Linear copositive Lyapunov functions for continuous-time positive switched systems

Ettore Fornasini Maria Elena Valcher

***Index Terms*—Switched system, positive linear system, asymptotic stability, linear (quadratic) copositive Lyapunov function.**

***Abstract*—Continuous-time positive systems, switching among p subsystems, are introduced, and a complete characterization for the existence of a common linear copositive Lyapunov function for all the subsystems is provided. When the subsystems are obtained by applying different feedback control laws to the same continuous-time single-input positive system, the above characterization leads to a very easy checking procedure.**

I. Introduction

By a continuous-time positive switched system (CPSS) we mean a dynamic system consisting of a family of positive state-space models [?], [?] and a switching law, specifying when and how the switching among the various subsystems takes place. Switched positive systems deserve investigation both for practical applications and for theoretical reasons. Indeed, switching among different models naturally arises as a way to formalize the fact that the behavior of a system changes under different operating conditions, and is therefore represented by different mathematical structures. On the other hand, the positivity constraint is pervasive in engineering practice as well as in chemical, biological and economic modeling.

In the context of CPSSs, stability analysis captured wide attention [?], [?], [?], [?], [?], [?], and mainly focused on the search for conditions ensuring that the family of positive subsystems a CPSS switches among shares either a linear copositive or a quadratic Lyapunov function. In particular, the existence of a common linear copositive function has been investigated in detail [?], [?], [?], thus leading to deeper insights into the properties the subsystems family must be endowed with.

This note is centered around a certain geometric object - the convex hull generated by the (columns) of the subsystems matrices - and aims at exploiting its structure for analyzing the existence of common linear copositive functions. In detail, the paper is organized as follows: in section II, CPSSs, switching among a finite family of subsystems, are introduced, and preliminary conditions for the existence of a common linear copositive Lyapunov function for all subsystems are provided. In section III, by resorting to two technical lemmas, a more complete characterization is given. Finally, in section IV, an elementary checking procedure is derived, for the special class of CPSSs whose subsystems are obtained by applying different feedback control laws to a continuous-time single-input positive system.

Dip. di Ingegneria dell'Informazione, Univ. di Padova, via Gradenigo 6/B, 35131 Padova, Italy, phone: +39-049-827-7605 - fax: +39-049-827-7614, e-mail:fornasini@dei.unipd.it

Author for correspondence: Dip. di Ingegneria dell'Informazione, Univ. di Padova, via Gradenigo 6/B, 35131 Padova, Italy, phone: +39-049-827-7795 - fax: +39-049-827-7614, e-mail:meme@dei.unipd.it

Before proceeding, we introduce some notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers and, for any positive integer k, $\langle k \rangle$ is the set of integers $\{1, 2, \dots, k\}$. The (ℓ, j)th entry of a matrix A will be denoted by $[A]_{\ell j}$, the ℓth entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_\ell$, and the jth column of a matrix A by $\mathrm{col}_j(A)$.

A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so we adopt the notation $A \geq 0$. If, in addition, A has at least one positive entry, the matrix is *positive* ($A > 0$), while if all its entries are positive, it is *strictly positive* ($A \gg 0$). A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative.

Given any real (not necessarily square) matrix A, with n columns, we define its *positive kernel*, $\ker_+(A)$, as the set of nonnegative vectors which belong to the kernel of A, namely

$$\ker_+(A) := \{\mathbf{v} \geq 0 : A\mathbf{v} = 0\} = \ker(A) \cap \mathbb{R}^n_+.$$

A set $\mathcal{K} \subset \mathbb{R}^n$ is a *cone* if $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$. Basic definitions and results about cones may be found, for instance, in [?]. We recall here only those facts that will be used within this paper. A cone is *convex* if it contains, with any two points, the line segment between them. A convex cone $\mathcal{K}$ is *solid* if it includes at least one interior point, and it is *pointed* if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$. A cone $\mathcal{K}$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of vectors, called *generating vectors*; if the generating vectors are the columns of a matrix A, we adopt the notation $\mathcal{K} = \mathrm{Cone}(A)$. The *dual cone* of a cone $\mathcal{K} \subset \mathbb{R}^n$ is

$$\mathcal{K}^* := \{\mathbf{v} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{v} \geq 0, \ \forall\, \mathbf{x} \in \mathcal{K}\}.$$

A closed convex cone $\mathcal{K}$ is pointed (solid) [polyhedral] if and only if $\mathcal{K}^*$ is solid (pointed) [polyhedral].

Given a family of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ in $\mathbb{R}^n$, the *convex hull* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ is the set of vectors

$$\left\{ \sum_{i=1}^{s} \alpha_i \mathbf{v}_i : \alpha_i \geq 0, \sum_{i=1}^{s} \alpha_i = 1 \right\}.$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ are *affinely (in)dependent* if $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_s - \mathbf{v}_1$ are linearly (in)dependent. A *simplex* in $\mathbb{R}^n$ is the convex hull of a set of $s \leq (n+1)$ affinely independent vectors.

