On the stabilizability of discrete-time positive switched systems

Ettore Fornasini and Maria Elena Valcher

***Abstract*— In this paper we consider the class of discrete-time systems switching between an arbitrary number p of autonomous positive subsystems. Necessary and sufficient conditions for the existence of (either linear or quadratic) copositive Lyapunov functions, whose values can be decreased in every positive state, by suitably choosing one of p subsystems, are obtained. When these conditions are fulfilled, state-dependent switching strategies, which prove to be stabilizing, can be adopted. Finally, the performances of these Lyapunov based strategies are compared.**

I. Introduction

A discrete-time positive switched system (DPSS) consists of a family of positive state-space models [5], [13] and a switching law, specifying when and how the switching among the various subsystems takes place. This class of systems has some interesting practical applications. They have been adopted for describing networks employing TCP and other congestion control applications [21], for modeling consensus and synchronization problems [12], and, quite recently, to describe the viral mutation dynamics under drug treatment [10].

In the context of positive switched systems, most of the research results about stability and stabilizabilty have been derived in the continuous-time case [2], [9], [14], [16], [17], [18], [24]. While results based on linear copositive functions find a straightforward extension to the discrete-time case, this is not true when dealing with quadratic stability and stabilizability, and at our knowledge the only contributions on this subject are [7], [15].

In this paper we consider the the stabilizability property of DPSS switching between p (unstable) susbsystems. We focus on state-feedback switching strategies, which are based on (either linear or quadratic) copositive Lyapunov functions, and prove that they stabilize the system under the simple condition that at each time instant the Lyapunov function decreases for (at least) one of the p subsystems. Equivalent conditions for the existence of stabilizing switching strategies based on linear copositive functions are provided, and it is shown that when any of these conditions is satisfied then stabilizing strategies, based either on positive definite quadratic functions or, more generally, on quadratic copositive functions, can be found.

Before proceeding, we introduce some notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so we adopt the notation $A \geq 0$. If, in addition, A has at least one positive entry the matrix is *positive* ($A > 0$), while if all its entries are positive it is *strictly positive* ($A \gg 0$).

A square matrix A is said to be *Metzler* if its off-diagonal entries are nonnegative. A square matrix A is *Schur* if all its eigenvalues lie within the unit circle, and it is *Hurwitz* if all its eigenvalues have negative real part. It is easily seen [5] that $A \in \mathbb{R}^{n\times n}$ is a positive Schur matrix if and only if $A - I_n$ is Metzler Hurwitz.

$\mathbf{1}_n$ is the n-dimensional vector with all entries equal to 1. Given a positive integer p, we set $[1,p] := \{1,2,\dots,p\}$.

A square symmetric matrix P is positive definite ($\succ 0$) if for every nonzero vector $\mathbf{x}$, of compatible dimension, $\mathbf{x}^\top P\mathbf{x} > 0$, and positive semidefinite ($\succeq 0$) if for every nonzero vector $\mathbf{x}$, of compatible dimension, $\mathbf{x}^\top P\mathbf{x} \geq 0$. P is negative (semi)definite ($\prec 0$) if $-P$ is positive (semi)definite.

II. Stabilizability

The class of *discrete-time positive switched systems* we consider in this paper is described by the following equation

$$\mathbf{x}(t+1) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{1}$$

where $\mathbf{x}(t)$ denotes the value of the n-dimensional state variable at time t, σ is an arbitrary switching sequence, taking values in the set $[1,p]$, and for each $i \in [1,p]$ the matrix A_i is the system matrix of a discrete-time positive system, which means that A_i is an $n \times n$ positive matrix. The initial condition $\mathbf{x}(0)$ is assumed to be nonnegative.

For this class of systems we introduce the concept of stabilizability, also known in the literature on (general) switched systems [22] as pointwise asymptotic stabilizablility.

Definition 1: The DPSS (1) is *stabilizable* if for every positive initial state $\mathbf{x}(0)$ there exists a switching sequence $\sigma : \mathbb{Z}_+ \to [1,p]$ such that the corresponding state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

Clearly, the stabilization problem is a non-trivial one only if all matrices A_i's are not Schur. So, in the following, we will steadily make this assumption.

