Stabilizability of discrete-time positive switched systems

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Abstract— **In this paper we consider the class of discrete-time switched systems switching between two autonomous positive subsystems. It is shown that if these systems are stabilizable, they can be stabilized by means of a periodic switching sequence, which asymptotically drives to zero every positive initial state. Necessary and sufficient conditions for the existence of state-dependent stabilizing switching laws, based on the values of a copositive (linear/quadratic) Lyapunov function, are investigated.**

I. INTRODUCTION

A discrete-time positive switched system (DPSS) consists of a family of positive state-space models [5], [12] and a switching law, specifying when and how the switching among the various subsystems takes place. This class of systems has some interesting practical applications. They have been adopted for describing networks employing TCP and other congestion control applications [20], for modeling consensus and synchronization problems [11], and, quite recently, to describe the viral mutation dynamics under drug treatment [9].

In the context of positive switched systems, most of the research results about stability and stabilizabilty have been derived in the continuous-time case [2], [8], [13], [15], [16], [17], [23]. While results based on linear copositive functions find a straightforward extension to the discrete-time case, this is not true when dealing with quadratic stability and stabilizability, and at our knowledge the only contribution on this subject is [14].

In this paper we consider the the stabilizability property of DPSS switching between two (unstable) susbystems. First, it is shown that a stabilizable DPSS can always be stabilized by resorting to a periodic switching sequence that ensures the state evolution convergence independently of the (positive) initial condition. Then we focus on state-feedback switching strategies, which are based on (either linear or quadratic) copositive Lyapunov functions, and prove that they stabilize the system under the simple condition that at each time instant the Lyapunov function decreases for (at least) one of the two subsystems. Equivalent conditions for the existence of stabilizing switching strategies based on linear copositive functions are provided, and it is shown that when any of these conditions is satisfied then stabilizing strategies, based either on positive definite quadratic functions or, more generally, on quadratic copositive functions, can be found.

Before proceeding, we introduce some notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so we adopt the notation $A \geq 0$. If, in addition, A has at least one positive entry the matrix is *positive* ($A > 0$), while if all its entries are positive it is *strictly positive* ($A \gg 0$). An $n \times n$ positive matrix A is *irreducible* if $\sum_{i=0}^{n-1} A^i \gg 0$. If A is irreducible and the vector $\mathbf{x} > 0$ has $k < n$ positive components, then $A\mathbf{x}$ has at least $k+1$ positive entries. Moreover, for every positive vector $\mathbf{x}$ there exist constant numbers $0 < C_1 < C_2$ such that $C_1\rho(A)^t \leq \|A^t\mathbf{x}\| \leq C_2\rho(A)^t$, for every $t \in \mathbb{Z}_+$, where $\rho(A)$ is the spectral radius of A.

A square matrix A is said to be *Metzler* if its off-diagonal entries are nonnegative. A square matrix A is *Schur* if all its eigenvalues lie within the unit circle, and it is *Hurwitz* if all its eigenvalues have negative real part. It is easily seen [5] that $A \in \mathbb{R}^{n\times n}$ is a positive Schur matrix if and only if $A - I_n$ is Metzler Hurwitz.

$\mathbf{1}_n$ is the n-dimensional vector with all entries equal to 1.

A square symmetric matrix P is positive definite ($\succ 0$) if for every nonzero vector $\mathbf{x}$, of compatible dimension, $\mathbf{x}^\top P\mathbf{x} > 0$, and negative definite ($\prec 0$) if $-P$ is positive definite.

II. STABILIZABILITY

The class of *discrete-time positive switched systems* (DPSS) we consider in this paper is described by the following equation

$$\mathbf{x}(t+1) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{1}$$

where $\mathbf{x}(t)$ denotes the value of the n-dimensional state variable at time t, σ is an arbitrary switching sequence, taking values in the set $\{1,2\}$, and for each $i \in \{1,2\}$ the matrix A_i is the system matrix of a discrete-time positive system, which means that A_i is an $n \times n$ positive matrix. The initial condition $\mathbf{x}(0)$ is assumed to be nonnegative.

For this class of systems we introduce the concept of stabilizability, also known in the literature on (general) switched systems [21] as pointwise asymptotic stabilizablility.

Definition 1: The DPSS (1) is *stabilizable* if for every positive initial state $\mathbf{x}(0)$ there exists a switching sequence $\sigma : \mathbb{Z}_+ \to \{1,2\}$ such that the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

Clearly, the stabilization problem is a non-trivial one only if both matrices A_i's are not Schur. So, in the following, we will steadily make this assumption. As remarked in the previous definition, the choice of the switching sequence σ may depend on the initial state $\mathbf{x}(0)$. A stronger definition

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of stabilizability requires that a stabilizing sequence exists that does not depend on the initial state [21].

Definition 2: The DPSS (1) is *consistently stabilizable* if there exists a switching sequence $\sigma : \mathbb{Z}_+ \to \{1,2\}$ such that, for every positive initial state $\mathbf{x}(0)$, the corresponding state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

Remark 1: It is clear that consistent stabilizability implies stabilizability. The natural question arises whether the converse is true. In the general case, i.e. when there is no positivity assumption, discrete-time switched systems can be found (see example at pages 112-113 in [21]) that are stabilizable, but not consistently stabilizable.