II. General positive switched systems: preliminary results

A *continuous-time positive switched system* is described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{1}$$

where $\mathbf{x}(t)$ denotes the value of the n-dimensional state variable at time t, σ is an arbitrary switching sequence, taking values in some set $\langle p \rangle$, and for each $i \in \langle p \rangle$, the matrix A_i is the system matrix of a continuous-time positive system, which means that A_i is an $n \times n$ Metzler matrix. We assume that the switching sequence is piece-wise continuous, and hence in every time interval $[0, t]$ there is a finite number of discontinuities, which correspond to a finite number of switching instants.

Definition 1: [**?**], [**?**] Given the CPSS (1) and an n-dimensional real vector $\mathbf{v}$, the function $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is a *linear copositive Lyapunov function* for the continuous-time positive ith subsystem

$$\dot{\mathbf{x}}(t) = A_i\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{2}$$

if for any positive vector $\mathbf{x} \in \mathbb{R}^n_+$,

$$\mathbf{v}^\top \mathbf{x} > 0 \quad \text{and} \quad \mathbf{v}^\top A_i \mathbf{x} < 0. \tag{3}$$

$V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is a *common linear copositive Lyapunov function* for the p subsystems (2) (or, equivalently, for the family $\mathcal{A} := \{A_1, A_2, \dots, A_p\}$ of $n \times n$ Metzler matrices), if it is a linear copositive Lyapunov function for each of them.

As it is well known [**?**] (and can be easily deduced from the fact that (3) must hold for every positive vector), $V(\mathbf{x})$ is a linear copositive Lyapunov function for the subsystem (2) if and only if $\mathbf{v} \gg 0$ and $\mathbf{v}^\top A_i \ll 0$. Consequently, it is a common linear copositive Lyapunov function (in the following, CLCLF) for the p subsystems (2) if and only if

$$\mathbf{v} \gg 0 \qquad \text{and} \qquad \mathbf{v}^\top A_i \ll 0, \ \forall \ i \in \langle p \rangle. \tag{4}$$

When dealing with the single ith subsystem (2), the existence of a linear copositive function is equivalent to its asymptotic stability [**?**], [**?**], by this meaning that for every $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n_+$, the state evolution $\mathbf{x}(t)$ asymptotically converges to zero. Asymptotic stability, in turn, is equivalent to the fact that the system matrix A_i is (Metzler) Hurwitz, i.e., all its eigenvalues lie in the open left half-plane $\mathbb{C}_- := \{s \in \mathbb{C} : \mathrm{Re}(s) < 0\}$. On the other hand, when dealing with a CPSS (1), asymptotic stability amounts to the convergence to zero of every state trajectory, independently of the nonnegative initial condition and for every choice of the switching sequence $\sigma : \mathbb{R}_+ \to \langle p \rangle$. This requires each single subsystem to be asymptotically stable, namely each matrix in $\mathcal{A}$ to be Hurwitz (and, of course, Metzler, by the positivity assumption). However, this is only a necessary condition, and examples have been given of CPSSs which are not asymptotically stable, even though all their subsystems are [**?**]. On the other hand, the existence of a CLCLF for the subsystems of the positive switched system (1) is sufficient for asymptotic stability, but it is not necessary.

Example 1: Consider the 2-dimensional CPSS (1), with $p = 2$ and matrices

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1/2 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 1/2 \\ 1 & -1 \end{bmatrix}.$$

By a result of Akar *et al.* [**?**], the CPSS is asymptotically stable. However it is easily seen that no CLCLF for A_1 and A_2 can be found. Indeed, if $\mathbf{v} = [\, v_1 \quad v_2 \,]^\top \gg 0$, then $\mathbf{v}^\top A_1$ implies $v_1 < v_2$, while $\mathbf{v}^\top A_2$ implies $v_2 < v_1$. So, a strictly positive vector $\mathbf{v}$ satisfying (4) does not exist.

The interest in the existence of a CLCLF for the subsystems of system (1) is motivated by its computational tractability. In fact, checking whether there exists a vector $\mathbf{v}$ such that (4) holds just amounts to solve a family of LMIs, and this can be done by using standard numerical software. However, since this is a stronger condition with respect to the asymptotic stability of the positive switched system (1), we are interested in characterizing, within the class of asymptotically stable CPSS, those admitting a CLCLF.

A first characterization is provided by the following proposition.