As clarified in the previous definition, the stabilizing switching sequence σ (depends on the initial state $\mathbf{x}(0)$, and) is a function of time. So, it can be thought of as an open-loop control action that we apply to the system in order to ensure the converge to zero of its state evolution. In [7] it has been shown that such a switching sequence can always be

Ettore Fornasini and Maria Elena Valcher are with the Dipartimento di Ingegneria dell'Informazione, Università di Padova, via Gradenigo 6/B, 35131 Padova, Italy, {fornasini,meme}@dei.unipd.it.

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periodic and independent of the initial state. Such a solution, however, is not robust and an alternative solution can be that of searching for a stabilizing switching sequence whose value at time t depends on the specific value that a suitable Lyapunov function takes on the state $\mathbf{x}(t)$.

More specifically, we search for a copositive Lyapunov function $V(\mathbf{x})$ (by this meaning a function that takes positive values on the positive states, and is zero in the origin) such that

$$\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) < 0, \qquad \forall\ \mathbf{x} > 0, \tag{2}$$

where

$$\Delta V_i(\mathbf{x}) := V(A_i\mathbf{x}) - V(\mathbf{x}). \tag{3}$$

This idea will be explored in the next two sections.

III. Lyapunov functions for DPSSs

In this section, we want to investigate what conditions on the positive matrices $A_i, i \in [1,p]$, ensure the existence of different kinds of copositive Lyapunov functions for the DPSS (1) that satisfy (2). In detail, we will focus on quadratic copositive functions $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$, P being a symmetric matrix, and on linear copositive functions $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$, $\mathbf{v}$ a vector that is necessarily strictly positive. Clearly, quadratic positive definite functions are a subset of quadratic copositive functions.

Proposition 1: Let $\mathcal{A} := \{A_1, A_2, \dots, A_p\}$ be a set of $n\times n$ positive matrices. If

a) there exist a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ and $\alpha_i \in [0,1]$, with $\sum_{i=1}^p \alpha_i = 1$, such that for every $\mathbf{x} > 0$

$$\sum_{i=1}^p \alpha_i \Delta V_i(\mathbf{x}) = \sum_{i=1}^p \alpha_i \mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} < 0,$$

then any of the following equivalent facts holds:

b1) $\exists\ \alpha_i \in [0,1]$, with $\sum_{i=1}^p \alpha_i = 1$, such that $\sum_{i=1}^p \alpha_i A_i$ is Schur;

b2) there exists a linear copositive function $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$ and $\alpha_i \in [0,1]$, with $\sum_{i=1}^p \alpha_i = 1$, such that for all $\mathbf{x} > 0$

$$\Delta V_\alpha(\mathbf{x}) := \left[\mathbf{v}^\top \left(\sum_{i=1}^p \alpha_i A_i\right) - \mathbf{v}^\top\right]\mathbf{x} < 0;$$

b3) there exists a linear copositive function $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$ such that for every $\mathbf{x} > 0$

$$\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]}\ \mathbf{v}^\top (A_i - I_n)\mathbf{x} < 0;$$

b4) there exists a quadratic copositive function of rank 1 $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ (by this meaning that $\mathrm{rank} P = 1$) such that for every $\mathbf{x} > 0$,

$$\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]}\ \mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} < 0.$$

If any of the equivalent conditions b1) ÷ b4) holds, then

c) there exists a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ such that for every $\mathbf{x} > 0$

$$\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]}\ \mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} < 0.$$

If c) holds, then

d) there exists a quadratic copositive function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ such that for every $\mathbf{x} > 0$

$$\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]}\ \mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} < 0.$$

Proof: a) ⇒ b1) The proof follows the same reasoning adopted in [8]. If $P \succ 0$, then

$$\begin{bmatrix} A_i^\top P A_i & A_i^\top P \\ P A_i & P \end{bmatrix} = \begin{bmatrix} A_i^\top \\ I_n \end{bmatrix} P \begin{bmatrix} A_i & I_n \end{bmatrix} \succeq 0,\ \forall\ i \in [1,p].$$