In [21] (see Theorem 3.5.4) it is also shown that for discrete-time switched systems, without positivity constraints, consistent stabilizability is equivalent to the existence of a periodic switching sequence that asymptotically drives to zero the state evolution for every $\mathbf{x}(0) \in \mathbb{R}^n$. As we will see, when dealing with positive switched systems (1), consistent stabilizability and stabilizability are equivalent properties, and they are both equivalent to the possibility of stabilizing the system by means of a periodic switching sequence, independent of the positive initial state.

Proposition 1: Given a DPSS (1), the following facts are equivalent:

i) the system is stabilizable;
ii) the system is consistently stabilizable;
iii) there exist $N > 0$ and indices $i_0, i_1, \dots, i_{N-1} \in \{1,2\}$, such that the matrix product $A_{i_{N-1}} A_{i_{N-2}} \cdots A_{i_1} A_{i_0}$ is a positive Schur matrix;
iv) the system is consistently stabilizable by means of a periodic switching sequence.

Proof: i) $\Rightarrow$ ii) Let σ be a switching sequence asymptotically driving to zero the state evolution originated in $\hat{\mathbf{x}}(0) = \mathbf{1}_n$. Then this same sequence drives to zero every other positive state $\mathbf{x}(0)$. Indeed, let $\hat{\mathbf{x}}(t)$ and $\mathbf{x}(t), t \in \mathbb{Z}_+$, be the state evolutions, corresponding to the switching sequence σ, originated from $\hat{\mathbf{x}}(0)$ and $\mathbf{x}(0)$, respectively. A positive number M can be found such that $0 < \mathbf{x}(0) \leq M\mathbf{1}_n$, and the positivity assumption on the matrices A_i's implies that, at each time $t \in \mathbb{Z}_+$, $0 \leq \mathbf{x}(t) \leq M\hat{\mathbf{x}}(t)$, thus ensuring that $\mathbf{x}(t)$ goes to zero as $t \to +\infty$. So, the system is consistently stabilizable.

ii) $\Rightarrow$ iii) Let σ be the switching sequence that makes the state evolution go to zero, independently of the initial state. Set $\mathbf{x}(0) = \mathbf{1}_n$ and $\varepsilon \in (0,1)$. Then a positive integer N can be found such that

$$\mathbf{x}(N) = A_{\sigma(N-1)} \cdots A_{\sigma(1)} A_{\sigma(0)} \mathbf{1}_n < \varepsilon \mathbf{1}_n.$$

This ensures (see Theorem 1.1, Chapter II, in [18]) that the spectral radius of the positive matrix $A_{\sigma(N-1)} \cdots A_{\sigma(1)} A_{\sigma(0)}$ is smaller than $\varepsilon < 1$ and hence the matrix is Schur. So, iii) holds for $i_k = \sigma(k), k \in \{0,1,\dots,N-1\}$.

iii) $\Rightarrow$ iv) If $A := A_{i_{N-1}} A_{i_{N-2}} \cdots A_{i_1} A_{i_0}$ is a positive Schur matrix, then A^k converges to zero as k goes to infinity. Consequently, the switching sequence

$$\sigma(t) = i_{(t \bmod N)}$$

drives to zero the state evolution corresponding to every positive initial state.

iv) $\Rightarrow$ i) is obvious. ■

Remark 2: It is worthwhile remarking that the equivalence of ii) and iv) in Proposition 1 could be alternatively proved by showing that a DPSS is consistently stabilizable (by means of a periodic σ) if and only if it is so when considered in the general setting, i.e. by removing the positivity assumption on the initial condition. Indeed, for every nonzero state $\mathbf{x}(0) \in \mathbb{R}^n$, nonnegative states $\mathbf{x}_+(0), \mathbf{x}_-(0) \in \mathbb{R}^n_+$ can be found such that $\mathbf{x}(0) = \mathbf{x}_+(0) - \mathbf{x}_-(0)$. If σ is the switching sequence that drives to zero the state evolution corresponding to every positive initial state, σ ensures that $\mathbf{x}_+(t)$ and $\mathbf{x}_-(t)$, the state trajectories corresponding to $\mathbf{x}_+(0)$ and $\mathbf{x}_-(0)$, converge to zero. But then, $\mathbf{x}_+(t) - \mathbf{x}_-(t)$ converges to zero, too. So, σ can drive to zero the evolution corresponding to every initial state. The converse is obvious.

Remark 3: Assuming that the DPSS (1) is stabilizable, one may wonder whether corresponding to some positive initial state $\mathbf{x}(0)$ there exists a switching sequence σ that leads to zero the state evolution and eventually takes a constant value $\ell \in \{1,2\}$.

If both matrices A_i's are irreducible, this is not possible. Indeed, if σ takes the value ℓ for every $t \geq N$, then the state evolution can be described as

$$\mathbf{x}(t) = A_\ell^{t-N} \mathbf{x}(N), \qquad \forall\ t \geq N.$$

However, as A_ℓ is a positive (non-Schur) irreducible matrix, then $A_\ell^{t-N}\mathbf{x}(N)$ asymptotically converges to zero only if $\mathbf{x}(N) = 0$. On the other hand, the equation $0 = \mathbf{x}(N) = A_{i_{N-1}} A_{i_{N-2}} \cdots A_{i_1} A_{i_0} \mathbf{x}(0)$ does not admit positive solutions, as all matrices A_{i_k} are irreducible. Consequently, for every choice of $\mathbf{x}(0) > 0$, the switching sequences that lead the state to zero change value an infinite number of times.

