LEMMA 3.2. *Suppose that the DPSS (2.2), with $A \in \mathbb{R}^{n\times n}_+, b_i \in \mathbb{R}^n_+$, $i \in [1, p]$, is monomially reachable. Then*

i) the $n \times (n + p)$ matrix $[\, A \quad b_1 \quad b_2 \quad \dots \quad b_p\,]$ includes an $n \times n$ monomial matrix, and at least one of the vectors $b_i, i \in [1, p]$, is monomial;

ii) if, in addition, the system is reachable and $n > 1$, A is nonzero,

iii) furthermore, if the system is reachable along a single switching sequence σ, there exists $i \in [1, p]$ such that b_i is monomial and $[\, A \quad b_i\,]$ includes an $n \times n$ monomial matrix.

Proof. i) If (2.2) is monomially reachable, then, by Proposition 3.1, the (non-switched) positive system (1.1) is reachable. This implies [21] that $[\, A \quad b_1 \quad b_2 \quad \dots \quad b_p\,]$ includes an $n \times n$ monomial matrix. On the other hand, if none of the b_i's would be monomial, A should be a monomial matrix, and $|\overline{\mathrm{ZP}}(A^k b_i)| = |\overline{\mathrm{ZP}}(b_i)| > 1$, for any $k \in \mathbb{Z}_+$ and $i \in [1, p]$. So, none of the columns of $\mathcal{R}_n(A, B)$ could be monomial, a contradiction.

ii) If system (2.2) is reachable and the dimension of the system is greater than 1, A cannot be zero, otherwise only the vectors $\alpha b_i,\ \alpha \in \mathbb{R}_+,\ i \in [1, p]$, could be reached.

iii) Suppose that $\mathcal{R}_k(\sigma) = [\, A^{k-1} b_{i_0} \quad \dots \quad A b_{i_{k-2}} \quad b_{i_{k-1}}\,]$, with $k > 0$, and $i_0, \dots, i_{k-2}, i_{k-1} \in [1, p]$, includes n linearly independent monomial vectors. If $b_{i_{k-1}}$ is monomial, then $n - 1$ monomial vectors linearly independent of $b_{i_{k-1}}$ must be obtained in the form $A^h b_i$, for some

$h > 0$ and $i \in [1, p]$. But this implies that each of these vectors is a column of A and hence $[\, A \quad b_{i_{k-1}} \,]$ includes an $n \times n$ monomial matrix. On the other hand, if $b_{i_{k-1}}$ is not monomial, then A is an $n \times n$ monomial matrix. In addition, by part i) there exists $i \in [1, p]$ such that b_i is monomial. So, $[\, A \quad b_i \,]$ includes an $n \times n$ monomial matrix. □

4. Reachability along a single switching sequence

In this section we investigate under which conditions a switching sequence σ (of suitable length k) can be found, such that $\mathcal{R}_k(\sigma)$ includes an $n \times n$ monomial submatrix. This is clearly equivalent to the possibility of reaching every positive state along the switching sequence σ. To avoid redundancy, we assume that all p subsystems are necessary in order to find such a switching sequence σ. We address, first, the case $p = 2$.

THEOREM 4.1. *Consider a DPSS (2.2), with $p = 2$, $A \in \mathbb{R}_+^{n \times n}, b_1, b_2 \in \mathbb{R}_+^n$, and suppose that neither (A, b_1) nor (A, b_2) is reachable. A necessary and sufficient condition for the existence of $k > 0$ and of a sequence σ such that $\mathcal{R}_k(\sigma)$ includes an $n \times n$ monomial submatrix is that there exist $r > 0$, a permutation matrix Π such that (possibly exchanging b_1 and b_2)*

$$\Pi^\top A \Pi = \left[\begin{array}{cc|c} S_r & v_1 & 0_{r \times (n-r)} \\ \hline 0_{(n-r) \times (r-1)} & v_2 & C_{n-r} \end{array} \right] \tag{4.1}$$

$$\Pi^\top B = \Pi^\top [\, b_1 \quad b_2 \,] \sim \begin{bmatrix} \mathbf{e}_1 & b_1^{(2)} \\ 0 & \mathbf{e}_1 \end{bmatrix}, \qquad b_1^{(2)} \in \mathbb{R}_+^r, \tag{4.2}$$

where $v_1 \in \mathbb{R}_+^r$ and $v_2 \in \mathbb{R}_+^{n-r}$. Moreover, if $b_1^{(2)} > 0$, then v_1 and v_2 are zero vectors, while if $b_1^{(2)} = 0$, then $|\overline{\mathrm{ZP}}(v_2)| = 1$ implies $v_1 \neq 0$. If so, k can always be chosen equal to n.

Proof. [Sufficiency] Assume that (4.1) and (4.2) hold, and set $k = n$, $\sigma(0) = \sigma(1) = \ldots = \sigma(n-r-1) = 2$ and $\sigma(n-r) = \sigma(n-r+1) = \ldots = \sigma(n-1) = 1$. Then $\mathcal{R}_n(\sigma)$ satisfies

$$\Pi^\top \mathcal{R}_n(\sigma) = \Pi^\top [\, A^{n-1} b_2 \quad \ldots \quad A^r b_2 \quad | \quad A^{r-1} b_1 \quad \ldots \quad b_1 \,] \sim \left[\begin{array}{c|c} 0_{r \times (n-r)} & \mathbf{e}_r \quad \ldots \quad \mathbf{e}_1 \\ \hline \tilde{\Pi} & 0_{(n-r) \times r} \end{array} \right],$$

for some permutation matrix $\tilde{\Pi}$, and hence it is an $n \times n$ monomial matrix. This immediately proves also the final statement of the theorem.

[Necessity] By Lemma 3.2, we can assume that b_1 is monomial and $[\, A \quad b_1 \,]$ includes an $n \times n$ monomial matrix. Consider the sequence $\{A^k b_1, k \in \mathbb{Z}_+\}$. Set

$$r := \max\{k \geq 0 : b_1, Ab_1, \ldots, A^{k-1} b_1 \text{ are linearly independent monomial vectors}\},$$

and assume w.l.o.g. that $A^k b_1 \sim \mathbf{e}_{k+1}, k \in [0, r-1]$. This implies

$$A = \left[\begin{array}{cc|c} S_r & v_1 & * \\ \hline 0_{(n-r) \times (r-1)} & v_2 & * \end{array} \right],$$

where $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A\mathbf{e}_r \sim A^r b_1 \not\sim \mathbf{e}_\ell, \ell \in [r+1, n]$, (otherwise r could be increased) and $*$ represents a block whose structure will be ascertained. As $[\,A \quad b_1\,]$ includes an $n \times n$ monomial matrix, the last $n - r$ columns of A are independent ℓth monomial vectors, $\ell \in [r+1, n]$. Thus

$$A \sim \left[\begin{array}{cc|c} S_r & v_1 & 0_{r\times(n-r)} \\ \hline 0_{(n-r)\times(r-1)} & v_2 & \tilde{\Pi} \end{array}\right],$$

where $\tilde{\Pi}$ is an $(n-r) \times (n-r)$ permutation matrix. Since $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \not\sim \mathbf{e}_\ell, \ell \in [r+1, n]$, the following cases may occur: (1) $v_1 \neq 0$; (2) $v_1 = 0$ and $|\overline{\mathrm{ZP}}(v_2)| > 1$; (3) $v_1 = 0$ and $v_2 = 0$. In all cases, for every $k \geq r$, $A^k b_1 \not\sim \mathbf{e}_\ell, \ell \in [r+1, n]$. Indeed, in case (1) $\overline{\mathrm{ZP}}(A^k b_1) \cap [1, r] \neq \emptyset, \forall\ k \geq r$. In case (2) $|\overline{\mathrm{ZP}}(A^k b_1)| \geq 2, \forall k \geq r$. In case (3) $A^k b_1 = 0, \forall\ k \geq r$. So, by the reachability assumption, for every $\ell \in [r+1, n]$ there must be $k \geq 0$ such that $A^k b_2 \sim \mathbf{e}_\ell$.

To conclude the proof, we analyze what nonzero patterns of A and b_2 make this possible.

- If $|\overline{\mathrm{ZP}}(b_2)| > 1$, in cases (1) and (2) one finds $A^k b_2 \not\sim \mathbf{e}_\ell$, $\ell \in [r+1, n]$. So, $|\overline{\mathrm{ZP}}(b_2)| > 1$ is compatible only with case (3), and it must be $|\overline{\mathrm{ZP}}(b_2) \cap [r+1, n]| = 1$.

- If $|\overline{\mathrm{ZP}}(b_2)| = 1$, it must be $\overline{\mathrm{ZP}}(b_2) \subseteq [r+1, n]$.