Consequently

$$\sum_{i=1}^p \alpha_i \begin{bmatrix} A_i^\top P A_i & A_i^\top P \\ P A_i & P \end{bmatrix} = \begin{bmatrix} \left(\sum_{i=1}^p \alpha_i A_i^\top P A_i\right) & \left(\sum_{i=1}^p \alpha_i A_i^\top\right) P \\ P\left(\sum_{i=1}^p \alpha_i A_i\right) & P \end{bmatrix} \succeq 0.$$

By the Schur complement's formula, this implies that for every $\mathbf{x}$, and hence, in particular, for every $\mathbf{x} > 0$,

$$\mathbf{x}^\top \left[\left(\sum_{i=1}^p \alpha_i A_i^\top P A_i\right) - \left(\sum_{i=1}^p \alpha_i A_i^\top\right) P \left(\sum_{i=1}^p \alpha_i A_i\right)\right]\mathbf{x} \ge 0,$$

namely

$$\mathbf{x}^\top \left[\sum_{i=1}^p \alpha_i (A_i^\top P A_i - P)\right]\mathbf{x} \ge \mathbf{x}^\top \left[\left(\sum_{i=1}^p \alpha_i A_i^\top\right) P \left(\sum_{i=1}^p \alpha_i A_i\right) - P\right]\mathbf{x}.$$

As the left hand-side is negative for every $\mathbf{x} > 0$, so is the right hand-side. But this implies that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic copositive function such that $V((\sum_{i=1}^p \alpha_i A_i)\mathbf{x}) < V(\mathbf{x})$ for every $\mathbf{x} > 0$. So, $\sum_{i=1}^p \alpha_i A_i$ is a Schur matrix.

We now prove that b1) ÷ b4) are equivalent conditions.

b1) ⇔ b2) Set $A := \sum_{i=1}^p \alpha_i A_i$ and notice that $\sum_{i=1}^p \alpha_i I_n = I_n$. The equivalence is based on two facts: (1) A is positive Schur if and only if $\tilde{A} := A - I_n$ is a Metzler Hurwitz matrix; (2) a Metzler matrix $\tilde{A}$ is Hurwitz if and only if [3], [11] there exists a vector $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top \tilde{A} \ll 0$.

b2) ⇒ b3) From b2) it follows that, for every positive vector $\mathbf{x}$, one gets

$$\left[\mathbf{v}^T \sum_{i=1}^p \alpha_i (A_i - I_n)\right]\mathbf{x} = \sum_{i=1}^p \alpha_i \left[\mathbf{v}^T (A_i - I_n)\mathbf{x}\right] < 0,$$

whence $\min_{i\in[1,p]}\ \mathbf{v}^\top (A_i - I_n)\mathbf{x} < 0$.

b3) $\Rightarrow$ b2) By assumption, there exists a strictly positive vector $\mathbf{v}$ such that for every $\mathbf{x} > 0$ the vector

$$\begin{bmatrix} \mathbf{v}^\top (A_1 - I_n) \\ \vdots \\ \mathbf{v}^\top (A_p - I_n) \end{bmatrix} \mathbf{x} \in \mathbb{R}^{p\times 1}$$

has at least one negative entry. So, once we set

$$W := \begin{bmatrix} \mathbf{v}^\top (A_1 - I_n) \\ \vdots \\ \mathbf{v}^\top (A_p - I_n) \end{bmatrix},$$

we can claim that no positive vector $\mathbf{x}$ can be found such that $W\mathbf{x} \geq 0$. But then, by Lemma 1, in the Appendix, a positive vector $\mathbf{y}$ exists such that $\mathbf{y}^T W \ll 0$. As it entails no loss of generality rescaling $\mathbf{y}$ so that its entries sum up to 1, this means that nonnegative coefficients α_i exist, with $\sum_{i=1}^p \alpha_i = 1$, such that

$$0 \gg [\,\alpha_1 \quad \dots \quad \alpha_p\,]\, W = \mathbf{v}^T \sum_{i=1}^p \alpha_i (A_i - I_n),$$

thus proving b2).