Proposition 1: Given a family $\mathcal{A} = \{A_1, A_2, \dots, A_p\}$ of $n \times n$ Metzler matrices, the following are equivalent:

i) there exists a CLCLF for the family $\mathcal{A}$, i.e. there exists a vector $\mathbf{v} \in \mathbb{R}^n_+$ such that (4) holds;

ii) $\ker_+ [\, I_n \quad -A_1 \quad -A_2 \quad \dots \quad -A_p \,] = \{0\}$;

iii) the convex hull of the vector family $\mathcal{C}_\mathcal{A} := \{\mathrm{col}_j(A_i) : j \in \langle n \rangle, i \in \langle p \rangle\}$ does not intersect the positive orthant $\mathbb{R}^n_+$.

Proof: i) $\Leftrightarrow$ ii) Notice, first, that

$$\{\mathbf{v} \gg 0 : A_i^\top \mathbf{v} \ll 0, \forall \ i \in \langle p \rangle\} = \left\{ \mathbf{v} \in \mathbb{R}^n_+ : \begin{bmatrix} I_n \\ -A_1^\top \\ \vdots \\ -A_p^\top \end{bmatrix} \mathbf{v} \gg 0 \right\}.$$

On the other hand, the set on the right-hand side in the previous identity is the interior of the closed convex cone

$$\mathcal{K}^* := \left\{ \mathbf{v} \in \mathbb{R}^n_+ : \begin{bmatrix} I_n \\ -A_1^\top \\ \vdots \\ -A_p^\top \end{bmatrix} \mathbf{v} \geq 0 \right\},$$

which, in turn, is the dual cone of the polyhedral cone

$$\mathcal{K} := \mathrm{Cone}\, [\, I_n \quad -A_1 \quad \dots \quad -A_p \,] .$$

So, the set $\{\mathbf{v} \gg 0 : A_i^\top \mathbf{v} \ll 0, \forall \ i \in \langle p \rangle\}$ is nonempty if and only if the dual cone $\mathcal{K}^*$ is solid, and this happens if and only if [**?**] the cone $\mathcal{K}$ is pointed (by this meaning that if both $\mathbf{v}$ and $-\mathbf{v}$ belong to $\mathcal{K}$, then $\mathbf{v} = 0$). However, as $[\, I_n \quad -A_1 \quad \dots \quad -A_p \,]$ is devoid of zero columns, it is easily seen that $\mathcal{K}$ is pointed if and only if the only nonnegative vector in the kernel of $[\, I_n \quad -A_1 \quad \dots \quad -A_p \,]$ is the zero vector. So, we have proved that i) and ii) are equivalent statements.

ii) $\Leftrightarrow$ iii) There exists a positive vector in $\ker [\, I_n \quad -A_1 \quad -A_2 \quad \dots \quad -A_p \,]$ if and only if there exist nonnegative vectors $\mathbf{y}, \mathbf{x}_1, \ldots, \mathbf{x}_p$, not all of them equal to zero, such that

$$\mathbf{y} = \sum_{i=1}^{p} A_i \mathbf{x}_i = \sum_{i=1}^{p} \sum_{j=1}^{n} \mathrm{col}_j(A_i)[\mathbf{x}_i]_j.$$

Possibly rescaling $\mathbf{y}$ and the various nonnegative coefficients $[\mathbf{x}_i]_j$, we can assume $\sum_{i=1}^{p} \sum_{j=1}^{n} [\mathbf{x}_i]_j = 1$, which amounts to saying that the convex hull of the family of vectors $\mathcal{C}_{\mathcal{A}}$ includes a nonnegative vector. Therefore, also ii) and iii) are equivalent. ∎

When $p = 1$, the following corollary of Proposition 1 provides a set of equivalent conditions for the asymptotic stability of a continuous-time positive linear system (as previously remarked, the equivalence of the first two items is well-known [**?**], [**?**]).

Corollary 1: For a continuous-time positive system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, the following are equivalent:

i) the system is asymptotically stable, namely A is a (Metzler) Hurwitz matrix;
ii) the system admits a linear copositive Lyapunov function, i.e. there exists $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top A \ll 0$;
iii) $\ker_+ [I_n \quad -A] = \{0\}$;
iv) the convex hull of the vector family $\mathcal{C}_A := \{\mathrm{col}_j(A) : j \in \langle n \rangle\}$ does not intersect the positive orthant $\mathbb{R}^n_+$.

III. CLCLFs for positive switched systems

In order to provide additional characterizations of CPSSs admitting a CLCLF, we need a technical lemma. Preliminarily, we remark that a Metzler Hurwitz matrix A satisfies

$$[A]_{\ell j} \begin{cases} < 0, & \text{if } \ell = j, \\ \geq 0, & \text{if } \ell \neq j. \end{cases}$$

Indeed, A is Metzler Hurwitz if and only if $-A$ is an M-matrix, and the properties of the M-matrices can be found in [**?**]. As a consequence, every column vector $\mathrm{col}_j(A)$ has the jth entry which is negative and the remaining ones which are nonnegative. In the following, for the sake of simplicity, we will denote the orthant of $\mathbb{R}^n$ including vectors with all nonnegative entries except for the jth, which is negative, as $\mathcal{O}_{j-}$. Notice that this is not a closed set.