In both situations, we may assume w.l.o.g. that $\overline{\mathrm{ZP}}(b_2) \cap [r+1, n] = \{r+1\}$, and it is easy to see that the only way to generate all ℓth monomial vectors, $\ell \in [r+1, n]$, is by imposing to $\tilde{\Pi}$ a cyclic structure, thus getting the matrix (4.1). □

Remark 4.2. *In order to reach all monomial vectors along a single switching sequence, we need to find n* distinct *indices $h_1, h_2, \ldots, h_n \in \mathbb{Z}_+$ such that $\overline{\mathrm{ZP}}(A^{h_i} b_{j_i}) = \{i\}$ for suitable $j_i \in [1, 2]$, $i \in [1, n]$. In graph theoretic terms, this amounts to saying that in $\mathcal{D}(A, B)$ all vertices $i \in [1, n]$ must be reached through deterministic s-paths of distinct lengths. Theorem 4.1 states that such paths can be found if and only if (1) the graph $\mathcal{D}(A, 0)$ can be partitioned into an elementary circuit and a chain, whose last vertex can have an arbitrary number of outgoing arcs; (2) one source, corresponding to b_1, has a single outgoing arc to the first vertex of the chain, while the other source, associated with b_2, accesses one (and only one) of the vertices of the elementary circuit. In addition, if the last vertex of the chain has no outgoing arcs, then this second source may also have an arbitrary number of arcs connecting them to the vertices of the chain (see Fig. 4.1 and Fig. 4.2).*

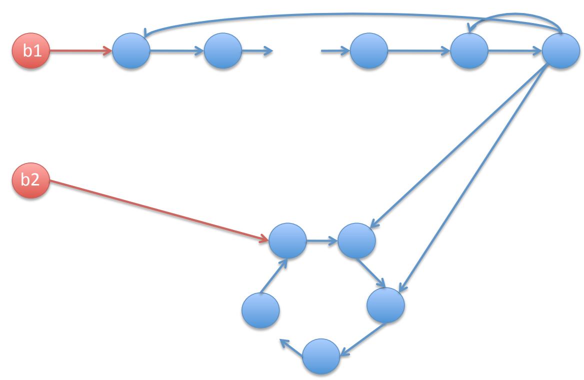

FIG. 4.1. Theorem 4.1, Case $b_1^{(2)} = 0$.

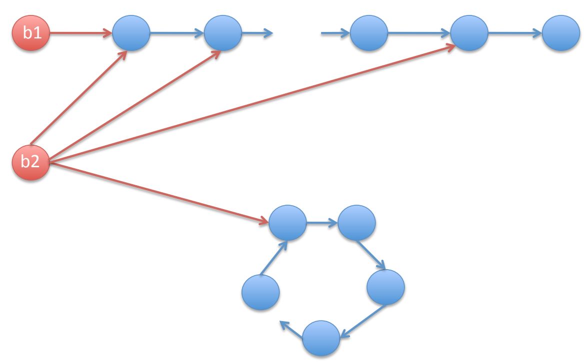

FIG. 4.2. Theorem 4.1, Case $b_1^{(2)} \neq 0$ and $v_1 = 0, v_2 = 0$.

At the light of the previous graph interpretation, the extension of Theorem 4.1 to the case $p > 2$ is rather intuitive, and therefore the proof is omitted.

THEOREM 4.3. *Consider a DPSS (2.2), with* $A \in \mathbb{R}_+^{n\times n}, b_i \in \mathbb{R}_+^n, i \in [1,p]$, *and suppose that the system is not reachable along a switching sequence taking values in a proper subset of* $[1,p]$. *A necessary and sufficient condition for the existence of* $k > 0$ *and a sequence* σ, *such that* $\mathcal{R}_k(\sigma)$ *includes an* $n \times n$ *monomial submatrix, is that there exist* $r_1, r_2, \ldots, r_p > 0$, *with*

$\sum_{i=1}^{p} r_i = n$*, a permutation matrix* Π*, and possibly a relabeling of the* b_i*'s, such that*[2]

$$\Pi^\top A \Pi = \begin{bmatrix} S_{r_1} & v_1 & | & & & \\ \hline & v_2 & | & C_{r_2} & & \\ & \vdots & | & & \ddots & \\ & v_{p-1} & | & & & C_{r_{p-1}} & \\ & v_p & | & & & & C_{r_p} \end{bmatrix}, \qquad v_i \in \mathbb{R}_+^{r_i},$$

$$\Pi^\top B \;=\; \Pi^\top [\, b_1 \quad b_2 \quad \dots \quad b_p \,] \sim \begin{bmatrix} \mathbf{e}_1 & b_1^{(2)} & b_1^{(3)} & \dots & b_1^{(p)} \\ & \mathbf{e}_1 & & & \\ & & \mathbf{e}_1 & & \\ & & & \ddots & \\ & & & & \mathbf{e}_1 \end{bmatrix}, \quad b_1^{(j)} \in \mathbb{R}_+^{r_1}.$$

If $b_1^{(j)} > 0$ *for some* j*, then* $v_i = 0$ *for all* i*, while if* $b_1^{(j)} = 0$ *for all* j*, then* $\sum_{i=2}^{p} |\overline{\mathrm{ZP}}(v_i)| = 1$ *implies* $v_1 \neq 0$*. If the above equivalent conditions hold,* k *can always be chosen equal to* n*.*

5. Reachability analysis: preliminary results

The case discussed in the previous section, when all positive vectors can be reached along a single switching sequence, is very restrictive. So, we want to derive general conditions for reachability of DPSS (2.2), without constraints on the number of switching sequences. In this section we derive two technical lemmas which provide necessary conditions for reachability. By Lemma 3.2, we know that reachability ensures that the matrix $[\, A \quad b_1 \quad b_2 \quad \dots \quad b_p \,]$ includes n linearly independent monomial vectors. So, if m of them can be found among the b_i's, the remaining $n-m$ are columns of A. Lemma 5.1 below shows that reachability constrains also the nonzero patterns of the other m columns of A.

LEMMA 5.1. *Consider a DPSS (2.2), with* $A \in \mathbb{R}_+^{n\times n}, b_i \in \mathbb{R}_+^n, i \in [1,p]$*. Let* m *be a positive integer in* $[1,p]$*, and suppose that the* b_i*'s,* $i \in [1,m]$*, are linearly independent monomial vectors, while the* b_i*'s,* $i \in [m+1,p]$*, are nonzero nonmonomial vectors. Set* $V_B := \{\overline{\mathrm{ZP}}(b_i) : i \in [1,m]\}$ *and* $\overline{V}_B = [1,n] \setminus V_B$*. If the system (2.2) is reachable, then*

- *i) for every* $\ell \in \overline{V}_B$*, a column of* A *can be selected which is an* ℓ*th monomial vector;*
- *ii) the remaining* m *columns of* A *satisfy one of the following conditions*[3] *(possibly up to a reordering the* b_i*'s,* $i \in [1,m]$*):*
  - *a)* $m-1$ *columns have the same nonzero patterns as the monomial vectors* $b_i, i \in [2,m]$*;*

[2]In the following we assume that the not specified off-diagonal blocks of the matrices are zero.
[3]If $m = 2$, only conditions a) and b) below are meaningful, while for $m = 1$ no constraints are imposed.

b) $m-2$ columns have the same nonzero patterns as the monomial vectors $b_i, i \in [3,m]$, and one column has the same nonzero pattern as b_1+b_2;

c) $m-3$ columns have the same nonzero patterns as the monomial vectors $b_i, i \in [4,m]$, and three columns have the same nonzero patterns as b_1+b_2, b_1+b_3, and b_2+b_3.

Proof. i) From Lemma 3.2, it follows that under the reachability assumption, the matrix $[\,A \quad b_1 \quad b_2 \quad \dots \quad b_p\,]$, and hence $[\,A \quad b_1 \quad b_2 \quad \dots \quad b_m\,]$, includes an $n \times n$ monomial matrix. So, we can select $n-m$ columns of A which are linearly independent ℓth monomial vectors, $\ell \in \overline{V}_B$, and denote by Θ_A the set of indices of the remaining m columns of A.

ii) Let (j_1, j_2), with $j_1 < j_2$, be an arbitrary pair in $V_B \times V_B$. By the reachability assumption, all positive vectors x, with $\overline{\rm ZP}(x) = \{j_1, j_2\}$, must be reachable. However, not all of them can be reached in a single step. This is obvious if $m = p$, namely all the b_i's are monomial. On the other hand, if $\overline{\rm ZP}(b_i) = \{j_1, j_2\}$ for some $i \in [m+1, p]$, then only positive multiples of b_i can be reached in a single step. Consequently, there exist $k > 0$ and $i \in [1,p]$ such that $\emptyset \neq \overline{\rm ZP}(A^k b_i) \subseteq \{j_1, j_2\}$. Since $k > 0$, this implies that there exists a nonzero column of A with nonzero pattern included in $\{j_1, j_2\}$.

We first prove that, to meet this constraint for every $(j_1, j_2) \in V_B \times V_B$, with $j_1 < j_2$, at least $m-3$ columns with indices in Θ_A must be linearly independent ℓth monomial vectors, $\ell \in V_B$. Assume there are only $m-q$ monomial columns of this kind, with $q > 3$. The set

$$M := \{\ell \in V_B : \text{ no column of } A \text{ is an } \ell\text{th monomial vector } \} \subseteq V_B$$

has cardinality q, and we want to prove that there exists at least one pair $(j_1, j_2) \in M \times M, j_1 < j_2$, such that no nonzero column of A can be found with nonzero pattern included in $\{j_1, j_2\}$. This follows trivially from the fact that, for $q > 3$, the number q of the remaining columns with indices in Θ_A is less than $\binom{q}{2} = \frac{q(q-1)}{2}$, the number of pairs $(j_1, j_2) \in M \times M$, with $j_1 < j_2$.

If the linearly independent ℓth monomial columns of A, $\ell \in V_B$, are exactly $m-3$, it entails no loss of generality assuming that their nonzero patterns coincide with those of $b_i, i \in [4,m]$. In this case the only way to ensure that for every $1 \leq i_1 < i_2 \leq 3$ there exists a nonzero column of A with nonzero pattern included in $\overline{\rm ZP}(b_{i_1} + b_{i_2})$ is to fulfill condition c).

If the linearly independent ℓth monomial columns of A, $\ell \in V_B$, are $m-2$, and we assume that their nonzero patterns coincide with those of $b_i, i \in [3,m]$, to ensure that there exists a nonzero column of A with nonzero pattern included in $\overline{\rm ZP}(b_1 + b_2)$ we have to fulfill condition b). Finally, if A has (at least) $m-1$ linearly independent ℓth monomial columns, $\ell \in V_B$, (case a)), all the inclusions are satisfied. □

The following lemma shows that if the DPSS (2.2) is reachable by resorting only to the monomial columns b_i's, then in the digraph $\mathcal{D}(A,B)$ there exist non-intersecting chains, stemming from the sources corresponding to these b_i's, and passing through all vertices. If reachability requires also non-monomial b_i's, then the previous chains do not cover all vertices, one chain ends without outgoing arcs and the final vertices of the others satisfy certain constraints on the outgoing arcs.