b3) $\Rightarrow$ b4) Let $\mathbf{v}$ be a strictly positive vector such that for every $\mathbf{x} > 0$ condition $\mathbf{v}^\top A_i \mathbf{x} < \mathbf{v}^\top \mathbf{x}$ holds for at least one index $i \in [1,p]$. This implies that for every $\mathbf{x} > 0$, condition

$$\mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top A_i \mathbf{x} = |\mathbf{v}^\top A_i \mathbf{x}|^2 < |\mathbf{v}^\top \mathbf{x}|^2 = \mathbf{x}^\top \mathbf{v}\mathbf{v}^\top \mathbf{x}$$

holds for at least one index $i \in [1,p]$. So, b4) is satisfied for $P := \mathbf{v}\mathbf{v}^\top$.

b4) $\Rightarrow$ b3) If rank $P = 1$ and $P = P^\top$, then P can be expressed as $P = \mathbf{v}\mathbf{v}^\top$, for some vector $\mathbf{v}$. As $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} > 0$, it follows that $\mathbf{v}$ has entries which are all nonzero and of the same sign. So, it entails no loss of generality assuming that they are all positive. On the other hand, from the fact that at every point $\mathbf{x} > 0$ there exists an index $i \in [1,p]$ such that

$$\mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x} = (\mathbf{x}^\top A_i^\top \mathbf{v})(\mathbf{v}^\top A_i \mathbf{x}) - (\mathbf{x}^\top \mathbf{v})(\mathbf{v}^\top \mathbf{x}) < 0$$

namely

$$|\mathbf{v}^\top A_i \mathbf{x}|^2 = (\mathbf{x}^\top A_i^\top \mathbf{v})(\mathbf{v}^\top A_i \mathbf{x}) < (\mathbf{x}^\top \mathbf{v})(\mathbf{v}^\top \mathbf{x}) = |\mathbf{v}^\top \mathbf{x}|^2,$$

and by the nonnegativity of both $\mathbf{v}^\top A_i \mathbf{x}$ and $\mathbf{v}^\top \mathbf{x}$, it follows that $\mathbf{v}^\top A_i \mathbf{x} < \mathbf{v}^\top \mathbf{x}$. This proves that condition b3) holds.

b4) $\Rightarrow$ c) Assume w.l.o.g. that the matrix P that makes b4) satisfied is expressed as $P = \mathbf{v}\mathbf{v}^\top$ for some $\mathbf{v} \gg 0$. Set $\tilde{P} := P + \varepsilon I_n$, with $\varepsilon > 0$. Clearly, $\tilde{P}$ is symmetric. We want to show that $\tilde{P}$ is positive definite for every choice of $\varepsilon > 0$. Indeed, for every $\mathbf{x} \neq 0$,

$$\mathbf{x}^\top \tilde{P} \mathbf{x} = \mathbf{x}^\top P \mathbf{x} + \varepsilon \|\mathbf{x}\|_2^2 = (\mathbf{v}^\top \mathbf{x})^2 + \varepsilon \|\mathbf{x}\|_2^2 > 0.$$

Consider, now, the compact set $\mathcal{S} := \{\mathbf{x} \in \mathbb{R}^n_+ : \|\mathbf{x}\|_2 = 1\}$. The two functions

$$\begin{aligned} f(\mathbf{x}) &= \min_{i\in[1,p]} \mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x}, \\ g(\mathbf{x}) &= \max_{i\in[1,p]} |\mathbf{x}^\top [A_i^\top A_i - I_n]\mathbf{x}|, \end{aligned}$$

are continuous functions in $\mathcal{S}$. So, by Weierstrass' theorem, both functions have maximum in $\mathcal{S}$ and it is easily seen that

$$\max_{\mathbf{x}\in\mathcal{S}} g(\mathbf{x}) = \max_{\mathbf{x}\in\mathcal{S}} \max_{i\in[1,p]} |\mathbf{x}^\top [A_i^\top A_i - I_n]\mathbf{x}| = M \geq 0,$$

while, by the assumption b4),

$$\max_{\mathbf{x}\in\mathcal{S}} f(\mathbf{x}) = \max_{\mathbf{x}\in\mathcal{S}} \min_{i\in[1,p]} \mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x} = -\delta < 0.$$