Finally, if A_1 is irreducible and A_2 is not, then, by the previous reasoning, the switching sequence σ may eventually take the value 1 only if $\mathbf{x}(0)$ can be driven to zero in a finite number of steps, and the value 2 only if $\mathbf{x}(0)$ can be driven (again, in a finite number of steps) to the generalized eigenspace of A_2 corresponding to the eigenvalues with modulus smaller than 1.

III. Lyapunov functions for DPSSs

In the previous section we introduced the general stabilization problem for the class of DPSS (1). According to Definition 1, the stabilizing switching sequence σ is a function of time, and hence can be thought of as an open-loop control action that we apply to the system in order to ensure the converge to zero of its state evolution. An alternative solution can be that of searching for a stabilizing switching sequence whose value at time t depends on the specific value that a suitable Lyapunov function takes on the state $\mathbf{x}(t)$.

More specifically, we search for a copositive Lyapunov function $V(\mathbf{x})$ (by this meaning a function that takes positive values on the positive states, and is zero in the origin) such that

$$\min_{i\in\{1,2\}} \Delta V_i(\mathbf{x}) < 0, \qquad \forall\ \mathbf{x} > 0, \tag{2}$$

where

$$\Delta V_i(\mathbf{x}) := V(A_i\mathbf{x}) - V(\mathbf{x}). \tag{3}$$

In this section, we want to investigate what conditions on the positive matrix pairs ensure the existence of different kinds of copositive Lyapunov functions for the DPSS (1) that satisfy (2). In detail, we will focus on quadratic copositive functions $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$, P being a symmetric matrix, and on linear copositive functions $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$, $\mathbf{v}$ a vector that is necessarily strictly positive. Clearly, quadratic positive definite functions are a subset of quadratic copositive functions.

Proposition 2: Let A_1 and A_2 be $n \times n$ positive matrices. The following facts are equivalent:

a1) there exists a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ such that for every $\mathbf{x} \neq 0$,
$$\min_{i\in\{1,2\}} \Delta V_i(\mathbf{x}) = \min_{i\in\{1,2\}}\ \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0;$$

a2) there exists a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ and $\varepsilon > 0$ such that for every $\mathbf{x} \neq 0$,
$$\min_{i\in\{1,2\}} \Delta V_i(\mathbf{x}) = \min_{i\in\{1,2\}}\ \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < -\varepsilon\mathbf{x}^\top\mathbf{x};$$

a3) there exists a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ and $\alpha \in [0,1]$ such that
$$\begin{aligned}&\alpha\Delta V_1(\mathbf{x}) + (1-\alpha)\Delta V_2(\mathbf{x})\\ &= \alpha\mathbf{x}^\top(A_1^\top PA_1 - P)\mathbf{x} + (1-\alpha)\mathbf{x}^\top(A_2^\top PA_2 - P)\mathbf{x}\end{aligned}$$
is negative definite.

If any of the conditions a1) $\div$ a3) holds, then anyone of the following equivalent conditions holds:

b1) $\exists\ \alpha \in [0,1]$ such that $\alpha A_1 + (1-\alpha)A_2$ is Schur;

b2) there exists a linear copositive function $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$ and $\alpha \in [0,1]$ such that for all $\mathbf{x} > 0$
$$\Delta V_\alpha(\mathbf{x}) := \left[\mathbf{v}^\top(\alpha A_1 + (1-\alpha)A_2) - \mathbf{v}^\top\right]\mathbf{x} < 0;$$

b3) there exists a linear copositive function $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$ such that for every $\mathbf{x} > 0$
$$\min_{i\in\{1,2\}} \Delta V_i(\mathbf{x}) = \min_{i\in\{1,2\}}\ \mathbf{v}^\top(A_i - I_n)\mathbf{x} < 0;$$

b4) there exists a quadratic copositive function of rank 1 $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ (by this meaning that $\mathrm{rank}P = 1$) such that for every $\mathbf{x} > 0$,
$$\min_{i\in\{1,2\}} \Delta V_i(\mathbf{x}) = \min_{i\in\{1,2\}}\ \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0.$$

If any of the equivalent conditions b1) $\div$ b4) holds, then

c) there exists a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ such that for every $\mathbf{x} > 0$
$$\min_{i\in\{1,2\}} \Delta V_i(\mathbf{x}) = \min_{i\in\{1,2\}}\ \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0.$$

If c) holds, then

d) there exists a quadratic copositive function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ such that for every $\mathbf{x} > 0$
$$\min_{i\in\{1,2\}} \Delta V_i(\mathbf{x}) = \min_{i\in\{1,2\}}\ \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0.$$

Proof: a1) $\Rightarrow$ a2) Both $\Delta V_1(\mathbf{x})$ and $\Delta V_2(\mathbf{x})$ are continuous and so is $f(\mathbf{x}) := \min_{i\in\{1,2\}}\Delta V_i(\mathbf{x})$. By Weierstrass' theorem, being $f(\mathbf{x})$ a negative and continuous function on the compact set $\partial\mathcal{B}_1 := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$, it follows that
$$\max_{\mathbf{x}\in\partial\mathcal{B}_1} f(\mathbf{x}) < -\varepsilon < 0.$$
This implies that, for every $\mathbf{x} \neq 0$, $f(\mathbf{x}) < -\varepsilon\mathbf{x}^\top\mathbf{x}$.