LEMMA 5.2. *Consider a reachable DPSS (2.2), with* $A \in \mathbb{R}_+^{n\times n}, b_i \in \mathbb{R}_+^n, i \in [1,p]$ *and* $m \in [1,p]$ *as in Lemma 5.1. Set*

r_1:=max$\{k \geq 0 : b_1, Ab_1, \dots, A^{k-1}b_1$ *are lin. independent monomial vectors*$\}$,
r_2:=max$\{k \geq 0 : b_1, Ab_1, \dots, A^{r_1-1}b_1, b_2, \dots, A^{k-1}b_2$ *are lin. independent monomial vectors*$\}$,
$\vdots$
r_m:=max$\{k \geq 0 : b_1, Ab_1, \dots, A^{r_2-1}b_2, b_m, \dots, A^{k-1}b_m$ *are lin. independent monomial vectors*$\}$.

Then

i) if the system (2.2) is reachable by switching only among the first m subsystems, endowed with a monomial b_i*, then* $\sum_{i=1}^m r_i = n$*;*

ii) otherwise $\sum_{i=1}^m r_i < n$*, the* m *vectors* $A^{r_i}b_i, i \in [1,m]$*, satisfy a) or b) of Lemma 5.1, and one of them is zero.*

Proof. It entails no loss of generality assuming that each r_i is positive. If not, we simply discard the corresponding b_i. If we set $r := \sum_{i=1}^m r_i$, and define $V_B := \{\overline{\mathrm{ZP}}(b_i) : i \in [1,m]\}$ and $V := \{\overline{\mathrm{ZP}}(A^k b_i) : i \in [1,m], k \in [0, r_i - 1]\}$, then
• the sets V_B and V have cardinalities m and r, respectively;
• for every $i \in [1,m]$ and $k \in [0, r_i - 1]$, $A^k b_i \sim \mathbf{e}_\ell$ for some $\ell \in V$;
• the vectors $A^k b_i$, $i \in [1,m]$, $k \in [1, r_i]$, are (up to scalar multiples) distinct columns of A;
• the m vectors $A^{r_i}b_i$, $i \in [1,m]$, cannot be ℓth monomial vectors for some $\ell \in \overline{V}$ (otherwise r_i could be increased).

Basing on these remarks, we can claim that A has $r - m$ columns that are ℓth monomials, $\ell \in V \setminus V_B$, and m columns (that correspond to $A^{r_i}b_i, i \in [1,m]$, and whose set of indices we denote by Θ_A) which are not ℓth monomial vectors, $\ell \in \overline{V}$. So, if $r < n$, there are still $n - r$ columns of A whose nonzero patterns have to be determined. We want to show that if $r < n$, then $A^k b_i \in \mathcal{C}_V := \mathrm{Cone}(\{e_i, i \in V\}) = \mathrm{Cone}(\{A^k b_i : i \in [1,m], k \in [0, r_i - 1]\})$ for every choice of $k > 0$ and $i \in [1,m]$, and there exists $q \in [1,m]$ such that $A^{r_q}b_q = 0$.

So, assume $r < n$. By the reachability assumption and Lemma 3.2, part i), the remaining $n - r$ columns of A are linearly independent ℓth monomial vectors, $\ell \in \overline{V}$. Therefore, by referring to Lemma 5.1, the m columns of A indexed on Θ_A are just the m columns for which one of the conditions a), b) or c) holds. Notice that at least $m-1$ of them (m in case c)) have nonzero patterns included in V_B and hence in V. Reachability property ensures that for every $\ell \in \overline{V}$ there exist $k > 0$ and $i \in [1,p]$ such that $A^k b_i \sim \mathbf{e}_\ell$. Set

$$K := \min\{k > 0 : \exists\ i \in [1,p], \ell \in \overline{V},\ \text{such that}\ A^k b_i \sim \mathbf{e}_\ell\}.$$

If $A^K b_I = A(A^{K-1}b_I) \sim e_L$, for some $L \in \overline{V}$, the nonzero vector $A^{K-1}b_I$ cannot be an ℓth monomial for some $\ell \in \overline{V}$, otherwise the minimality of K would be contradicted, nor an ℓth monomial for some $\ell \in V$, otherwise $A^{K-1}b_I \sim A^k b_i$ for some $i \in [1,m]$ and $k \in [0, r_i - 1]$, thus preventing $A^K b_I$ to be an Lth monomial vector. So, $|\overline{\mathrm{ZP}}(A^{K-1}b_I)| \geq 2$ and its nonzero entries must correspond to columns of A that are either Lth monomials or zero vectors. Since A has only one Lth monomial column, one of the m columns indexed on Θ_A must be zero, which amounts to saying that $A^{r_q} b_q = 0$, for some index $q \in [1, m]$. If so, only cases a) and b) are possible for those m columns and all of them have nonzero patterns included in V. Since $A^{r_i} b_i \in \mathcal{C}_V$ for every $i \in [1, m]$, $\mathcal{C}_V$ is an A-invariant cone, and hence $A^k b_i \in \mathcal{C}_V$ for every choice of the indices k and $i \in [1, m]$. So, $A^K b_I \sim e_L$ implies $I \in [m + 1, p]$.

As a consequence, if we assume that the DPSS (2.2) is reachable by switching only among the first m subsystems, it must be $V = [1, n]$ and $r = n$. Conversely, if the system is not reachable by switching only among the first m subsystems, all ℓth monomial vectors, $\ell \in \overline{V}$, must be obtained in the form $A^k b_i$, for some $k > 0$ and some $i \in [m + 1, p]$. This implies that $V \subsetneq [1, n]$, $r < n$, the m columns of A corresponding to the vectors $A^{r_i} b_i$, $i \in [1, m]$, satisfy a) or b) of Lemma 5.1, and one of them is zero. □

6. Reachability analysis: the case when all the b_i's are monomial

In this section, we consider the class of DPSS (2.2) switching among p subsystems whose b_i's are linearly independent monomial (and hence w.l.o.g. canonical) vectors. Proposition 6.1 provides a necessary condition for the reachability of this class of systems, which proves to be also sufficient in the cases $p = 2$ (Proposition 6.3) and $p = 3$ (Proposition 6.4). By making use of these partial results, we finally show, in Theorem 6.7, that the necessary condition obtained in Proposition 6.1 is also sufficient (for every p).

Proposition 6.1. *Consider a DPSS (2.2), with $A \in \mathbb{R}_+^{n \times n}, b_i \in \mathbb{R}_+^n, i \in [1, p]$, and suppose that all the b_i's are linearly independent canonical vectors. If the system (2.2) is reachable, there exist a permutation matrix Π, an integer $m \in [1, p]$, a selection matrix $S \in \mathbb{R}_+^{p \times m}$ and positive integers $r_1, r_2, \dots, r_m, \sum_{i=1}^m r_i = n$, such that*

$$\Pi^\top A \Pi = \left[\begin{array}{cc|cc|cc|ccc}
S_{r_1} & v_1^{(1)} & 0_{r_1\times(r_2-1)} & v_1^{(2)} & 0_{r_1\times(r_3-1)} & v_1^{(3)} & & & \\ \hline
0_{r_2\times(r_1-1)} & v_2^{(1)} & S_{r_2} & v_2^{(2)} & 0_{r_2\times(r_3-1)} & v_2^{(3)} & & & \\ \hline
0_{r_3\times(r_1-1)} & v_3^{(1)} & 0_{r_3\times(r_2-1)} & v_3^{(2)} & S_{r_3} & v_3^{(3)} & & & \\ \hline
 & v_4^{(1)} & & & & & C_{r_4} & & \\
 & \vdots & & & & & & \ddots & \\
 & v_m^{(1)} & & & & & & & C_{r_m}
\end{array}\right] \tag{6.1}$$

$$\Pi^\top BS = \left[\begin{array}{ccc|ccc} \mathbf{e}_1 & & & & & \\ & \mathbf{e}_1 & & & & \\ & & \mathbf{e}_1 & & & \\ \hline & & & \mathbf{e}_1 & & \\ & & & & \ddots & \\ & & & & & \mathbf{e}_1 \end{array}\right] = [\,\mathbf{e}_1 \quad \mathbf{e}_{r_1+1} \quad \mathbf{e}_{r_1+r_2+1} \quad | \quad \dots \quad \mathbf{e}_{r_1+r_2+\dots+r_{m-1}+1}\,],$$

where $v_j^{(i)} \in \mathbb{R}_+^{r_j}$, *and the vectors* $v^{(i)} := \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \\ v_3^{(i)} \end{bmatrix}, i \in [1,3]$, *and* $z^{(1)} := \begin{bmatrix} v_4^{(1)} \\ \vdots \\ v_m^{(1)} \end{bmatrix}$ *satisfy one of the following conditions:*

α) $\begin{bmatrix} v^{(1)} \\ z^{(1)} \end{bmatrix} \not\sim \mathbf{e}_j, \forall\, j \in \{r_1+1, r_1+r_2+1, \dots, r_1+\dots+r_{m-1}+1\}$, $v^{(2)} \sim \mathbf{e}_{r_1+1}$ *and* $v^{(3)} \sim \mathbf{e}_{r_1+r_2+1}$;

β) $\begin{bmatrix} v^{(1)} \\ z^{(1)} \end{bmatrix} \not\sim \mathbf{e}_j, \forall\, j \in \{r_1+1, r_1+r_2+1, \dots, r_1+\dots+r_{m-1}+1\}$, $v^{(2)} \sim \mathbf{e}_1 + \mathbf{e}_{r_1+1}$ *and* $v^{(3)} \sim \mathbf{e}_{r_1+r_2+1}$;

γ) $\{v^{(1)}, v^{(2)}, v^{(3)}\} \sim \{\mathbf{e}_1 + \mathbf{e}_{r_1+1}, \mathbf{e}_1 + \mathbf{e}_{r_1+r_2+1}, \mathbf{e}_{r_1+1} + \mathbf{e}_{r_1+r_2+1}\}$ *and* $z^{(1)} = 0$.