Let ε be a positive number such that $\varepsilon M < \delta$. It is easily seen that, for every $\mathbf{x} \in \mathcal{S}$,

$$\begin{aligned} & \min_{i\in[1,p]} \mathbf{x}^\top (A_i^\top \tilde{P} A_i - \tilde{P})\mathbf{x} \\ = \; & \min_{i\in[1,p]} \left[\mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} + \varepsilon \left(\mathbf{x}^\top (A_i^\top A_i - I_n)\mathbf{x}\right)\right] \\ \leq \; & \max_{\mathbf{x}\in\mathcal{S}} \min_{i\in[1,p]} \left[\mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} \right. \\ & \left. + \varepsilon \left(\mathbf{x}^\top (A_i^\top A_i - I_n)\mathbf{x}\right)\right] \\ \leq \; & \max_{\mathbf{x}\in\mathcal{S}} \min_{i\in[1,p]} \left[\mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x}\right] \\ & + \varepsilon \cdot \max_{\mathbf{x}\in\mathcal{S}} \max_{i\in[1,p]} |(\mathbf{x}^\top (A_i^\top A_i - I_n)\mathbf{x}| \\ = \; & -\delta + \varepsilon M < 0. \end{aligned}$$

Clearly, for every $\mathbf{x} > 0$, one finds

$$\min_{i\in[1,p]} \mathbf{x}^\top (A_i^\top \tilde{P} A_i - \tilde{P})\mathbf{x} \leq (-\delta + \varepsilon M)\|x\|_2^2 < 0,$$

and hence the result is proved.
c) $\Rightarrow$ d) is obvious. ∎

Remark 1: While condition c) implies d), the converse is not true, as shown by the following example.

Example 1: Consider the positive matrices

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

It is easy to verify that the symmetric matrix of rank 2

$$P = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

is such that for every $\mathbf{x} \in \mathbb{R}^2_+, \mathbf{x} > 0$, we have $\mathbf{x}^\top P \mathbf{x} > 0$ and either

$$\mathbf{x}^\top (A_1^\top P A_1 - P)\mathbf{x} = 6x_1^2 - 6x_1x_2 - 2x_2^2 < 0$$

or

$$\mathbf{x}^\top (A_2^\top P A_2 - P)\mathbf{x} = -2x_1^2 - 6x_1x_2 + 6x_2^2 < 0.$$

On the other hand, no symmetric positive definite matrix P can be found such that in every point $\mathbf{x} > 0$ either $\mathbf{x}^\top (A_1^\top P A_1 - P)\mathbf{x}$ or $\mathbf{x}^\top (A_2^\top P A_2 - P)\mathbf{x}$ is negative. Indeed, if such a matrix would exist, it could be described w.l.o.g. in the form

$$P = \begin{bmatrix} 1 & c \\ c & b \end{bmatrix},$$

with $b > c^2$, and in every nonzero point either one of the following inequalities would be satisfied:

$$\begin{aligned} \mathbf{x}^\top (A_1^\top P A_1 - P)\mathbf{x} &= 3x_1^2 - 2cx_1x_2 - bx_2^2 < 0 \\ \mathbf{x}^\top (A_2^\top P A_2 - P)\mathbf{x} &= -x_1^2 - 2cx_1x_2 + 3bx_2^2 < 0. \end{aligned}$$

Since for $x_1 = 0$ the first equation is obviously satisfied, we assume now $x_1 \neq 0$ and set $y := x_2/x_1$. So, the previous inequalities become:

$$p_1(y) := -by^2 - 2cy + 3 \quad < \quad 0 \qquad (4)$$
$$p_2(y) := 3by^2 - 2cy - 1 \quad < \quad 0, \qquad (5)$$

Upon observing that $b = c^2 + \varepsilon$ for some $\varepsilon > 0$, our goal is that of proving that, for every choice of $c \in \mathbb{R}$ and $\varepsilon > 0$, there exists $y > 0$ such that both $-by^2 - 2cy + 3 \geq 0$ and $3by^2 - 2cy - 1 \geq 0$. Indeed, the two zeros of the polynomial $p_1(y)$ are

$$\lambda_{-,+} := \frac{-2c \pm \sqrt{4c^2 + 12b}}{2b} = \frac{-c \pm \sqrt{4c^2 + 3\varepsilon}}{c^2 + \varepsilon},$$

and it is easy to prove that $\lambda_- < 0 < \lambda_+$. On the other hand, polynomial $p_2(y)$ has zeros

$$\mu_{-,+} := \frac{2c \pm \sqrt{4c^2 + 12b}}{6b} = \frac{c \pm \sqrt{4c^2 + 3\varepsilon}}{3(c^2 + \varepsilon)}.$$