Lemma 1: Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ be a family of $s \leq n+1$ vectors in $\mathbb{R}^n$, each of them belonging to some orthant $\mathcal{O}_{j-}, j \in \langle n \rangle$. Suppose that there exists a positive convex combination

$$\mathbf{y} = \sum_{j=1}^{s} \mathbf{v}_j c_j, \qquad c_j > 0, \quad \sum_{j=1}^{s} c_j = 1,$$

such that $\mathbf{y}$ is a nonnegative vector . If (at least) two vectors of the family, say $\mathbf{v}_1$ and $\mathbf{v}_2$, belong to the same orthant $\mathcal{O}_{j-}$, then a nonnegative vector, possibly different from $\mathbf{y}$, can be obtained as a convex combination of a subfamily of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$, where either $\mathbf{v}_1$ or $\mathbf{v}_2$ has been removed.

Proof: Without loss of generality (w.l.o.g.) we assume that $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{O}_{1-}$, i.e., that their negative entry is the first one. We prove the result by induction on n.

The first elementary case is $n = 2$. If so, we necessarily deal with $s = 3$ vectors and $\mathbf{v}_3$ must belong to $\mathcal{O}_{2-}$ (as the convex combination of vectors in $\mathcal{O}_{1-}$ cannot belong to $\mathbb{R}^2_+$). As the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ belong to the second orthant in $\mathbb{R}^2$, while $\mathbf{v}_3$ belongs to the fourth orthant, if the convex hull of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ intersects $\mathbb{R}^2_+$ in $\mathbf{y}$, then either the line segment of vertices $\mathbf{v}_1$ and $\mathbf{v}_3$, or the line segment of vertices $\mathbf{v}_2$ and $\mathbf{v}_3$, intersects the positive orthant $\mathbb{R}^2_+$. This amounts to saying that there exist $\alpha \in]0,1[$ and $i \in \langle 2 \rangle$ such that $\alpha\mathbf{v}_i + (1-\alpha)\mathbf{v}_3 \geq 0$.

We assume now, inductively, that the result is true if we deal with vectors of size smaller than n, and we want to prove that it holds (when dealing with $s \leq n+1$ vectors, with the aforementioned pattern properties) in $\mathbb{R}^n$.

Suppose that $\mathbf{y} = \sum_{j=1}^{s} \mathbf{v}_j c_j > 0$, with $c_j > 0, \sum_{j=1}^{s} c_j = 1$, and that $\mathbf{v}_1$ and $\mathbf{v}_2$ belong to the same orthant $\mathcal{O}_{1-}$. We distinguish three cases:

[Case 1] If $s \leq n$, then (since $\mathbf{v}_1$ and $\mathbf{v}_2$ belong to the same orthant) the number of distinct orthants the s vectors belong to is at most $s-1$. As a consequence, there exist (at least) $n-s+1$ indices corresponding to entries which are nonnegative in all vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$. We assume, for the sake of simplicity, that these positions are the last ones. Let $\hat{\mathbf{v}}_j$, $j \in \langle s \rangle$, and $\hat{\mathbf{y}} \geq 0$ be the $(s-1)$-dimensional vectors obtained from $\mathbf{v}_j$, $j \in \langle s \rangle$, and $\mathbf{y}$, respectively, by deleting the last $n-s+1$ entries. Obviously,

$$\hat{\mathbf{y}} = \sum_{j=1}^{s} \hat{\mathbf{v}}_j c_j, \qquad c_j > 0, \quad \sum_{j=1}^{s} c_j = 1.$$

Since these vectors belong to a vector space of dimension smaller than n, and $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ belong to the same orthant, by the inductive assumption we can obtain a nonnegative vector as a convex combination of a subset of the family of vectors $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \ldots, \hat{\mathbf{v}}_s\}$, where either $\hat{\mathbf{v}}_1$ or $\hat{\mathbf{v}}_2$ does not appear. In other words, there exist $i \in \langle 2 \rangle, \alpha_j \geq 0, \alpha_i + \sum_{j=3}^{s} \alpha_j = 1$, such that

$$\alpha_i \hat{\mathbf{v}}_i + \sum_{j=3}^{s} \alpha_j \hat{\mathbf{v}}_j \geq 0.$$

