# On the stability and stabilizability of a class of continuous-time positive switched systems with rank one difference

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***Abstract*— Given a single-input continuous-time positive system, described by a pair $(A, \mathbf{b})$, with $A$ a diagonal matrix, we investigate under what conditions there exist state-feedback laws $u(t) = \mathbf{c}^\top \mathbf{x}(t)$ that make the resulting controlled system positive and asymptotically stable, namely $A + \mathbf{b}\mathbf{c}^\top$ Metzler and Hurwitz. In the second part of the paper we assume that the state-space model switches among different state-feedback laws $\mathbf{c}_i^\top, i = 1, 2, \dots, p$, each of them ensuring the positivity, and show that the asymptotic stability of the switched system is equivalent to the asymptotic stability of all the subsystems, while its stabilizability is equivalent to the existence of an asymptotically stable subsystem.**

## I. INTRODUCTION

Recent years have seen a growing interest in systems that are subject to a positivity constraint on their dynamical variables. There are several motivations for this interest, coming from different domains of science and technology. In fact, the positivity assumption is a natural one when describing physical, biological or economical processes whose variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc. [3].

By a continuous-time positive switched system (CPSS) we mean a dynamic system consisting of a family of continuous-time positive state-space models and a switching law, specifying when and how the switching takes place. Switching among different positive subsystems naturally arises as a way to formalize the fact that the behavior of a positive system changes under different operating conditions, and is therefore represented by different mathematical structures.

Recently, CPSSs have been the object of an intense research activity, mainly focused on stability [4], [7], [9], [11], [12], [13] and stabilizability [1], [2], [19]. Special attention has been devoted to the class of CPSSs that switch among subsystems whose matrices differ by a rank one matrix [8], [12], [13], [14], [15], [16]. The reason for the interest in these systems is twofold. On the one hand, they can be thought of as the possible configurations one obtains from a given single-input system, when applying different state-feedback laws that ensure the positivity of the resulting closed-loop system. For this reason, the subsystem matrices can be denoted by $A + \mathbf{b}\mathbf{c}_i^\top, i \in \{1, 2, \dots, p\}$. On the other hand, interesting connections have been highlighted [15] between the quadratic stability of CPSSs, switching between two subsystems of matrices $A$ and $A + \mathbf{b}\mathbf{c}^\top$, and the SISO circle criterion for the transfer function $\mathbf{c}^\top(sI_n - A)^{-1}\mathbf{b}$.

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In this paper we investigate stability and stabilizability of this class of CPSSs, under the additional assumption that the matrix $A$ is a diagonal one. We show that asymptotic stability is equivalent to the Hurwitz property of all the matrices $A + \mathbf{b}\mathbf{c}_i^\top$, while stabilizability is equivalent to the existence of an index $i$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz. As a preliminary step, in section II, we consider a continuous-time single-input state space model with diagonal state space matrix $A$, and address the problem of ensuring the positivity and the asymptotic stability of the resulting system $\dot{\mathbf{x}}(t) = (A + \mathbf{b}\mathbf{c}^\top)\mathbf{x}(t)$.

**Notation.** $\mathbb{R}_+$ is the semiring of nonnegative real numbers and, for any pair of positive integers $k, n$ with $k \le n$, $[k, n]$ is the set of integers $\{k, k+1, \dots, n\}$. The $i$th entry of a vector $\mathbf{v}$ is denoted by $[\mathbf{v}]_i$. We denote by $\mathbf{1}_n$ the $n$-dimensional vector with all unitary entries, and by $\mathbf{e}_i$ the $i$th canonical vector in $\mathbb{R}^n$ ($n$ being clear from the context). A matrix (in particular, a vector) $A$ with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so we adopt the notation $A \ge 0$. If, in addition, $A$ has at least one positive entry, the matrix is *positive* ($A > 0$), while if all its entries are positive, it is *strictly positive* ($A \gg 0$). A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. A square matrix $A$ is Hurwitz if all its eigenvalues have negative real part. In particular, a Metlzer matrix is Hurwitz if and only if the coefficients of its characteristic polynomial are all positive [3].

## II. DIAGONAL SYSTEMS AND POSITIVITY PRESERVING STABILIZING STATE-FEEDBACK LAWS

Consider a single-input state-space model

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}u(t), \qquad t \in \mathbb{R}_+, \tag{1}$$

where $\mathbf{x}(t)$ and $u(t)$ are the $n$-dimensional state variable and the scalar input, respectively, at time $t$. We assume

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \qquad \lambda_i \in \mathbb{R}.$$

Our first goal is to investigate under what conditions the state feedback law

$$u(t) = \mathbf{c}^\top \mathbf{x}(t)$$

makes the resulting autonomous system positive, by this meaning that the matrix $A + \mathbf{b}\mathbf{c}^\top$ is Metzler. To this end, it is convenient to assume that

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_+ \\ \mathbf{b}_- \\ \mathbf{0} \end{bmatrix}, \qquad \text{with } \mathbf{b}_+ \gg 0 \text{ and } \mathbf{b}_- \ll 0. \quad (2)$$

The matrix $A$ and the vector $\mathbf{c}$ can be accordingly partitioned as follows:

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix} \qquad \mathbf{c}^\top = [\, \mathbf{c}_1^\top \quad \mathbf{c}_2^\top \quad \mathbf{c}_3^\top \,]. \quad (3)$$

Notice that we can always reduce ourselves to this situation by resorting to a suitable permutation of the state components. We set $n_+ := \dim \mathbf{b}_+$ and $n_- := \dim \mathbf{b}_-$. Clearly $n_+ + n_- \leq n$.