In order to ensure that in every $y \geq 0$ either (4) or (5) holds, it should be true that $\mu_- < 0$ and $\lambda_+ < \mu_+$. The first condition is easily proved to be verified, however condition $\lambda_+ < \mu_+$ amounts to

$$\frac{-c + \sqrt{4c^2 + 3\varepsilon}}{c^2 + \varepsilon} < \frac{c + \sqrt{4c^2 + 3\varepsilon}}{3(c^2 + \varepsilon)},$$

namely

$$2c > \sqrt{4c^2 + 3\varepsilon},$$

a condition that, of course, is never verified. So, for every choice of c and $\varepsilon > 0$ all positive pairs (x_1, x_2) such that $\mu_+ < x_2/x_1 < \lambda_+$ make both $\Delta V_1(\mathbf{x})$ and $\Delta V_2(\mathbf{x})$ positive.

IV. STATE-FEEDBACK STABILIZATION

In Section III we have investigated under which conditions on the positive matrices $A_i, i \in [1, p]$, a copositive Lyapunov function $V(\mathbf{x})$ can be found for which (2) holds, with $\Delta V_i(\mathbf{x})$ defined as in (3). This amounts to saying that a function $V(\mathbf{x})$ can be found such that, in every nonzero point $\mathbf{x}$ of the positive orthant, the difference $V(A_i\mathbf{x}) - V(\mathbf{x})$ is negative for at least one index $i \in [1, p]$. So, the function $V(\mathbf{x})$ represents an "energy function" that can always be decreased at every step along the state trajectories (within the positive orthant), by suitably choosing when and how to switch.

Accordingly, we can adopt a "min-projection" switching strategy [20] (also known as "variable structure control" [23])[1]

$$\sigma(\mathbf{x}(t)) := \min\{k : \Delta V_k(\mathbf{x}(t)) \leq \Delta V_i(\mathbf{x}(t)), \forall\ i \in [1, p]\}. \quad (6)$$

[1] It is worthwhile noticing that the switching law we chose assigns a unique value to $\sigma(\mathbf{x})$ in every point $\mathbf{x} > 0$. Indeed, whenever $\Delta V_1(\mathbf{x}) = \Delta V_2(\mathbf{x})$, the switching rule sets $\sigma(\mathbf{x}) = 1$. The opposite choice, or the choice of a conservative policy that keeps memory of the value of σ at the previous time instant, or even a random choice would not make any difference in terms of convergence.

It is rather intuitive that this strategy is stabilizing, by this meaning that it ensures that the state evolution converges to the origin for every choice of the positive initial state. We want to provide a formal proof for the classes of functions that we have considered in Proposition 1.

We consider first the case when $V(\mathbf{x})$ is a quadratic copositive Lyapunov function.

Proposition 2: Given a discrete-time positive switched system (1), if there exists a quadratic copositive Lyapunov function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ satisfying (2), then the state feedback switching rule (6) stabilizes the system.

Proof: The function $f(\mathbf{x}) := \min_{i\in[1,p]} \Delta V_i(\mathbf{x})$ is a continuous function that takes negative values in every point of the compact set

$$\mathcal{S} := \mathbb{R}^n_+ \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top P\mathbf{x} = 1\}.$$

So, by Weierstrass' Theorem, $\max_{\mathbf{x}\in\mathcal{S}} f(\mathbf{x}) = -\nu$, with $0 < \nu \leq 1$, and this ensures that for every positive state $\mathbf{x}$, $f(\mathbf{x}) \leq -\nu\mathbf{x}^\top P\mathbf{x}$. Consequently

$$\begin{aligned} V(\mathbf{x}(t+1)) &= V(\mathbf{x}(t)) + f(\mathbf{x}(t)) \leq (1-\nu)\mathbf{x}^\top(t)P\mathbf{x}(t) \\ &\leq\ (1-\nu)^{t+1}\mathbf{x}^\top(0)P\mathbf{x}(0), \end{aligned}$$

and hence $V(\mathbf{x}(t))$ converges to zero, thus guaranteeing that $\mathbf{x}(t)$ converges to zero in turn. ■

Remark 2: Both the result and the proof of Proposition 2 obviously extend to the class of quadratic positive definite Lyapunov functions $V(\mathbf{x})$.