Consequently,

$$\alpha_i \mathbf{v}_i + \sum_{j=3}^{s} \alpha_j \mathbf{v}_j \geq 0.$$

[Case 2] Suppose, now, that $s = n+1$, and there exists (at least) one index $h \in \langle n \rangle$ such that none of the vectors $\mathbf{v}_j$ belongs to $\mathcal{O}_{h-}$. This implies that there is a pair of indices $(i_1, i_2) \neq (1, 2), i_1, i_2 \in \langle n+1 \rangle$, such that $\mathbf{v}_{i_1}$ and $\mathbf{v}_{i_2}$ belong to the same orthant $\mathcal{O}_{k-}, k \neq h$. In this case, we may first replace $c_{i_1}\mathbf{v}_{i_1} + c_{i_2}\mathbf{v}_{i_2}$ with $(c_{i_1} + c_{i_2})\mathbf{v}_{i_1 i_2}$, where $\mathbf{v}_{i_1 i_2}$ is a suitable vector belonging again to $\mathcal{O}_{k-}$. In this way, $\mathbf{y}$ is expressed as the convex combination of n vectors in $\mathbb{R}^n_+$, and hence we may apply the same reasoning adopted in Case 1, thus getting a nonnegative vector as the convex combination of a proper subset of these n vectors, where either $\mathbf{v}_1$ or $\mathbf{v}_2$ has been removed. If $\mathbf{v}_{i_1 i_2}$ is weighted by a zero coefficient, we are done, otherwise we can replace $\mathbf{v}_{i_1 i_2}$ with its positive combination in terms of $\mathbf{v}_{i_1}$ and $\mathbf{v}_{i_2}$, and, after a suitable rescaling, find, in the nonnegative orthant, a convex combination of at most n vectors.

[Case 3] Now, we consider the case when $s = n+1$ and, w.l.o.g., $\mathbf{v}_j \in \mathcal{O}_{(j-1)-}$ for every $j \in \{3, 4, \ldots, n+1\}$. If the $n+1$ vectors are affinely dependent, then there exists a (nontrivial) linear combination, with coefficients summing up to zero, which gives the zero vector:

$$0 = \sum_{j=1}^{n+1} \mathbf{v}_j \gamma_j, \qquad \sum_{j=1}^{n+1} \gamma_j = 0.$$

Consider the set

$$T := \{\tau \in \mathbb{R} : \gamma_j \tau \geq -c_j, \forall\, j \in \langle n+1 \rangle\}.$$

This set is not empty (as $0 \in T$), and it is the intersection of a finite number of closed half-lines. Consequently, since at least one of the γ_j's is positive and at least one of them is negative, T is a closed interval in $\mathbb{R}$. By selecting one of its extremal points, say $\bar{\tau}$, we get

$$\mathbf{y} = \sum_{j=1}^{n+1} \mathbf{v}_j (\gamma_j \bar{\tau} + c_j),$$

where now the coefficients $\gamma_j \bar{\tau} + c_j$ are nonnegative for every $j \in \langle n+1 \rangle$, and at least one of them is zero. If both $\mathbf{v}_1$ and $\mathbf{v}_2$ are weighted by a positive coefficient, we get rid of one of them as in Case 1. Otherwise, we have obtained $\mathbf{y}$ as a convex combination of a subset of the original set of vectors, each of them belonging to a distinct orthant.

Finally, if the $n+1$ vectors $\mathbf{v}_j, j \in \langle n+1 \rangle$, are affinely independent, they generate a simplex in $\mathbb{R}^n$. So, if the simplex intersects $\mathbb{R}^n_+$, at least one of its faces does, which means that we can get rid of one of the vectors and still get a convex combination which is a nonnegative vector. If both $\mathbf{v}_1$ and $\mathbf{v}_2$ are weighted by a positive coefficient, we can apply, again, Case 1, and get rid of one of them. Otherwise, we have obtained $\mathbf{y}$ by combining a subset of the original set of vectors, each of them belonging to a distinct orthant. ■

Condition iii) of Proposition 1 pertains the convex hull generated by the columns of the $n \times pn$ matrix $[\, A_1 \quad A_2 \quad \ldots \quad A_p \,]$. Proposition 2, below, shows that the same condition can be expressed in terms of the convex hulls of a family of $n \times n$ matrices, namely those matrices one obtains by selecting the 1st column among the 1st columns of the matrices in $\mathcal{A}$, the 2nd column among the 2nd columns of the matrices in $\mathcal{A}$, etc.. As a result, point ii) in Proposition 2 provides a further characterization for the existence of a CLCLF for the matrices in $\mathcal{A}$.

Proposition 2: Given a family $\mathcal{A} = \{A_1, A_2, \ldots, A_p\}$ of $n \times n$ Metzler Hurwitz matrices, the following are equivalent:

i) the convex hull of the vector family $\mathcal{C}_\mathcal{A} := \{\mathrm{col}_j(A_i) : j \in \langle n \rangle, i \in \langle p \rangle\}$ does not intersect the positive orthant $\mathbb{R}^n_+$;
ii) for every map $\pi : \langle n \rangle \to \langle p \rangle$, the convex hull of the vector family $\mathcal{C}_\pi := \{\mathrm{col}_j(A_{\pi(j)}) : j \in \langle n \rangle\}$ does not intersect the positive orthant $\mathbb{R}^n_+$.