Notice that if $m = 3$ *then all blocks* $C_{r_4}, \dots, C_{r_m}$ *and* $z^{(1)}$ *are empty, and if* $m = 2$ *only conditions* α) *and* β) *are possible, and can be rewritten as:*

α′) $v^{(1)} \not\sim \mathbf{e}_{r_1+1}$ *and* $v^{(2)} \sim \mathbf{e}_{r_1+1}$;

β′) $v^{(1)} \not\sim \mathbf{e}_{r_1+1}$ *and* $v^{(2)} \sim \mathbf{e}_1 + \mathbf{e}_{r_1+1}$.

Proof. Let $r_1, r_2, \dots, r_p$ be defined as in Lemma 5.2, and assume that each of them is positive. If $r_i = 0$, we simply remove the column b_i. Set $P := p$. We recall that

i) the vectors $A^k b_i$, $i \in [1, P]$ and $k \in [0, r_i - 1]$, are n linearly independent monomial vectors;

ii) the vectors $A^k b_i$, $i \in [1, P]$, $k \in [1, r_i]$, coincide (up to scaling factors) with the n distinct columns of A;

iii) each vector $A^{r_i} b_i$, $i \in [1, P]$, is not a monomial vector linearly independent of the vectors $A^k b_j$ that precede it in the selection procedure.

If $A^{r_i} b_i \sim b_j$ for some $1 \le j < i \le P$, then the whole chain $b_j, Ab_j, \dots, A^{r_j-1} b_j$ can be appended to the chain stemming from b_i. So, if we remove b_j and repeat the procedure of Lemma 5.2, to obtain the chains and the indices r_i's, we end up with $P \le p-1$ positive indices and chains that still satisfy conditions i) ÷ iii). By proceeding in this way, in a finite number of steps, we end up with $P := m$ columns $b_i, i \in \mathcal{M} \subseteq [1, p], |\mathcal{M}| = m$, and m chains for which conditions i) ÷ iii) hold and, in addition, no $i, j \in \mathcal{M}, i < j$, can be found such that $A^{r_i} b_i \sim b_j$. Clearly, by Lemma 5.1, the m columns of A corresponding to the vectors $A^{r_i} b_i, i \in \mathcal{M}$, must satisfy one of the conditions a), b) or c), but this implies that every time $A^{r_i} b_i$ is an ℓth monomial vector, for some $\ell \in \{\overline{\mathrm{ZP}}(b_i) : i \in \mathcal{M}\}$, it must be $A^{r_i} b_i \sim b_i$. So, after the selection of the b_i's (described by the matrix S), we can permute the entries (by means of the matrix $\Pi^\top$) of the vectors $A^k b_i$ of the m final chains, so that $\Pi^\top A \Pi$ and $\Pi^\top BS$ take the form given in the statement, and conditions a), b) and c) impose α), β) and γ), respectively. □

REMARK 6.2. *The necessary conditions given in Proposition 6.1 can be enlightened in graph theoretic terms. Figures 6.1 and 6.2, below, are the digraphs of two DPSS (2.2) that satisfy such conditions. Indeed, (1) the graph $\mathcal{D}(A, 0)$ can be partitioned into three chains and $m-3$ elementary circuits, with $m \le p$; (2) the outgoing arcs of the last vertices of the chains are constrained according to either condition β) or condition γ); (3) among the p sources associated with the b_i's, one can select $m-3$ so that each of them accesses one (and only one) of the vertices of a distinct elementary circuit. Moreover, for each chain there exists a source with a single outgoing arc to the first vertex of the chain.*

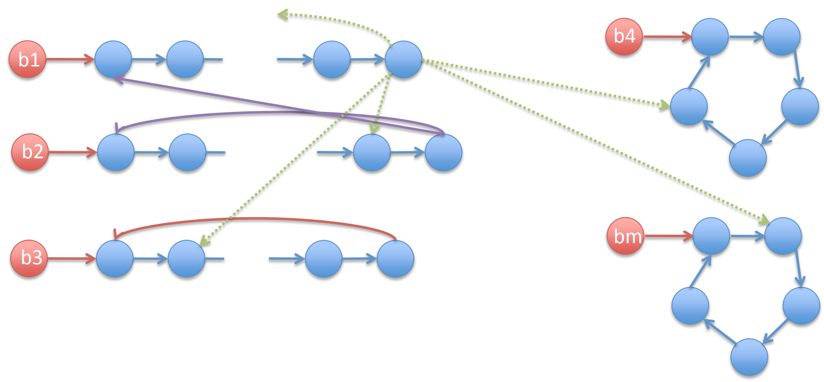

FIG. 6.1. *Proposition 6.1, Case β.*

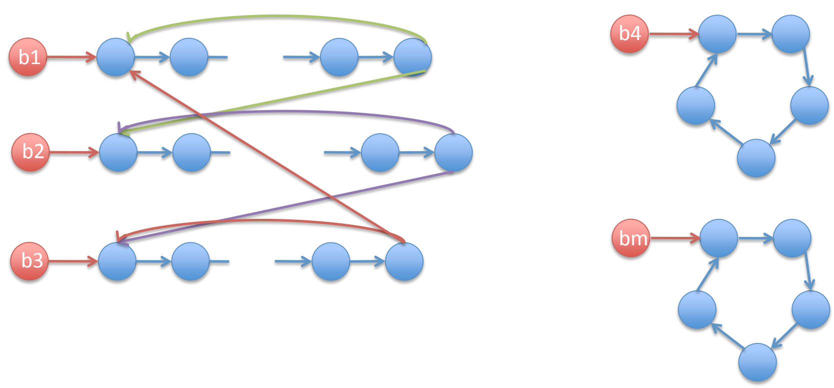

FIG. 6.2. *Proposition 6.1, Case γ.*

PROPOSITION 6.3. *Consider a DPSS (2.2), with $A \in \mathbb{R}_+^{n\times n}$, $p=2$, and b_1 and b_2 linearly independent canonical vectors in $\mathbb{R}_+^n$. A sufficient condition for the system to be reachable is that there exist a positive integer r, and a permutation matrix Π, such that*

$$\Pi^\top A \Pi = \left[\begin{array}{cc|cc} S_r & v_1^{(1)} & 0_{r\times(n-r-1)} & v_1^{(2)} \\ \hline 0_{(n-r)\times(r-1)} & v_2^{(1)} & S_{n-r} & v_2^{(2)} \end{array}\right], \qquad \Pi^\top [\, b_1 \quad b_2 \,] = [\, \mathbf{e}_1 \quad \mathbf{e}_{r+1} \,], \tag{6.2}$$

where the two vectors $v^{(i)} := \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \end{bmatrix}, i \in [1,2]$, satisfy either one of the following conditions:

α') $v^{(1)} \not\sim \mathbf{e}_{r+1}$ and $v^{(2)} \sim \mathbf{e}_{r+1}$;
β') $v^{(1)} \not\sim \mathbf{e}_{r+1}$ and $v^{(2)} \sim \mathbf{e}_1 + \mathbf{e}_{r+1}$.

If so, $\mathcal{I}_R = n$.

Proof. Suppose, for the sake of simplicity, that $\Pi = I_n$. Three cases possibly occur:
(1) $v^{(1)}$ and $v^{(2)}$ meet condition α');
(2) $v^{(1)}$ and $v^{(2)}$ meet condition β') and $r \leq n-r$;
(3) $v^{(1)}$ and $v^{(2)}$ meet condition β') and $r > n-r$.

Case (1): By Theorem 4.1, there exists a sequence σ of length n, such that the reachability matrix $\mathcal{R}_n(\sigma)$ is an $n \times n$ monomial matrix. This ensures reachability.
Case (2): We preliminarily notice that $A^k b_1, k \in [0, r-1]$, and $A^k b_2, k \in [0, n-1]$, satisfy:

$$\begin{array}{lll} b_1 \sim \mathbf{e}_1, & b_2 \sim \mathbf{e}_{r+1}, & A^{n-r}b_2 \sim \mathbf{e}_1 + \mathbf{e}_{r+1}, \\ Ab_1 \sim \mathbf{e}_2, & Ab_2 \sim \mathbf{e}_{r+2}, & A^{n-r+1}b_2 \sim \mathbf{e}_2 + \mathbf{e}_{r+2}, \\ \vdots & \vdots & \vdots \\ A^{r-1}b_1 \sim \mathbf{e}_r, & A^{r-1}b_2 \sim \mathbf{e}_{2r}, & A^{n-1}b_2 \sim \mathbf{e}_r + \mathbf{e}_{2r} \\ & A^r b_2 \sim \mathbf{e}_{2r+1} & \\ & \vdots & \\ & A^{n-r-1}b_2 \sim \mathbf{e}_n, & \end{array}$$