a2) $\Rightarrow$ a3) Set $T_i := A_i^\top PA_i - P + \varepsilon I_n$ for $i \in \{1,2\}$. Clearly, condition a2) implies that for every $\mathbf{x} \neq 0$ such that $\mathbf{x}^\top T_1\mathbf{x} \geq 0$, one has $\mathbf{x}^\top T_2\mathbf{x} < 0$. So, once we prove that there exists $\bar{\mathbf{x}} \neq 0$ such that $\bar{\mathbf{x}}^\top T_1\bar{\mathbf{x}} > 0$, by making using of the S-procedure in the Appendix we can claim that there exists $\tau \geq 0$ such that $\tau T_1 + T_2 = \tau(A_1^\top PA_1 - P + \varepsilon I_n) + (A_2^\top PA_2 - P + \varepsilon I_n)$ is negative definite, and this immediately implies a3) for $\alpha = \tau/(1+\tau) \in [0,1[$. To prove that there is a nonzero vector $\bar{\mathbf{x}}$ such that $\bar{\mathbf{x}}^\top T_1\bar{\mathbf{x}} > 0$, we observe that A_1 is not Schur, and hence there exists $\bar{\mathbf{x}} \neq 0$ such that $\bar{\mathbf{x}}^\top(A_1^\top PA_1 - P)\bar{\mathbf{x}} \geq 0$. Consequently, $\bar{\mathbf{x}}^\top T_1\bar{\mathbf{x}} > 0$.

a3) $\Rightarrow$ a1) is obvious.

a2) $\Rightarrow$ b1) Condition a2) implies that for every $\mathbf{x} \neq 0$ either
$$\begin{aligned}&\mathbf{x}^\top[(A_1 - I_n)^\top P + P(A_1 - I_n)]\mathbf{x} <\\ &\quad < -\mathbf{x}^\top(A_1 - I_n)^\top P(A_1 - I_n)\mathbf{x} - \varepsilon\mathbf{x}^\top\mathbf{x}\end{aligned}$$
or
$$\begin{aligned}&\mathbf{x}^\top[(A_2 - I_n)^\top P + P(A_2 - I_n)]\mathbf{x} <\\ &\quad < -\mathbf{x}^\top(A_2 - I_n)^\top P(A_2 - I_n)\mathbf{x} - \varepsilon\mathbf{x}^\top\mathbf{x}.\end{aligned}$$
As P is positive definite, $\mathbf{x}^\top(A_i - I_n)^\top P(A_i - I_n)\mathbf{x} \geq 0$ for $i \in \{1,2\}$. Therefore, for every $\mathbf{x} \neq 0$ either
$$\mathbf{x}^\top[(A_1 - I_n)^\top P + P(A_1 - I_n)]\mathbf{x} < -\varepsilon\mathbf{x}^\top\mathbf{x}$$
or
$$\mathbf{x}^\top[(A_2 - I_n)^\top P + P(A_2 - I_n)]\mathbf{x} < -\varepsilon\mathbf{x}^\top\mathbf{x}.$$
By Theorem 2.2 in [6], this implies that there exists $\alpha \in [0,1]$ such that
$$\alpha(A_1 - I_n) + (1-\alpha)(A_2 - I_n) = [\alpha A_1 + (1-\alpha)A_2] - I_n$$
is a Hurwitz matrix. This implies that $\alpha A_1 + (1-\alpha)A_2$ is Schur.

b1) $\Leftrightarrow$ b2) is based on two facts: (1) A is nonnegative Schur if and only if $\tilde{A} := A - I$ is a Metzler Hurwitz matrix; (2)

a Metzler matrix $\tilde{A}$ is Hurwitz if and only if [3], [10] there exists a vector $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top \tilde{A} \ll 0$.

b2) $\Rightarrow$ b3) From b2) it follows that, for every positive vector $\mathbf{x}$, one gets

$$\begin{aligned} & \mathbf{v}^\top \left[\alpha(A_1 - I_n) + (1-\alpha)(A_2 - I_n)\right] \mathbf{x} \\ = \;& \alpha \left[\mathbf{v}^\top (A_1 - I_n)\mathbf{x}\right] + (1-\alpha)\left[\mathbf{v}^\top (A_2 - I_n)\mathbf{x}\right] < 0, \end{aligned}$$

whence $\min_{i\in\{1,2\}} \mathbf{v}^\top (A_i - I_n)\mathbf{x} < 0$.

b3) $\Rightarrow$ b2) By assumption b3), there exists a strictly positive vector $\mathbf{v}$ such that for every $\mathbf{x} > 0$ the vector

$$\begin{bmatrix} \mathbf{v}^\top (A_1 - I_n) \\ \mathbf{v}^\top (A_2 - I_n) \end{bmatrix} \mathbf{x} \in \mathbb{R}^{2\times 1}$$

has at least one negative entry. So, once we set

$$W := \begin{bmatrix} \mathbf{v}^\top (A_1 - I_n) \\ \mathbf{v}^\top (A_2 - I_n) \end{bmatrix},$$

we can claim that no positive vector $\mathbf{x}$ can be found such that $W\mathbf{x} \geq 0$. But then, by Lemma 2, in the Appendix, a positive vector $\mathbf{y}$ exists such that $\mathbf{y}^T W \ll 0$. As it entails no loss of generality rescaling $\mathbf{y}$ so that its entries sum up to 1, this means that a nonnegative coefficient α exists, such that

$$0 \gg [\,\alpha \quad (1-\alpha)\,]\, W = \alpha[\mathbf{v}^\top (A_1 - I_n)] + (1-\alpha)[\mathbf{v}^\top (A_2 - I_n)],$$

thus proving b2).