Proof: We first notice that each vector in the aforementioned families, being a column of a Metzler Hurwitz matrix, belongs to some orthant $\mathcal{O}_{j-}$, for some $j \in \langle n \rangle$. The proof of i) $\Rightarrow$ ii) is obvious.

ii) $\Rightarrow$ i) We proceed by showing that $\overline{i)}$ implies $\overline{ii)}$. Consider a nonnegative vector $\mathbf{y} \in \mathbb{R}^n_+$ obtained as the convex combination of the vectors of $\mathcal{C}_\mathcal{A}$. By the Caratheodory's theorem [**?**], there exist $s \leq n+1$ vectors, say $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ in $\mathbb{R}^n$, such that

$$\mathbf{y} = \sum_{j=1}^{s} \mathbf{v}_j c_j, \qquad c_j > 0, \quad \sum_{j=1}^{s} c_j = 1.$$

Starting from the above combination and repeatedly applying Lemma 1, we reduce ourselves to the situation when we have vectors, say $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$, with $r \leq \min\{s, n\}$, endowed with the following properties:

- each of them belongs to $\mathcal{C}_\mathcal{A}$;
- for every pair of distinct indices $i, j \in \langle r \rangle$, $\mathbf{w}_i$ and $\mathbf{w}_j$ belong to distinct orthants;
- there exists a convex combination of the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ that gives a nonnegative vector in $\mathbb{R}^n_+$.

If $r < n$, we complete the r-tuple above by introducing $n-r$ vectors of $\mathcal{C}_A$, each of them belonging to one of the orthants which are not represented by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$. So, in any case, we end up with an n-tuple of columns $\{\mathrm{col}_j(A_{\pi(j)}), j \in \langle n \rangle\}$, that corresponds to a suitable map π, and produces, via convex combination, a nonnegative vector in $\mathbb{R}^n_+$. This contradicts ii). ■

Theorem 1, below, encompasses the four characterizations, given in Propositions 1 and 2, of CPSSs whose p subsystems admit a CLCLF, and provides additional two. Condition v) has been recently obtained by Knorn, Mason and Shorten in [**?**], by means of different techniques, and it extends a preliminary result by Mason and Shorten [**?**]. Condition vi) relates the existence of a common linear copositive function to the existence of another type of common Lyapunov function, which we now introduce.

Definition 2: Given an $n \times n$ symmetric real matrix P, the function $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is a *common quadratic copositive Lyapunov function* (CQCLF) for the p subsystems (2) (for the p matrices in $\mathcal{A}$), if it is a quadratic copositive Lyapunov function for each of them, by this meaning that, for each index $i \in \langle p \rangle$ and for positive vector $\mathbf{x} \in \mathbb{R}^n_+$, one has

$$\mathbf{x}^\top P \mathbf{x} > 0 \qquad \text{and} \qquad \mathbf{x}^\top [A_i P + P A_i] \mathbf{x} < 0.$$

Theorem 1: Given a family $\mathcal{A} = \{A_1, A_2, \ldots, A_p\}$ of $n \times n$ Metzler Hurwitz matrices, the following are equivalent:

i) there exists a CLCLF for the family $\mathcal{A}$, i.e. there exists a vector $\mathbf{v} \in \mathbb{R}^n_+$ such that (4) holds;
ii) $\ker_+ [\, I_n \quad -A_1 \quad -A_2 \quad \ldots \quad -A_p \,] = \{0\}$;
iii) the convex hull of the vector family $\mathcal{C}_\mathcal{A} = \{\mathrm{col}_j(A_i) : j \in \langle n \rangle, i \in \langle p \rangle\}$ does not intersect the positive orthant $\mathbb{R}^n_+$;
iv) for every map $\pi : \langle n \rangle \to \langle p \rangle$, the convex hull of the vector family $\mathcal{C}_\pi = \{\mathrm{col}_j(A_{\pi(j)}) : j \in \langle n \rangle\}$ does not intersect the positive orthant $\mathbb{R}^n_+$;
v) for every map $\pi : \langle n \rangle \to \langle p \rangle$, the square matrix

$$A_\pi := [\, \mathrm{col}_1(A_{\pi(1)}) \quad \mathrm{col}_2(A_{\pi(2)}) \quad \ldots \quad \mathrm{col}_n(A_{\pi(n)}) \,]$$

is a (Metzler) Hurwitz matrix;

vi) there exists $P = P^\top$, with rank $P = 1$, such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a CQCLF for the p matrices in $\mathcal{A}$.