where we exploited the fact that $r \leq n-r$. Every vector $x > 0$ can be expressed as $x = \sum_{p=1}^{r}([x]_p \mathbf{e}_p + [x]_{p+r}\mathbf{e}_{p+r}) + \sum_{i=2r+1}^{n}[x]_i \mathbf{e}_i$. On the other hand, each vector $[x]_p\mathbf{e}_p + [x]_{p+r}\mathbf{e}_{p+r}$, $p \in [1, r]$, belongs either to $\mathrm{Cone}([A^{p-1}b_1 \;\; A^{n-r+p-1}b_2])$ or to $\mathrm{Cone}([A^{p-1}b_2 \;\; A^{n-r+p-1}b_2])$, depending on the specific values of its nonzero entries. Consequently, x belongs to the cone of a reachability matrix that, once reordered in the most convenient way, is thus composed:

$$\left[b_{j_1} \quad A^{n-r}b_2 \;\middle|\; Ab_{j_2} \quad A^{n-r+1}b_2 \;\middle|\; \dots \quad \middle|\; A^{r-1}b_{j_r} \quad A^{n-1}b_2 \;\middle\|\; A^r b_2 \;\middle|\; \dots \;\middle|\; A^{n-r-1}b_2 \right]$$

where each index $j_p \in [1, 2]$ depends on the specific values of $[x]_p$ and $[x]_{p+r}$.
Case (3): If $r > s := n-r$, the vectors of the two sequences $\{A^{j-1}b_1, j \in [1, r]\}$ and $\{A^{j-1}b_2, j \in [1, n]\}$ have the following nonzero patterns:

$$A^{p-1+ts}b_1 \sim \mathbf{e}_{p+ts} \qquad\qquad A^{p-1+ts}b_2 \sim \mathbf{e}_{p+r} + \sum_{h=0}^{t-1} \mathbf{e}_{p+hs}$$

where we exploited the fact that every $j \in [1, n]$ can be expressed as $j = p + ts$, for some $t \in \mathbb{Z}_+$ and $p \in [1, s]$. Let $x^{[p]}, p \in [1, s]$, be a positive vector with $\overline{\mathrm{ZP}}(x^{[p]}) \subseteq \{p, p+s, p+2s, \dots, p+ks; p+r\}$, where $k \geq 0$ and $p + ks \leq r$. We want to prove for every $k \geq 0$:

Claim 1 If $\overline{\mathrm{ZP}}(x^{[p]}) \subsetneq \{p, p+s, p+2s, \dots, p+ks; p+r\}$, then there exist indices $j_0, j_1, \dots, j_k \in [1, 2]$ such that $x^{[p]}$ belongs to the cone generated by a (reachability sub)matrix of the following type: $[\, A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \dots \quad A^{p-1+ks}b_{j_k} \,]$;

Claim 2 If $\overline{\mathrm{ZP}}(x^{[p]}) = \{p, p+s, p+2s, \dots, p+ks; p+r\}$, then there exist indices $j_0, j_1, \dots, j_{k+1} \in [1, 2]$ such that $x^{[p]} \in \mathrm{Cone}\,([\, A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \dots \quad A^{p-1+(k+1)s}b_{j_{k+1}} \,])$.

We proceed by induction on $k \geq 0$. The case $k = 0$ corresponds to $\overline{\mathrm{ZP}}(x^{[p]}) \subseteq \{p; p+r\}$. So, if $\overline{\mathrm{ZP}}(x^{[p]}) \subsetneq \{p; p+r\}$, then $x^{[p]} \sim \mathbf{e}_p$ implies $x^{[p]} \in \mathrm{Cone}\left(A^{p-1}b_1\right)$, while $x^{[p]} \sim \mathbf{e}_{p+r}$ implies $x^{[p]} \in \mathrm{Cone}\left(A^{p-1}b_2\right)$. On the other hand, if $\overline{\mathrm{ZP}}(x^{[p]}) = \{p; p+r\}$, then $x^{[p]}$ belongs either to $\mathrm{Cone}\,([\,A^{p-1}b_1 \quad A^{p-1+s}b_2\,])$ or to $\mathrm{Cone}\,([\,A^{p-1}b_2 \quad A^{p-1+s}b_2\,])$, depending on the specific values of its two nonzero entries. So, the claims hold for $k = 0$.

Suppose, now, that the two claims are verified for every $x^{[p]} > 0$ with $\overline{\mathrm{ZP}}(x^{[p]}) \subseteq \{p, p+s, p+2s, \ldots, p+(k-1)s; p+r\}$. We want to prove that the results extend to all vectors $x^{[p]} > 0$ with $\overline{\mathrm{ZP}}(x^{[p]}) \subseteq \{p, p+s, p+2s, \ldots, p+ks; p+r\}$.

To this end, suppose, first, that $\overline{\mathrm{ZP}}(x^{[p]}) \subsetneq \{p, p+s, p+2s, \ldots, p+ks; p+r\}$. If $p+ks \notin \overline{\mathrm{ZP}}(x^{[p]})$, then $\overline{\mathrm{ZP}}(x^{[p]}) \subseteq \{p, p+s, p+2s, \ldots, p+(k-1)s; p+r\}$, and hence by the inductive assumption, $x^{[p]}$ belongs either to $\mathrm{Cone}([\,A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \ldots \quad A^{p-1+(k-1)s}b_{j_{k-1}}\,])$ or to $\mathrm{Cone}([\,A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \ldots \quad A^{p-1+ks}b_{j_k}\,])$. So, in both cases, we may claim that $x^{[p]} \in \mathrm{Cone}\,([\,A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \ldots \quad A^{p-1+ks}b_{j_k}\,])$ for suitable indices. On the other hand, if $p+ks \in \overline{\mathrm{ZP}}(x^{[p]})$, condition $A^{p-1+ks}b_1 \sim \mathbf{e}_{p+ks}$ implies that there exists $\alpha > 0$ such that

$$z^{[p]} = x^{[p]} - \alpha A^{p-1+ks}b_1$$

satisfies $\overline{\mathrm{ZP}}(z^{[p]}) \subsetneq \{p, p+s, \ldots, p+(k-1)s; p+r\}$, and hence, again by the inductive assumption, $x^{[p]} \in \mathrm{Cone}([\,A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \ldots \quad A^{p-1+(k-1)s}b_{j_{k-1}} \quad A^{p-1+ks}b_1\,])$.

Assume, finally, $\overline{\mathrm{ZP}}(x^{[p]}) = \{p, p+s, p+2s, \ldots, p+ks; p+r\}$. As $\overline{\mathrm{ZP}}(A^{p-1+(k+1)s}b_2) = \{p, p+s, p+2s, \ldots, p+ks; p+r\}$, there exists $\alpha > 0$ such that

$$z^{[p]} = x^{[p]} - \alpha A^{p-1+(k+1)s}b_2$$

satisfies $\overline{\mathrm{ZP}}(z^{[p]}) \subsetneq \{p, p+s, p+2s, \ldots, p+ks; p+r\}$. So, by the previous part of the proof, we can claim that $z^{[p]}$ belongs to the cone generated by some matrix

$$[\,A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \ldots \quad A^{p-1+ks}b_{j_k}\,],$$

and this implies that $x^{[p]} \in \mathrm{Cone}\,([\,A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \ldots \quad A^{p-1+ks}b_{j_k} \quad A^{p-1+(k+1)s}b_2\,])$, thus completing the proof by induction of our Claims.

To conclude, now that we have shown that every $x^{[p]} > 0, p \in [1, s]$, with $\overline{\mathrm{ZP}}(x^{[p]}) \subseteq \{p, p+s, p+2s, \ldots, p+ks; p+r\}$, k a suitable nonnegative number, belongs to some

$$\mathrm{Cone}\,([\,A^{p-1}b_{j_0} \quad A^{p-1+s}b_{j_1} \quad \ldots \quad A^{p-1+(k+1)s}b_{j_{k+1}}\,]),$$

(where $p-1+(k+1)s \leq n-1$), we notice that every $x \in \mathbb{R}^n_+$ can be expressed as $x = \sum_{p=1}^{s} x^{[p]}$, and each $x^{[p]}$ is obtained by combining columns where different powers of A appear. So, indices $i_0, i_1, \ldots, i_{n-1} \in [1, 2]$ can be found so that $x \in \mathrm{Cone}\,([\,A^{n-1}b_{i_0} \quad \ldots \quad A^2 b_{i_{n-3}} \quad Ab_{i_{n-2}} \quad b_{i_{n-1}}\,])$.

Clearly, in all previous cases, $\mathcal{I}_R = n$. ☐

Proposition 6.4. *Consider a DPSS (2.2), with $A \in \mathbb{R}^{n\times n}_+$, $p = 3$, and b_1, b_2 and b_3 linearly independent canonical vectors in $\mathbb{R}^n_+$. A sufficient condition for the system (2.2) to be* *reachable is that there exist* $r_1, r_2, r_3 > 0$*, with* $\sum_{i=1}^3 r_i = n$*, and a permutation matrix* Π*, s.t.*

$$(6.3) \qquad \Pi^\top A \Pi = \left[\begin{array}{cc|cc|cc} S_{r_1} & v_1^{(1)} & 0_{r_1\times(r_2-1)} & v_1^{(2)} & 0_{r_1\times(r_3-1)} & v_1^{(3)} \\ \hline 0_{r_2\times(r_1-1)} & v_2^{(1)} & S_{r_2} & v_2^{(2)} & 0_{r_2\times(r_3-1)} & v_2^{(3)} \\ \hline 0_{r_3\times(r_1-1)} & v_3^{(1)} & 0_{r_3\times(r_2-1)} & v_3^{(2)} & S_{r_3} & v_3^{(3)} \end{array}\right],$$

$$(6.4) \quad \Pi^\top [\, b_1 \quad b_2 \quad b_3 \,] = [\, \mathbf{e}_1 \quad \mathbf{e}_{r_1+1} \quad \mathbf{e}_{r_1+r_2+1} \,],$$

where the vectors $v^{(i)} := \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \\ v_3^{(i)} \end{bmatrix}$, $i \in [1,3]$*, satisfy anyone of the following conditions:*

α) $v^{(1)} \not\sim \mathbf{e}_j, \forall\, j \in \{r_1+1, r_1+r_2+1\}$, $v^{(2)} \sim \mathbf{e}_{r_1+1}$ *and* $v^{(3)} \sim \mathbf{e}_{r_1+r_2+1}$;

β) $v^{(1)} \not\sim \mathbf{e}_j, \forall\, j \in \{r_1+1, r_1+r_2+1\}$, $v^{(2)} \sim \mathbf{e}_1 + \mathbf{e}_{r_1+1}$ *and* $v^{(3)} \sim \mathbf{e}_{r_1+r_2+1}$;

γ) $\{v^{(1)}, v^{(2)}, v^{(3)}\} \sim \{\mathbf{e}_1 + \mathbf{e}_{r_1+1},\ \mathbf{e}_1 + \mathbf{e}_{r_1+r_2+1},\ \mathbf{e}_{r_1+1} + \mathbf{e}_{r_1+r_2+1}\}$.