b3) $\Rightarrow$ b4) Let $\mathbf{v}$ be a strictly positive vector such that for every $\mathbf{x} > 0$ condition $\mathbf{v}^\top A_i \mathbf{x} < \mathbf{v}^\top \mathbf{x}$ holds for at least one index $i \in \{1,2\}$. This implies that for every $\mathbf{x} > 0$, condition

$$\mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top A_i \mathbf{x} = |\mathbf{v}^\top A_i \mathbf{x}|^2 < |\mathbf{v}^\top \mathbf{x}|^2 = \mathbf{x}^\top \mathbf{v}\mathbf{v}^\top \mathbf{x}$$

holds for at least one index $i \in \{1,2\}$. So, b4) is satisfied for $P := \mathbf{v}\mathbf{v}^\top$.

b4) $\Rightarrow$ b3) If rank $P = 1$ and $P = P^\top$, then P can be expressed as $P = \mathbf{v}\mathbf{v}^\top$, for some vector $\mathbf{v}$. As $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} > 0$, it follows that $\mathbf{v}$ has entries which are all nonzero and of the same sign. So, it entails no loss of generality assuming that they are all positive. On the other hand, from the fact that at every point $\mathbf{x} > 0$ there exists an index $i \in \{1,2\}$ such that

$$\mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x} = (\mathbf{x}^\top A_i^\top \mathbf{v})(\mathbf{v}^\top A_i \mathbf{x}) - (\mathbf{x}^\top \mathbf{v})(\mathbf{v}^\top \mathbf{x}) < 0$$

namely

$$|\mathbf{v}^\top A_i \mathbf{x}|^2 = (\mathbf{x}^\top A_i^\top \mathbf{v})(\mathbf{v}^\top A_i \mathbf{x}) < (\mathbf{x}^\top \mathbf{v})(\mathbf{v}^\top \mathbf{x}) = |\mathbf{v}^\top \mathbf{x}|^2,$$

and by the nonnegativity of both $\mathbf{v}^\top A_i \mathbf{x}$ and $\mathbf{v}^\top \mathbf{x}$, it follows that $\mathbf{v}^\top A_i \mathbf{x} < \mathbf{v}^\top \mathbf{x}$. This proves that condition b3) holds.

b4) $\Rightarrow$ c) Assume w.l.o.g. that the matrix P that makes b4) satisfied is expressed as $P = \mathbf{v}\mathbf{v}^\top$ for some $\mathbf{v} \gg 0$. Set $\tilde{P} := P + \varepsilon I_n$, with $\varepsilon > 0$. Clearly, $\tilde{P}$ is symmetric. We want to show that $\tilde{P}$ is positive definite for every choice of $\varepsilon > 0$. Indeed, for every $\mathbf{x} \neq 0$,

$$\mathbf{x}^\top \tilde{P} \mathbf{x} = \mathbf{x}^\top P \mathbf{x} + \varepsilon \|\mathbf{x}\|_2^2 = (\mathbf{v}^\top \mathbf{x})^2 + \varepsilon \|\mathbf{x}\|_2^2 > 0.$$

Consider, now, the compact set $\mathcal{S} := \{\mathbf{x} \in \mathbb{R}^n_+ : \|\mathbf{x}\|_2 = 1\}$. The two functions

$$\begin{aligned} f(\mathbf{x}) &= \min_{i\in\{1,2\}} \mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x}, \\ g(\mathbf{x}) &= \max_{i\in\{1,2\}} |\mathbf{x}^\top [A_i^\top A_i - I_n]\mathbf{x}|, \end{aligned}$$

are continuous functions in $\mathcal{S}$. So, by Weierstrass' theorem, both functions have maximum in $\mathcal{S}$ and it is easily seen that

$$\max_{\mathbf{x}\in\mathcal{S}} g(\mathbf{x}) = \max_{\mathbf{x}\in\mathcal{S}} \max_{i\in\{1,2\}} |\mathbf{x}^\top [A_i^\top A_i - I_n]\mathbf{x}| = M \geq 0,$$

while, by the assumption b4),

$$\max_{\mathbf{x}\in\mathcal{S}} f(\mathbf{x}) = \max_{\mathbf{x}\in\mathcal{S}} \min_{i\in\{1,2\}} \mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x} = -\delta < 0.$$