Proof: The equivalence of points i), ii) and iii) is Proposition 1. The equivalence of iii) and iv) has been shown in Lemma 2. Finally, the equivalence of iv) and v) is just a restatement of the equivalence of points i) and iv) in Corollary 1, upon replacing the Metzler matrix A with any Metzler matrix A_π. We now prove the equivalence of i) and vi).

i) $\Rightarrow$ vi) Set $P = \mathbf{v}\mathbf{v}^\top$. Clearly, $P = P^\top$ has rank 1 and it is immediately seen that, being $\mathbf{v} \gg 0$, $V(\mathbf{x}) = (\mathbf{x}^\top\mathbf{v})(\mathbf{v}^\top\mathbf{x}) = (\mathbf{v}^\top\mathbf{x})^2 > 0$ for every $\mathbf{x} > 0$. On the other hand, as in

$$\mathbf{x}^\top[A_i^\top P + PA_i]\mathbf{x} = (\mathbf{x}^\top A_i^\top\mathbf{v})(\mathbf{v}^\top\mathbf{x}) + (\mathbf{x}^\top\mathbf{v})(\mathbf{v}^\top A_i\mathbf{x})$$

$(\mathbf{v}^\top\mathbf{x}) > 0$ and $\mathbf{x}^\top A_i^\top\mathbf{v} < 0$, for every $\mathbf{x} > 0$ and every $i \in \langle p\rangle$, it follows that $\dot{V}(\mathbf{x}(t)) < 0$ along every positive trajectory of each ith subsystem, and hence $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a CQCLF for the p matrices in $\mathcal{A}$.

vi) $\Rightarrow$ i) If rank $P = 1$ and $P = P^\top$, then P can be expressed as

$$P = T\ \mathrm{diag}\{0, 0\ldots, 0, \lambda\}\ T^\top,$$

for some orthonormal matrix T and some $\lambda \neq 0$. Let $\mathbf{v}$ denote the nth column of P. Then $P = \lambda\mathbf{v}\mathbf{v}^\top$. Condition $\mathbf{x}^\top P\mathbf{x} > 0$ for every $\mathbf{x} > 0$ ensures that $\lambda > 0$ and $\mathbf{v}$ has entries which are all nonzero and of the same sign. So, it entails no loss of generality assuming that they are all positive. On the other hand, from

$$\mathbf{x}^\top[A_i^\top P + PA_i]\mathbf{x} = \lambda\left[(\mathbf{x}^\top A_i^\top\mathbf{v})(\mathbf{v}^\top\mathbf{x}) + (\mathbf{x}^\top\mathbf{v})(\mathbf{v}^\top A_i\mathbf{x})\right] < 0$$

it follows that

$$(\mathbf{x}^\top\mathbf{v})(\mathbf{v}^\top A_i\mathbf{x}) < 0,$$

and since $\mathbf{x}^\top\mathbf{v} > 0$ for every $\mathbf{x} > 0$, it must be $\mathbf{v}^\top A_i\mathbf{x} < 0$ for every $\mathbf{x} > 0$. This ensures that $\mathbf{v}^\top A_i \ll 0$. As this is true for every $i \in \langle p\rangle$, condition i) holds. ∎

Example 2: Consider the 2-dimensional CPSS of Example 1. As we know, the system is asymptotically stable, but a CLCLF for A_1 and A_2 does not exist. Indeed, by making use of condition v) in Theorem 1, we easily see that the matrix

$$A_\pi = [\,\mathrm{col}_1(A_2) \quad \mathrm{col}_2(A_1)\,] = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

is not Hurwitz. Also, by Theorem 1, we can claim that no CQCLF of rank 1 can be found for A_1 and A_2. However, the quadratic positive definite function

$$\bar{V}(\mathbf{x}) = \mathbf{x}^\top \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

is a CQCLF of rank 2 for both matrices.

Example 3: Consider the 2-dimensional DPSS, with $p = 2$ and matrices

$$A_1 = \begin{bmatrix} 0 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/4 \end{bmatrix}.$$

It is easily seen that all four matrices A_π, as π varies within the set of maps from $[1, 2]$ to $[1, 2]$, are Schur, and hence a CLCLF for A_1 and A_2 exists. Indeed, each vector $\mathbf{v} = [\,v_1 \quad v_2\,]^\top$, with $0 < 0.25\ v_1 < v_2 < v_1$, defines a CLCLF for A_1 and A_2.

Remark 2: When dealing with 2-dimensional matrices, conditions iv) and v) of Theorem 1 can be tested on a single matrix A_π, instead of on p^2 matrices. Indeed, by resorting to a geometric reasoning, we may notice that first columns $\mathrm{col}_1(A_i), i \in \langle p\rangle$, belong to the second orthant of $\mathbb{R}^2$, while second columns $\mathrm{col}_2(A_i), i \in \langle p\rangle$, belong to the fourth orthant. Moreover, if we denote by α_i and by β_i the angles that $\mathrm{col}_1(A_i)$ and $\mathrm{col}_2(A_i)$, respectively, form with the half-line $\{x_1 \geq 0\}$, then $\alpha_i \in]\pi/2, \pi]$, while $\beta_i \in [-\pi/2, 0[$. So, there exists $\pi : \langle 2\rangle \rightarrow \langle p\rangle$, such that the convex hull of $\mathcal{C}_\pi$ intersects $\mathbb{R}^2_+$ if and only if the line segment connecting the extremal vertices of $\mathrm{col}_1(A_{i^*})$ and $\mathrm{col}_2(A_{j^*})$ does, where

$$i^* := \arg\ \min_i\ \alpha_i, \qquad j^* := \arg\ \max_i\ \beta_i.$$

Equivalently, condition v) holds if and only if the Metzler matrix $A_\pi = [\,\mathrm{col}_1(A_{i^*}) \quad \mathrm{col}_2(A_{j^*})\,]$ is Hurwitz.