If α) or β) apply, then $\mathcal{I}_R = n$*. If γ) applies, then* $\mathcal{I}_R \le \min\{\nu \in \mathbb{Z}_+ : A^\nu + A^{\nu+1} + \ldots + A^{\nu+D-1} \gg 0\} + D$*, where* $D := \text{g.c.d.}\{r_1, r_2, r_3\}$*.*

Proof. Suppose, for the sake of simplicity, that in (6.3) and (6.4) $\Pi = I_n$. In case α)

$$A = \left[\begin{array}{cc|c|c} S_{r_1} & v_1^{(1)} & 0_{r_1\times r_2} & 0_{r_1\times r_3} \\ \hline 0_{r_2\times(r_1-1)} & v_2^{(1)} & C_{r_2} & 0_{r_2\times r_3} \\ \hline 0_{r_3\times(r_1-1)} & v_3^{(1)} & 0_{r_3\times r_2} & C_{r_3} \end{array}\right].$$

So, by Theorem 4.3, the system is reachable (along a single switching sequence) and $\mathcal{I}_R = n$.

If we are in case β), then

$$A = \left[\begin{array}{cc|cc|c} S_{r_1} & v_1^{(1)} & 0_{r_1\times(r_2-1)} & v_1^{(2)} & 0_{r_1\times r_3} \\ \hline 0_{r_2\times(r_1-1)} & v_2^{(1)} & S_{r_2} & v_2^{(2)} & 0_{r_2\times r_3} \\ \hline 0_{r_3\times(r_1-1)} & v_3^{(1)} & 0_{r_3\times(r_2-1)} & 0_{r_3\times 1} & C_{r_3} \end{array}\right], \qquad \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \end{bmatrix} \sim \mathbf{e}_1 + \mathbf{e}_{r_1+1}.$$

If x is a positive vector with $\overline{\text{ZP}}(x) \subseteq [r_1+r_2+1, n]$, then $x \in \text{Cone}\,[\, A^{r_1+r_2} b_3 \quad A^{r_1+r_2+1} b_3 \quad \ldots \quad A^{n-1} b_3 \,]$. So, reachability is proved if we show that every vector x, with $\overline{\text{ZP}}(x) \subseteq [1, r_1+r_2]$, is reachable (in at most r_1+r_2 steps) by switching between subsystems (A, b_1) and (A, b_2). But this is proved along the same lines we followed within the proof of Proposition 6.3, case β'), since for reaching all such vectors we use only $\{A^k b_1, k \in [0, r_1-1]\}$ and $\{A^k b_2, k \in [0, r_1+r_2-1]\}$. This way, all vectors $x \in \mathbb{R}^n_+$ can be reached in at most n steps, and $\mathcal{I}_R = n$.

To prove reachability in case γ), we need a technical lemma.

Lemma 6.5. *If the assumptions of Proposition 6.4 hold, with* $\Pi = I_n$ *and* $\{v^{(1)}, v^{(2)}, v^{(3)}\} \sim \{\mathbf{e}_1 + \mathbf{e}_{r_1+1},\ \mathbf{e}_1 + \mathbf{e}_{r_1+r_2+1},\ \mathbf{e}_{r_1+1} + \mathbf{e}_{r_1+r_2+1}\} = \{b_1 + b_2, b_1 + b_3, b_2 + b_3\}$*, then*

i) *A is an irreducible matrix with imprimitivity index* $D := \text{g.c.d.}\{r_1, r_2, r_3\}$*, and we can denote its imprimitivity classes as*

$$\mathcal{C}_p := \{p, p+D, p+2D, \dots, p+LD\}, \qquad p \in [1, D], \qquad L := (n/D) - 1.$$

If p is any number in $[1, D]$*, then*

ii) $\forall\ k \in \mathbb{Z}_+$ *and* $j \in [1, 3]$, $\overline{\text{ZP}}(A^{p-1+kD} b_j) \subseteq \mathcal{C}_p$*;*

iii) $\forall\ k \in \mathbb{Z}_+$ *and* $1 \leq j_1 < j_2 \leq 3$*, there exists* $h > k$ *and* $j_3 \in [1, 3]$*, such that*

$$A^{p-1+hD} b_{j_3} \sim A^{p-1+kD} b_{j_1} + A^{p-1+kD} b_{j_2}.$$

If $x^{[p]}$ *is any vector in* $\mathbb{R}^n_+$*, with* $\overline{\text{ZP}}(x^{[p]}) \subsetneq \mathcal{C}_p$*, and we define*[4]

$$\bar{k} := \max\{k \in \mathbb{Z}_+ : \exists\ j \in [1, 3],\ s.t.\ \overline{\text{ZP}}(A^{p-1+kD} b_j) \subseteq \overline{\text{ZP}}(x^{[p]})\},$$

then

iv) $A^{p-1+\bar{k}D} b_{j_1} \sim A^{p-1+\bar{k}D} b_{j_2}$ *implies* $j_1 = j_2$*, and* $\bar{k} \geq |\overline{\text{ZP}}(x^{[p]})| - 1$.

Proof of Lemma. i) The irreducibility of A follows from the fact that every pair of vertices in $\mathcal{D}(A)$ belongs to a circuit. In addition, in all the situations compatible with the assumption on the $v^{(i)}$'s, the g.c.d. of the lengths of all circuits coincides with g.c.d.$\{r_1, r_2, r_3\}$.
ii) For every $p \in [1, D]$, clearly $|\overline{\text{ZP}}(A^{p-1} b_j)| = 1$ and $\overline{\text{ZP}}(A^{p-1} b_j) \in \mathcal{C}_p$. On the other hand, as A is irreducible with imprimitivity index D, $\overline{\text{ZP}}(A^{kD} A^{p-1} b_j) \subseteq \mathcal{C}_p$ for every $k \in \mathbb{Z}_+$.
iii) By assumption, if $1 \leq j_1 < j_2 \leq 3$, there exists $i \in [1, 3]$ such that $v^{(i)} \sim b_{j_1} + b_{j_2}$. Since $v^{(i)} \sim A^{r_i} b_i$ and $r_i = m_i D$ for some positive integer m_i, it follows that

$$A^{p-1+kD+m_iD} b_i \sim A^{p-1+kD}[A^{r_i} b_i] \sim A^{p-1+kD}[b_{j_1} + b_{j_2}] \sim A^{p-1+kD} b_{j_1} + A^{p-1+kD} b_{j_2}.$$

So, the result holds for $h = m_i + k$ and $j_3 = i$.
iv) Suppose that $A^{p-1+\bar{k}D} b_{j_1} \sim A^{p-1+\bar{k}D} b_{j_2}$ and $j_1 \neq j_2$. By part iii), $\exists\ h > \bar{k}, \exists\ j_3 \in [1, 3]$, such that $A^{p-1+hD} b_{j_3} \sim A^{p-1+\bar{k}D} b_{j_1} \sim A^{p-1+\bar{k}D} b_{j_2}$, and therefore $\overline{\text{ZP}}(x^{[p]}) \supseteq \overline{\text{ZP}}(A^{p-1+hD} b_{j_3})$. But this contradicts the definition of $\bar{k}$.

To prove that $\bar{k} \geq |\overline{\text{ZP}}(x^{[p]})| - 1$, we proceed by induction on $s := |\overline{\text{ZP}}(x^{[p]})|$. If $s = 1$, then $x^{[p]}$ is a monomial vector and clearly $\bar{k} \geq s - 1 = 0$. Suppose, now, that the result holds for every vector $\tilde{x}^{[p]}$, with $\overline{\text{ZP}}(\tilde{x}^{[p]}) \subseteq \mathcal{C}_p$ and $|\overline{\text{ZP}}(\tilde{x}^{[p]})| \leq s < \frac{n}{D} - 1$. We want to prove that the result extends to all vectors $x^{[p]}$, with $\overline{\text{ZP}}(x^{[p]}) \subseteq \mathcal{C}_p$ and $|\overline{\text{ZP}}(x^{[p]})| = s + 1 < \frac{n}{D}$. Set

[4]The index $\bar{k}$ is well defined, since, when A is irreducible, $\overline{\text{ZP}}(A^{p-1+kD} b_j) = \mathcal{C}_p$ for sufficiently large k.