Let ε be a positive number such that $\varepsilon M < \delta$. It is easily seen that, for every $\mathbf{x} \in \mathcal{S}$,

$$\begin{aligned} & \min_{i\in\{1,2\}} \mathbf{x}^\top (A_i^\top \tilde{P} A_i - \tilde{P})\mathbf{x} \\ = \;& \min_{i\in\{1,2\}} \left[\mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} + \varepsilon\left(\mathbf{x}^\top (A_i^\top A_i - I_n)\mathbf{x}\right)\right] \\ \leq \;& \max_{\mathbf{x}\in\mathcal{S}} \min_{i\in\{1,2\}} \left[\mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} \right.\\ & \left. + \varepsilon\left(\mathbf{x}^\top (A_i^\top A_i - I_n)\mathbf{x}\right)\right] \\ \leq \;& \max_{\mathbf{x}\in\mathcal{S}} \min_{i\in\{1,2\}} \left[\mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x}\right] \\ & + \varepsilon \cdot \max_{\mathbf{x}\in\mathcal{S}} \max_{i\in\{1,2\}} |(\mathbf{x}^\top (A_i^\top A_i - I_n)\mathbf{x}| \\ = \;& -\delta + \varepsilon M < 0. \end{aligned}$$

Clearly, for every $\mathbf{x} > 0$, one finds

$$\min_{i\in\{1,2\}} \mathbf{x}^\top (A_i^\top \tilde{P} A_i - \tilde{P})\mathbf{x} \leq (-\delta + \varepsilon M)\|x\|_2^2 < 0,$$

and hence the result is proved.

c) $\Rightarrow$ d) is obvious. ■

Remark 4: a3) $\Rightarrow$ b1) has been proved in [7] for general discrete-time switched systems, by different arguments.

Remark 5: The question whether conditions b1) implies a1) is still an open one. An intuitive approach to solve this problem would be that of considering the set

$$\begin{aligned} \mathcal{P}_\alpha := \{ & P \succ 0 : [\alpha A_1 + (1-\alpha)A_2]^\top P [\alpha A_1 + (1-\alpha)A_2] \\ & - P \prec 0\}, \end{aligned}$$

for the specific value of α that makes b1) satisfied, and check whether the elements of $\mathcal{P}_\alpha$ fulfill condition a1). As a matter of fact, this intuition proves to be true when dealing with the analogous problem in the continuous-time case. In fact, as shown in [22], if $\exists\, \alpha \in [0,1]$ such that $\alpha A_1 + (1-\alpha)A_2$ is Hurwitz, then for every $P \succ 0$ such that

$$[\alpha A_1 + (1-\alpha)A_2]^\top P + P[\alpha A_1 + (1-\alpha)A_2] \prec 0,$$

one finds that in every point $\mathbf{x} \neq 0$

$$\begin{aligned} 0 &> \mathbf{x}^\top \left\{[\alpha A_1 + (1-\alpha)A_2]^\top P + P[\alpha A_1 + (1-\alpha)A_2]\right\} \mathbf{x} \\ &= \alpha \mathbf{x}^\top (A_1^\top P + A_1 P)\mathbf{x} + (1-\alpha)\mathbf{x}^\top (A_2^\top P + A_2 P)\mathbf{x}, \end{aligned}$$

and hence either $\mathbf{x}^\top(A_1^\top P + A_1 P)\mathbf{x} < 0$ or $\mathbf{x}^\top(A_2^\top P + A_2 P)\mathbf{x} < 0$.

In the discrete-time case, examples can be given of positive matrix pairs for which a Schur convex combination can be found, but not every matrix in $\mathcal{P}_\alpha$ makes a1) satisfied.

Example 1: Consider the positive matrices

$$A_1 = \begin{bmatrix} 6/5 & 0 \\ 0 & 3/5 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 3/5 & 0 \\ 0 & 6/5 \end{bmatrix}.$$

Since $A := 0.5A_1 + 0.5A_2 = \begin{bmatrix} 9/10 & 0 \\ 0 & 9/10 \end{bmatrix}$ is a Schur scalar matrix, $A^\top P A - P \prec 0$ for every positive definite matrix $P \succ 0$.

If we choose, for instance, $P = \begin{bmatrix} 1 & 1 \\ 1 & 10 \end{bmatrix}$, the set of inequalities

$$\begin{aligned} \mathbf{x}^\top(A_1^\top P A_1 - P)\mathbf{x} &= 11x_1^2 - 14x_1x_2 - 160x_2^2 \geq 0, \\ \mathbf{x}^\top(A_2^\top P A_2 - P)\mathbf{x} &= -16x_1^2 - 14x_1x_2 + 110x_2^2 \geq 0, \end{aligned}$$

has only the zero solution, and hence a1) is verified. On the other hand, corresponding to $P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{x} = [\,5 \quad -3\,]^\top$, we easily verify that $\mathbf{x}^\top(A_i^\top P A_i - P)\mathbf{x} > 0$ for both indices i.

Remark 6: While condition c) implies d), the converse is not true, as shown by the following example.

Example 2: Consider the positive matrices

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

It is easy to verify that the symmetric matrix of rank 2

$$P = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

is such that for every $\mathbf{x} \in \mathbb{R}_+^2, \mathbf{x} > 0$, we have $\mathbf{x}^\top P \mathbf{x} > 0$ and either

$$\mathbf{x}^\top(A_1^\top P A_1 - P)\mathbf{x} = 6x_1^2 - 6x_1x_2 - 2x_2^2 < 0$$

or

$$\mathbf{x}^\top(A_2^\top P A_2 - P)\mathbf{x} = -2x_1^2 - 6x_1x_2 + 6x_2^2 < 0.$$