We consider, now, the case of linear copositive Lyapunov functions.

Proposition 3: Given a discrete-time positive switched system (1), if there exists a linear copositive Lyapunov function $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$ satisfying (2), then the state feedback switching rule (6) stabilizes the system.

Proof: The proof follows the same lines of the previous one upon assuming

$$\mathcal{S} := \mathbb{R}^n_+ \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{v}^\top\mathbf{x} = 1\}.$$

■

For the class of positive switched systems (1) for which a convex Schur combination of the matrices $A_i, i \in [1, p]$, can be found, we can apply different state feedback switching strategies. Indeed, as a consequence of Proposition 1, we may either resort to a linear copositive function or to a quadratic copositive function (or rank 1 or of higher rank) or to a quadratic positive definite function. So, it is natural to ask which of the available strategies ensures the best performances.

We first notice that the switching strategies based on linear copositive and on quadratic copositive functions of rank 1 are just the same. In fact, as clarified in the proof of Proposition

1, the matrices $P = P^\top$ of rank 1 that satisfy condition b4) of Proposition 1 are those and those only that can be expressed as $P = \mathbf{v}\mathbf{v}^\top$ for some vector $\mathbf{v}$ satisfying b3) of the same proposition. On the other hand, by the nonnegativity of the quantities involved,

$$\begin{aligned} & \min\{k : \mathbf{v}^\top(A_k - I_n)\mathbf{x} \leq \mathbf{v}^\top(A_i - I_n)\mathbf{x}, \forall\ i\} \\ = \ & \min\{k : \mathbf{v}^\top A_k\mathbf{x} \leq \mathbf{v}^\top A_i\mathbf{x}, \forall\ i\} \\ = \ & \min\{k : \mathbf{x}^\top A_k^\top \mathbf{v}\mathbf{v}^\top A_k\mathbf{x} \leq \mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top A_i\mathbf{x}, \forall\ i\} \\ = \ & \min\{k : \mathbf{x}^\top(A_k^\top \mathbf{v}\mathbf{v}^\top A_k - \mathbf{v}\mathbf{v}^\top)\mathbf{x} \\ & \qquad \leq \mathbf{x}^\top(A_i^\top \mathbf{v}\mathbf{v}^\top A_i - \mathbf{v}\mathbf{v}^\top)\mathbf{x}, \forall\ i\}, \end{aligned}$$

and hence the switching sequences (6) based on $\mathbf{v}^\top\mathbf{x}$ and on $\mathbf{x}^\top\mathbf{v}\mathbf{v}^\top\mathbf{x}$ are just the same.

On the other hand, we may design switching strategies based on the broader class of quadratic copositive Lyapunov functions (of arbitrary rank) fulfilling condition d). Clearly, this class of switching laws encompasses those based on quadratic copositive functions of rank 1, and hence it ensures convergence performances at least as good as the previous ones.

Similarly, since the class of positive definite functions is included in the class of quadratic copositive functions, the stabilizing switching laws described in Remark 2 are a subset of those described in Proposition 2. So, resorting to switching laws based on quadratic copositive functions allows to obtain better converge performances.

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APPENDIX

Lemma 1 (see [1], Corollary 3.49): Let W be an $n \times p$ real matrix. Then one and only one of the following alternatives holds:

a) $\exists\ \mathbf{y} > 0$ such that $\mathbf{y}^\top W \ll 0$;
c) $\exists\ \mathbf{x} > 0$ such that $W\mathbf{x} \geq 0$ (namely $W\mathbf{x} \in \mathbb{R}_+^{p\times 1}$).

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