$\mathcal{S} := \overline{\mathrm{ZP}}(x^{[p]})$ and denote by $\mathcal{S}_i, i \in [1, s+1]$, the distinct subsets of $\mathcal{S}$ obtained by removing a single index. Set

$$k_i := \max\{k \in \mathbb{Z}_+ : \exists\ j \in [1,3],\ \text{s.t. } \overline{\mathrm{ZP}}(A^{p-1+kD}b_j) \subseteq \mathcal{S}_i\}, \qquad i \in [1, s+1].$$

By the inductive assumption, $k_i \geq s-1$. If there exists $\hat{\imath} \in [1, s+1]$ such that $k_{\hat{\imath}} \geq s$, then we are set. If $k_i = s-1$ for all $i \in [1, s+1]$, condition $\mathcal{S}_i \subsetneq \mathcal{S}$ implies $\overline{\mathrm{ZP}}(A^{p-1+(s-1)D}b_j) \subsetneq \mathcal{S}$ for at least one index $j \in [1,3]$. On the other hand, there cannot be a single index $j \in [1,3]$ such that $\overline{\mathrm{ZP}}(A^{p-1+(s-1)D}b_j) \subsetneq \mathcal{S}$, otherwise it would be $\overline{\mathrm{ZP}}(A^{p-1+(s-1)D}b_j) \subseteq \cap_{i=1}^{s+1}\mathcal{S}_i = \emptyset$. So, there exist $1 \leq j_1 < j_2 \leq 3$ such that $\overline{\mathrm{ZP}}(A^{p-1+(s-1)D}b_{j_1}) \subsetneq \mathcal{S}$ and $\overline{\mathrm{ZP}}(A^{p-1+(s-1)D}b_{j_2}) \subsetneq \mathcal{S}$. By part iii) of this lemma, there exist $h > s-1$ and $j_3 \in [1,3]$ such that $\overline{\mathrm{ZP}}(A^{p-1+hD}b_{j_3}) \subseteq \mathcal{S}$. This ensures that $\bar{k} \geq h \geq s$. *End of the Proof of Lemma.*

By resorting to the previous lemma, we first show that every $x^{[p]} > 0$, with $\overline{\mathrm{ZP}}(x^{[p]}) \subseteq \mathcal{C}_p$, is reachable, and hence it can be expressed as the nonnegative combination of vectors $A^{p-1+hD}b_j$, for $h \in \mathbb{Z}_+$ and $j \in [1,3]$. If $\overline{\mathrm{ZP}}(x^{[p]}) \subsetneq \mathcal{C}_p$, set $x_1^{[p]} := x^{[p]}$ and

$$k_1 := \max\{k \in \mathbb{Z}_+ : \exists\ j \in [1,3],\ \text{s.t. } \overline{\mathrm{ZP}}(A^{p-1+kD}b_j) \subseteq \overline{\mathrm{ZP}}(x_1^{[p]})\}.$$

By Lemma 6.5, $k_1 \geq |\overline{\mathrm{ZP}}(x_1^{[p]})| - 1$ and there exists a unique $j_1 \in [1,3]$ such that $\overline{\mathrm{ZP}}(A^{p-1+k_1D}$ $b_{j_1}) \subseteq \overline{\mathrm{ZP}}(x_1^{[p]})$. Choose $\alpha_1 > 0$ so that $x_2^{[p]} := x_1^{[p]} - \alpha_1 A^{p-1+k_1D}b_{j_1}$ (is nonnegative and) satisfies $\overline{\mathrm{ZP}}(x_2^{[p]}) \subsetneq \overline{\mathrm{ZP}}(x_1^{[p]}) \subsetneq \mathcal{C}_p$. Define

$$k_2 := \max\{k \in \mathbb{Z}_+ : \exists\ j \in [1,3],\ \text{s.t. } \overline{\mathrm{ZP}}(A^{p-1+kD}b_j) \subseteq \overline{\mathrm{ZP}}(x_2^{[p]})\}.$$

Clearly, $k_2 < k_1$ (otherwise either the definition of k_1 or the uniqueness of j_1 would be contradicted) and $k_2 \geq |\overline{\mathrm{ZP}}(x_2^{[p]})| - 1$. We pick up $\alpha_2 > 0$ such that $x_3^{[p]} := x_2^{[p]} - \alpha_2 A^{p-1+k_2D}b_{j_2}$, j_2 a suitable index in $[1,3]$, satisfies $\overline{\mathrm{ZP}}(x_3^{[p]}) \subsetneq \overline{\mathrm{ZP}}(x_2^{[p]}) \subsetneq \mathcal{C}_p$. So, by further proceeding, after at most $|\overline{\mathrm{ZP}}(x^{[p]})|$ steps, we obtain for $x^{[p]}$ the following expression:

$$x^{[p]} = \sum_{\ell} \alpha_\ell A^{p-1+k_\ell D} b_{j_\ell}, \qquad \alpha_\ell > 0,\ k_1 > k_2 > \ldots .$$

On the other hand, if $\overline{\mathrm{ZP}}(x^{[p]}) = \mathcal{C}_p$, by the irreducibility of A, there exists $k_0 \in \mathbb{N}$ such that $\overline{\mathrm{ZP}}(A^{p-1+hD}b_j) = \overline{\mathrm{ZP}}(A^{hD} \cdot A^{p-1}b_j) = \mathcal{C}_p$, for every $h \geq k_0$ and every $j \in [1,3]$. Let α_0 be a positive integer such that $x_1^{[p]} := x^{[p]} - \alpha_0 A^{p-1+k_0D}b_1$ (is nonnegative and) has nonzero pattern properly included in $\mathcal{C}_p$. Then we may apply to $x_1^{[p]}$ the same algorithm we have previously described, and finally obtain $x^{[p]}$ as a finite positive combination of the following kind:

$$x^{[p]} = \sum_{\ell} \alpha_\ell A^{p-1+k_\ell D} b_{j_\ell}, \qquad \alpha_\ell > 0,\ k_0 > k_1 > k_2 > \ldots .$$

To conclude, it suffices to notice that every positive vector x can be expressed as $x = \sum_{p=1}^{D} x^{[p]}$ and hence as the nonnegative combination of the columns of a suitable reachability matrix.

As far as the reachability index is concerned, it is clear from the previous algorithm that $\mathcal{I}_R \leq \max_{p\in[1,D]}(p-1+k_0(p)D)+1$, where $k_0(p) := \min\{k \in \mathbb{Z}_+ : \overline{\mathrm{ZP}}(A^{p-1+kD}b_j) = \mathcal{C}_p, \forall\ j \in$

$[1,3]\}$. Set $\nu := \min\{h \geq 0 : \forall i \in [1,n],\ \overline{\text{ZP}}(A^h \mathbf{e}_i) = \mathcal{C}_p$, for some $p \in [1,D]\}$. Clearly, $\forall\ i \in [1,3], \forall\ h \in [0, D-1]$, we have $\overline{\text{ZP}}(A^\nu A^h b_i) = \mathcal{C}_p$ for some $p \in [1,D]$. This implies that $\max_{p\in[1,D]}(p-1+k_0(p)D) \leq \nu + D - 1$, and hence $\mathcal{I}_R \leq \nu + D$. Since the index ν coincides with $\min\{h \in \mathbb{Z}_+ : A^h + A^{h+1} + \ldots + A^{h+D-1} \gg 0\}$, the final upper bound on $\mathcal{I}_R$ is proved. □

Remark 6.6. *The algorithmic proof of reachability in case γ) provides a "reduction procedure" which is uniquely determined and always brings to a positive result, meaning that a reachability matrix such that x belongs to the cone generated by this matrix can be found. This method, however, does not generally bring to the most convenient solution, as illustrated in the following example.*

Example 2. Consider the DPSS (2.2), with

$$A = \left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \qquad [\, b_1 \quad b_2 \quad b_3\,] = [\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_4\,].$$

In this case, $r_1 = 1, r_2 = 2$ and $r_3 = 3$, so that $D = \text{g.c.d.}\{r_1, r_2, r_3\} = 1$ and A is primitive. The smallest index $\nu \in \mathbb{Z}_+$ such that $A^\nu \gg 0$ is $\nu = 8$, so that $\mathcal{I}_R \leq 9$. The algorithm illustrated within the previous proof allows to express the vector $x = [2\ 4\ 0\ 1\ 3\ 3]^\top$ in the form

$$x = b_2 + Ab_3 + 2A^2 b_3 + A^4 b_3 + A^6 b_2,$$

thus ensuring the reachability of x in 7 steps. However, x could be reached also in 5 steps, as

$$x = 3b_2 + 2Ab_3 + 3A^2 b_3 + A^3 b_3 + A^4 b_3. \qquad \diamondsuit$$

Theorem 6.7. *A DPSS (2.2), with $A \in \mathbb{R}_+^{n\times n}$, $b_i \in \mathbb{R}_+^n, i \in [1,p]$, linearly independent canonical vectors, is reachable if and only if there exist a permutation matrix Π, an integer $m \in [1,p]$, a selection matrix $S \in \mathbb{R}_+^{p\times m}$ and positive integers $r_1, r_2, \ldots, r_m$, with $\sum_{i=1}^m r_i = n$, such that the matrices $\Pi^\top A \Pi$ and $\Pi^\top BS$ have the structures given in (6.1) and satisfy conditions $\alpha) \div \gamma)$ of Proposition 6.1.*

Proof. Necessity has been proved in Proposition 6.1. As far as the sufficiency is concerned, it is easily seen that each positive vector x, with $\overline{\text{ZP}}(x) \subseteq [r_1 + r_2 + r_3 + 1, n]$, belongs to the cone generated by some matrix $[\, A^K b_{j_0} \quad A^{K+1} b_{j_1} \quad \ldots \quad A^{K+M-1} b_{j_{M-1}}\,]$, where $K \in \mathbb{Z}_+$ is arbitrarily large, $M := (\sum_{i=4}^p r_i)$, and the indices $j_h, h \in [0, M-1]$, belong to $[4,p]$. So, to prove reachability, it is sufficient to show that each vector x, with $\overline{\text{ZP}}(x) \subseteq [1, r_1 + r_2 + r_3]$, can be reached by making use of the subsystems with input-to-state matrices $b_i, i \in [1,3]$. But the proof of this part follows just the same lines as that of Proposition 6.4. □

7. Reachability analysis: the case when there exist nonmonomial b_i's

Suppose, now, that the system switches among p subsystems, and at least one b_i is not monomial. Notice, however, that the reachability assumption forces one b_i to be monomial.