On the other hand, no symmetric positive definite matrix P can be found such that in every point $\mathbf{x} > 0$ either $\mathbf{x}^\top(A_1^\top P A_1 - P)\mathbf{x}$ or $\mathbf{x}^\top(A_2^\top P A_2 - P)\mathbf{x}$ is negative. Indeed, if such a matrix would exist, it could be described w.l.o.g. in the form

$$P = \begin{bmatrix} 1 & c \\ c & b \end{bmatrix},$$

with $b > c^2$, and in every nonzero point either one of the following inequalities would be satisfied:

$$\begin{aligned} \mathbf{x}^\top(A_1^\top P A_1 - P)\mathbf{x} &= 3x_1^2 - 2cx_1x_2 - bx_2^2 < 0 \\ \mathbf{x}^\top(A_2^\top P A_2 - P)\mathbf{x} &= -x_1^2 - 2cx_1x_2 + 3bx_2^2 < 0. \end{aligned}$$

Since for $x_1 = 0$ the first equation is obviously satisfied, we assume now $x_1 \neq 0$ and set $y := x_2/x_1$. So that the previous inequalities become:

$$p_1(y) := -by^2 - 2cy + 3 \quad < \quad 0 \tag{4}$$
$$p_2(y) := 3by^2 - 2cy - 1 \quad < \quad 0, \tag{5}$$

Upon observing that $b = c^2 + \varepsilon$ for some $\varepsilon > 0$, our goal is that of proving that, for every choice of $c \in \mathbb{R}$ and $\varepsilon > 0$, there exists $y > 0$ such that both $-by^2 - 2cy + 3 \geq 0$ and $3by^2 - 2cy - 1 \geq 0$. Indeed, the two zeros of the polynomial $p_1(y)$ are

$$\lambda_{-,+} := \frac{-2c \pm \sqrt{4c^2 + 12b}}{2b} = \frac{-c \pm \sqrt{4c^2 + 3\varepsilon}}{c^2 + \varepsilon},$$

and it is easy to prove that $\lambda_- < 0 < \lambda_+$. On the other hand, polynomial $p_2(y)$ has zeros

$$\mu_{-,+} := \frac{2c \pm \sqrt{4c^2 + 12b}}{6b} = \frac{c \pm \sqrt{4c^2 + 3\varepsilon}}{3(c^2 + \varepsilon)}.$$

In order to ensure that in every $y \geq 0$ either (4) or (5) holds, it should be true that $\mu_- < 0$ and $\lambda_+ < \mu_+$. The first condition is easily proved to be verified, however condition $\lambda_+ < \mu_+$ amounts to

$$\frac{-c + \sqrt{4c^2 + 3\varepsilon}}{c^2 + \varepsilon} < \frac{c + \sqrt{4c^2 + 3\varepsilon}}{3(c^2 + \varepsilon)},$$

namely

$$2c > \sqrt{4c^2 + 3\varepsilon},$$

a condition that, of course, is never verified. So, for every choice of c and $\varepsilon > 0$ all positive pairs (x_1, x_2) such that $\mu_+ < x_2/x_1 < \lambda_+$ make both $\Delta V_1(\mathbf{x})$ and $\Delta V_2(\mathbf{x})$ positive.

IV. State-feedback stabilization

In Section III we have investigated under which conditions on the pair of positive matrices (A_1, A_2) a copositive Lyapunov function $V(\mathbf{x})$ can be found for which (2) holds, with $\Delta V_i(\mathbf{x})$ defined as in (3). This amounts to saying that a function $V(\mathbf{x})$ can be found such that, in every nonzero point $\mathbf{x}$ of the positive orthant, the difference $V(A_i\mathbf{x}) - V(\mathbf{x})$ is negative for at least one index $i \in \{1, 2\}$. So, the function $V(\mathbf{x})$ represents an "energy function" that can always be decreased at every step along the state trajectories (within the positive orthant), by suitably choosing whether to switch or not.

Accordingly, we can adopt a "min-projection" switching strategy [19] (also known as "variable structure control" [22])[1]

$$\sigma(\mathbf{x}(t)) := \min\{k : \Delta V_k(\mathbf{x}(t)) \leq \Delta V_i(\mathbf{x}(t)), \forall\, i \in \{1, 2\}\}. \tag{6}$$

It is rather intuitive that this strategy is stabilizing, by this meaning that it ensures that the state evolution converges to

[1] It is worthwhile noticing that the switching law we chose assigns a unique value to $\sigma(\mathbf{x})$ in every point $\mathbf{x} > 0$. Indeed, whenever $\Delta V_1(\mathbf{x}) = \Delta V_2(\mathbf{x})$, the switching rule sets $\sigma(\mathbf{x}) = 1$. The opposite choice, or the choice of a conservative policy that keeps memory of the value of σ at the previous time instant, or even a random choice would not make any difference in terms of convergence.

the origin for every choice of the positive initial state. We want to provide a formal proof for the classes of functions that we have considered in Proposition 2.

We consider first the case when $V(\mathbf{x})$ is a quadratic copositive Lyapunov function.

Proposition 3: Given a discrete-time positive switched system (1), if there exists a quadratic copositive Lyapunov function $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ satisfying (2), then the state feedback switching rule (6) stabilizes the system.