IV. CLCLFs for positive switched systems obtained by state-feedback

In this section we consider the situation when the CPSS (1) is obtained by applying, to the same continuous-time positive system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + bu(t), \qquad t \in \mathbb{R}_+, \tag{5}$$

a state-feedback law, switching among a finite number of possible configurations, $u(t) = K_{\sigma(t)}\mathbf{x}(t)$, $\sigma : \mathbb{R}_+ \rightarrow \langle p\rangle$. Here $\mathbf{x}(t)$ and $u(t)$ denote the values of the n-dimensional state variable and of the scalar input, respectively, at time t, A is an $n \times n$ Metzler matrix and b an n-dimensional positive vector. Consequently, the ith subsystem (2) is characterized by the Metzler matrix $A_i = A + bK_i$.

As it has been remarked in [?], we are not obliged to resort to positive matrices K_i to ensure that $A + bK_i$ is a Metzler matrix. In fact, the positivity constraint typically pertains only the state evolution and not the input signal, so we just need to ensure that for every positive initial condition the state evolution of the resulting feedback system remains in the positive orthant. So, in the sequel, we will only assume that the feedback matrices K_i's are real row vectors which ensure that $A + bK_i$ is Metzler. Conditions for this to happen have been provided in [?] (for the dual case of positive observers), and have been adapted to the feedback control case, for instance, in [?].

The following proposition shows that, when dealing with a positive switched system (1), obtained by means of different feedback connections, the existence of a CLCLF for the matrices $A + bK_i$ can be verified by checking the Hurwitz property of a single matrix.

Proposition 3: Consider the n-dimensional CPSS (1), obtained by switching among p controllers:

$$\dot{\mathbf{x}}(t) = (A + bK_i)\mathbf{x}(t), \qquad i \in \langle p\rangle,$$

where A is an $n \times n$ Metzler matrix, b an n-dimensional positive vector and each K_i an n-dimensional row vector such that $A + bK_i$ is a Metzler matrix. The p matrices

$A+bK_i, i \in \langle p \rangle$, admit a CLCLF if and only if the matrix $A+bK^*$ is Metzler Hurwitz, where

$$[K^*]_j := \max_{i \in \langle p \rangle} [K_i]_j, \qquad j \in \langle n \rangle.$$

Proof: We first observe that since $\mathrm{col}_j(A+bK^*) = \mathrm{col}_j(A+bK_{i_j})$ for some $i_j \in \langle p \rangle$, it follows that $\mathrm{col}_j(A+bK^*) \in \mathcal{O}_{j-}$. Since this holds for every index $j \in \langle n \rangle$, the columns of $A+bK^*$ have a structure which ensures that $A+bK^*$ is a Metzler matrix.

[Only if] By Theorem 1, if the p matrices $A+bK_i, i \in \langle p \rangle$, admit a CLCLF, then for every map $\pi : \langle n \rangle \to \langle p \rangle$, the square matrix

$$\begin{aligned} A_\pi &:= [\,\mathrm{col}_1(A+bK_{\pi(1)}) \quad \dots \quad \mathrm{col}_n(A+bK_{\pi(n)})\,] \\ &= A + b\,[\,[K_{\pi(1)})]_1 \dots \quad [K_{\pi(n)}]_n\,] \end{aligned}$$

is a (Metzler) Hurwitz matrix. But then, in particular, $A+bK^*$ is Metzler Hurwitz.

[If] Conversely, assume that $A+bK^*$ is Metzler Hurwitz. Then a vector $\mathbf{v} \gg 0$ can be found such that $\mathbf{v}^\top(A+bK^*) \ll 0$. But then, for every $i \in \langle p \rangle$, being $K_i \leq K^*$ and $\mathbf{v}^\top b > 0$, one finds

$$\begin{aligned} 0 &\gg \mathbf{v}^\top(A+bK^*) = \mathbf{v}^\top(A+bK_i) + (\mathbf{v}^\top b)(K^* - K_i) \\ &\geq \mathbf{v}^\top(A+bK_i). \end{aligned}$$

This proves that $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is a CLCLF for all matrices $A+bK_i, i \in \langle p \rangle$. ∎