THEOREM 7.1. *Consider a DPSS (2.2), with* $A \in \mathbb{R}_+^{n\times n}, b_i \in \mathbb{R}_+^n, i \in [1,p]$*. Let* m *be a positive integer in* $[1, p-1]$ *and suppose that the* b_i*'s,* $i \in [1,m]$*, are linearly independent canonical vectors, while the* b_i*'s,* $i \in [m+1,p]$*, are nonzero nonmonomial vectors. Assume that the system is not reachable when switching only among subsystems with canonical* b_i*'s. Then system (2.2) is reachable if and only if there exist a permutation matrix* Π*, integers* $\bar m$ *and* $\bar p$*, with* $1 \le \bar m \le m$ *and* $1 \le \bar p - \bar m \le p - m$*, a selection matrix* $S \in \mathbb{R}_+^{p\times \bar p}$ *and positive integers* $r_1, r_2, \dots, r_{\bar p}, \sum_{i=1}^{\bar p} r_i = n$*, such that*

$$\Pi^\top A \Pi = \left[\begin{array}{cc|cc|cccc} S_{r_1} & 0_{r_1\times 1} & 0_{r_1\times(r_2-1)} & a_{1,r_1+r_2}\mathbf{e}_1 & & & & \\ \hline \multicolumn{2}{c|}{0_{r_2\times r_1}} & \multicolumn{2}{c|}{C_{r_2}} & & & & \\ \hline & & & & C_{r_3} & & & \\ & & & & & \ddots & & \\ & & & & & & C_{r_{\bar p}} & \end{array}\right], \tag{7.1}$$

$$\Pi^\top B S \sim \begin{bmatrix} \mathbf{e}_1 & & & & \\ & \mathbf{e}_1 & & & \\ & & \ddots & & \\ & & & \mathbf{e}_1 & \\ & & & & \mathbf{e}_1 \end{bmatrix} + \begin{bmatrix} I_{r_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & b_1^{(\bar m+1)} & \dots & b_1^{(\bar p)} \end{bmatrix}. \tag{7.2}$$

where $a_{1,r_1+r_2} = 0$ *if* $\bar m = 1$*,* $b_1^{(j)} \in \mathbb{R}_+^{r_1}$ *is a positive vector, and each vector* $\mathbf{e}_1$ *in* $\Pi^\top BS$ *has the same dimension as the corresponding block in* $\Pi^\top A\Pi$*. Under these assumptions* $\mathcal{I}_R = n$*.*

Proof. [Sufficiency] Consider, first, the case $\bar m = 1$. If so,

$$\Pi^\top A \Pi = \begin{bmatrix} S_{r_1} \; 0_{r_1\times 1} & & & & \\ & C_{r_2} & & & \\ & & C_{r_3} & & \\ & & & \ddots & \\ & & & & C_{r_{\bar p}} \end{bmatrix}, \qquad \Pi^\top B S \sim \begin{bmatrix} \mathbf{e}_1 & b_1^{(2)} & b_1^{(3)} & \dots & b_1^{(\bar p)} \\ & \mathbf{e}_1 & & & \\ & & \mathbf{e}_1 & & \\ & & & \ddots & \\ & & & & \mathbf{e}_1 \end{bmatrix},$$

and, by Theorem 4.3, we can claim that there exists a single switching sequence (of length n) along which all monomial vectors can be reached. So, the system is reachable with $\mathcal{I}_R = n$.

Consider, now, the case $\bar m \ge 2$. If $a_{1,r_1+r_2} = 0$, then we fall in the same case we have just examined. If $a_{1,r_1+r_2} > 0$, then, by Proposition 6.3, all positive vectors x with $\overline{\mathrm{ZP}}(x) \subseteq [1, r_1+r_2]$, can be reached (in at most r_1+r_2 steps) by commuting between the two subsystems (A, b_1) and (A, b_2). On the other hand, each positive vector x, with $\overline{\mathrm{ZP}}(x) \subseteq [r_1 + r_2 + 1, n]$, belongs to the cone generated by the matrix

$$[\,A^{r_1+r_2}b_3 \;\dots\; A^{r_1+r_2+r_3-1}b_3 \;|\; A^{r_1+r_2+r_3}b_4 \;\dots\; A^{r_1+r_2+r_3+r_4-1}b_4 \;|\; \dots \;|\; A^{n-r_p}b_p \dots A^{n-1}b_p\,].$$

So, all positive vectors are reachable in at most n steps (and $\mathcal{I}_R = n$).

[Necessity] Introduce the indices $r_i, i \in [1,m]$, as in Lemma 5.2. Since the DPSS (2.2) is not reachable if we make use only of the canonical b_i's, one the vectors $A^{r_i}b_i$ is zero. So, after a suitable reordering of the columns $b_i, i \in [1,m]$, we can assume w.l.o.g. that

- all the r_i's are positive (if not, we simply eliminate the corresponding b_i);
- $A^{r_1} b_1 = 0$;
- $A^h b_1 \sim \mathbf{e}_{h+1}$, for $h \in [0, r_1 - 1]$, and $A^h b_i \sim \mathbf{e}_{(\sum_{\ell=1}^{i-1} r_\ell)+h+1}$, for $i \in [2,m], h \in [0, r_i - 1]$.

In addition, by Lemma 5.2, $r := \sum_{i=1}^m r_i < n$ and the $m-1$ vectors $A^{r_2}b_2, A^{r_3}b_3, \dots, A^{r_m}b_m$, which correspond to the columns of A of indices $r_1 + r_2$, $r_1 + r_2 + r_3$, ..., r, respectively, satisfy a) or b) of Lemma 5.1, and hence, in particular, have nonzero patterns included in $[1, r]$. By the reachability assumption and Lemma 3.2, the last $n - r$ columns of A must be distinct ℓth monomial vectors $\ell \in [r+1, n]$. Consequently, after a suitable permutation, we can assume

$$A \sim \left[\begin{array}{c|c} \begin{array}{cc|c|cc} S_{r_1} & 0_{r_1\times 1} & \dots & 0 & v_1^{(m)} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0_{r_m \times 1} & \dots & S_{r_m} & v_m^{(m)} \end{array} & \\ \hline & \tilde{\Pi} \end{array}\right], \quad B \sim \left[\begin{array}{ccc|c} \mathbf{e}_1 & & & \\ & \ddots & & B_{12} \\ & & \mathbf{e}_1 & \\ \hline & & & B_{22} \end{array}\right],$$

$\tilde{\Pi}$ a permutation matrix. If we apply to the vectors $A^{r_i}b_i, i \in [1,m]$, the same algorithm as in the proof of Proposition 6.1, we can ensure that, for a suitable $\bar{m} \in [1,m]$, a selection of the monomial b_i's, a change of the r_i's, and a suitable relabeling, we get

$$\Pi_1^\top A \Pi_1 \sim \left[\begin{array}{c|c} \begin{array}{c|ccc} \begin{array}{cc|cc} S_{r_1} & 0_{r_1\times 1} & 0_{r_1\times(r_2-1)} & a_{1,r_1+r_2}\mathbf{e}_1 \\ \hline 0_{r_2\times r_1} & & C_{r_2} & \end{array} & & & \\ \hline & C_{r_3} & & \\ & & \ddots & \\ & & & C_{r_{\bar{m}}} \end{array} & \\ \hline & \tilde{\Pi} \end{array}\right]$$

$$\Pi_1^\top B S_1 \sim \left[\begin{array}{ccc|c} \mathbf{e}_1 & & & \\ & \ddots & & \tilde{B}_{12} \\ & & \mathbf{e}_1 & \\ \hline & & & B_{22} \end{array}\right].$$

Finally, since $\tilde{\Pi}$ is a permutation matrix, it entails no loss of generality assuming that

$$\tilde{\Pi} = \begin{bmatrix} C_{q_1} & & \\ & \ddots & \\ & & C_{q_d} \end{bmatrix},$$

for a suitable d, with C_{q_j} a cyclic block of size q_j. In order to ensure that, for every $\ell \in [r+1, n]$, indices $k > 0$ and $i \in [m+1, p]$ can be found such that $\mathbf{e}_\ell \sim \Pi_1^\top A^k b_i$, it is necessary that

$|\overline{\mathrm{ZP}}(\Pi_1^\top b_i)\cap[r+1,n]|=1$ and $\overline{\mathrm{ZP}}(\Pi_1^\top b_i)\cap[1,r]\subseteq[1,r_1]$. If we preserve one single column $\Pi_1^\top b_i$ for each C_{q_j}, a selection matrix S_2 and a permutation matrix Π_2 can be found so that

$$\left[\begin{array}{c|c} I_r & \\ \hline & \Pi_2 \end{array}\right]\left[\begin{array}{c}\tilde{B}_{12}\\ \hline B_{22}\end{array}\right] S_2 = \left[\begin{array}{cccc} b_1^{(m+1)} & b_1^{(m+2)} & \dots & b_1^{(m+d)} \\ 0_{(r-r_1)\times 1} & 0_{(r-r_1)\times 1} & \dots & 0_{(r-r_1)\times 1} \\ \hline \mathbf{e}_1 & \mathbf{e}_{q_1+1} & \dots & \mathbf{e}_{q_1+\ldots+q_{d-1}+1}\end{array}\right], \quad b_1^{(j)}\in\mathbb{R}_+^{r_1}, b_1^{(j)}>0, \forall\, j.$$

□

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