Proof: The function $f(\mathbf{x}) := \min_{i \in \{1,2\}} \Delta V_i(\mathbf{x})$ is a continuous function that takes negative values in every point of the compact set

$$\mathcal{S} := \mathbb{R}^n_+ \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top P \mathbf{x} = 1\}.$$

So, by Weierstrass' Theorem, $\max_{\mathbf{x} \in \mathcal{S}} f(\mathbf{x}) = -\nu$, with $0 < \nu \leq 1$, and this ensures that for every positive state $\mathbf{x}$, $f(\mathbf{x}) \leq -\nu \mathbf{x}^\top P \mathbf{x}$. Consequently

$$\begin{aligned} V(\mathbf{x}(t+1)) = V(\mathbf{x}(t)) + f(\mathbf{x}(t)) &\leq (1-\nu)\mathbf{x}^\top(t) P \mathbf{x}(t) \\ &\leq (1-\nu)^{t+1} \mathbf{x}^\top(0) P \mathbf{x}(0), \end{aligned}$$

and hence $V(\mathbf{x}(t))$ converges to zero, thus guaranteeing that $\mathbf{x}(t)$ converges to zero in turn. ■

Remark 7: Both the result and the proof of Proposition 3 obviously extend to the class of quadratic positive definite Lyapunov functions $V(\mathbf{x})$.

We consider, now, the case of linear copositive Lyapunov functions.

Proposition 4: Given a discrete-time positive switched system (1), if there exists a linear copositive Lyapunov function $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ satisfying (2), then the state feedback switching rule (6) stabilizes the system.

Proof: The proof follows the same lines of the previous one upon assuming

$$\mathcal{S} := \mathbb{R}^n_+ \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{v}^\top \mathbf{x} = 1\}.$$

■

For the class of positive switched systems (1) for which a convex Schur combination of the matrices A_1 and A_2 can be found, we can apply different state feedback switching strategies. Indeed, as a consequence of Proposition 2, we may either resort to a linear copositive function or to a quadratic copositive function (or rank 1 or of higher rank) or to a quadratic positive definite function. So, it is natural to ask which of the available strategies ensures the best performances. We first notice that the switching strategies based on linear copositive and on quadratic copositive functions of rank 1 are just the same. In fact, as clarified in the proof of Proposition 2, the matrices $P = P^\top$ of rank 1 that satisfy condition b4) of Proposition 2 are those and those only that can be expressed as $P = \mathbf{v}\mathbf{v}^\top$ for some vector $\mathbf{v}$ satisfying b3) of the same proposition. On the other hand, by the nonnegativity of the quantities involved,

$$\begin{aligned} & \min\{k : \mathbf{v}^\top (A_k - I_n)\mathbf{x} \leq \mathbf{v}^\top (A_i - I_n)\mathbf{x}, \forall\, i\} \\ = \ & \min\{k : \mathbf{v}^\top A_k \mathbf{x} \leq \mathbf{v}^\top A_i \mathbf{x}, \forall\, i\} \\ = \ & \min\{k : \mathbf{x}^\top A_k^\top \mathbf{v}\mathbf{v}^\top A_k \mathbf{x} \leq \mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top A_i \mathbf{x}, \forall\, i\} \\ = \ & \min\{k : \mathbf{x}^\top (A_k^\top \mathbf{v}\mathbf{v}^\top A_k - \mathbf{v}\mathbf{v}^\top)\mathbf{x} \\ & \qquad \leq \mathbf{x}^\top (A_i^\top \mathbf{v}\mathbf{v}^\top A_i - \mathbf{v}\mathbf{v}^\top)\mathbf{x}, \forall\, i\}, \end{aligned}$$

and hence the switching sequences (6) based on $\mathbf{v}^\top \mathbf{x}$ and on $\mathbf{x}^\top \mathbf{v}\mathbf{v}^\top \mathbf{x}$ are just the same.

On the other hand, we may design switching strategies based on the broader class of quadratic copositive Lyapunov functions (of arbitrary rank) fulfilling condition d). Clearly, this class of switching laws encompasses those based on quadratic copositive functions of rank 1, and hence it ensures convergence performances at least as good as the previous ones.

Similarly, since the class of positive definite functions is included in the class of quadratic copositive functions, the stabilizing switching laws described in Remark 7 are a subset of those described in Proposition 3. So, resorting to switching laws based on quadratic copositive functions allows to obtain better converge performances.

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APPENDIX

The following lemma provides a restatement of the S-procedure, as it can be found, for instance, in [4], which is particularly convenient for the proof of a2) ⇒ a3) in Proposition 2.

Lemma 1 (S-procedure): Let T_1 and $T_2 \in \mathbb{R}^{n \times n}$ be two symmetric matrices, and suppose that there exists $\bar{\mathbf{x}} \neq 0$ such that $\bar{\mathbf{x}}^\top T_1 \bar{\mathbf{x}} > 0$. Then, the following facts are equivalent ones:

i) for every $\mathbf{x} \neq 0$ such that $\mathbf{x}^\top T_1 \mathbf{x} \geq 0$, one finds $\mathbf{x}^\top T_2 \mathbf{x} < 0$;
ii) there exists $\tau \geq 0$ such that $T_2 + \tau T_1$ is negative definite.

Lemma 2 (see [1], Corollary 3.49): Let W be an $n \times p$ real matrix. Then one and only one of the following alternatives holds:

a) $\exists\ \mathbf{y} > 0$ such that $\mathbf{y}^\top W \ll 0$;
b) $\exists\ \mathbf{x} > 0$ such that $W\mathbf{x} \geq 0$ (namely $W\mathbf{x} \in \mathbb{R}^{p \times 1}_+$).

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