Since $A$ is diagonal and the Metzler property is a constraint on the off-diagonal entries, $A + \mathbf{b}\mathbf{c}^\top$ is Meztler if and only if $\mathbf{b}\mathbf{c}^\top$ is.

*Lemma 1:* Given two vectors $\mathbf{b}$ and $\mathbf{c} \in \mathbb{R}^n$, partitioned as in (2) and (3), the matrix $\mathbf{b}\mathbf{c}^\top$ is Metzler if and only if the following conditions hold:

Case 1: $[n_+ \geq 1$ and $n_- \geq 1]$

- if $n_+ = 1$ then $\mathbf{c}_1 \leq 0$, otherwise $\mathbf{c}_1 = 0$;
- if $n_- = 1$ then $\mathbf{c}_2 \geq 0$, otherwise $\mathbf{c}_2 = 0$;
- if $n_+ + n_- < n$ then $\mathbf{c}_3 = 0$.

Case 2: $[n_+ \geq 1$ and $n_- = 0]$

- if $n_+ = 1$ then $\mathbf{c}_1$ is arbitrary, otherwise $\mathbf{c}_1 \geq 0$;
- if $n_+ < n$ then $\mathbf{c}_3 \geq 0$.

Case 3: $[n_+ = 0$ and $n_- \geq 1]$

- if $n_- = 1$ then $\mathbf{c}_2$ is arbitrary, otherwise $\mathbf{c}_2 \leq 0$;
- if $n_- < n$ then $\mathbf{c}_3 \leq 0$.

*Proof:* We only prove Case 1, as the others are obvious. [Necessity] Consider the matrix

$$\mathbf{b}\mathbf{c}^\top = \begin{bmatrix} \mathbf{b}_+\mathbf{c}_1^\top & \mathbf{b}_+\mathbf{c}_2^\top & \mathbf{b}_+\mathbf{c}_3^\top \\ \mathbf{b}_-\mathbf{c}_1^\top & \mathbf{b}_-\mathbf{c}_2^\top & \mathbf{b}_-\mathbf{c}_3^\top \\ 0 & 0 & 0 \end{bmatrix}.$$

In order for $\mathbf{b}\mathbf{c}^\top$ to be Metzler, it is necessary that $\mathbf{c}_1 \leq 0, \mathbf{c}_2 \geq 0$. Moreover, the following conditions must simultaneously hold: $\mathbf{c}_3 \geq 0$ and $\mathbf{c}_3 \leq 0$, which implies $\mathbf{c}_3 = 0$. On the other hand, $\mathbf{b}_+\mathbf{c}_1^\top \leq 0$ must be Metzler in turn. So, if $n_+ > 1$ this is possible only if $\mathbf{c}_1 = 0$. By the same reasoning, if $n_- > 1$, then $\mathbf{b}_-\mathbf{c}_2^\top \leq 0$ can be Metzler only if $\mathbf{c}_2 = 0$.

[Sufficiency] The proof is obvious. ∎

We now investigate under what conditions the state-feedback law makes the resulting system not only positive but also asymptotically stable, which means that $A + \mathbf{b}\mathbf{c}^\top$ is both Metzler and Hurwitz. To this end, we first address Case 1 of Lemma 1. We distinguish the following subcases:

a) $[n_+ > 1$ and $n_- > 1]$. If so, $A + \mathbf{b}\mathbf{c}^\top$ is Metzler if and only if $\mathbf{c} = 0$, and hence $A + \mathbf{b}\mathbf{c}^\top$ coincides with $A$. This implies that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler Hurwitz if and only if $A$ is Hurwitz.

b) $[n_+ > 1$ and $n_- = 1]$ or $[n_+ = 1$ and $n_- > 1]$. In the first case, $\mathbf{c}_1 = 0$, $\mathbf{c}_3 = 0$ and $c_2$ is a nonnegative scalar. Consequently,

$$A + \mathbf{b}\mathbf{c}^\top = \begin{bmatrix} A_1 & \mathbf{b}_+ c_2 & 0 \\ 0 & A_2 + b_- c_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

is a upper triangular matrix, which is Hurwitz if and only if $A_1, A_2 + b_- c_2$ and $A_3$ are Hurwitz. The second case is symmetric and leads to a lower triangular matrix

$$A + \mathbf{b}\mathbf{c}^\top = \begin{bmatrix} A_1 + b_+ c_1 & 0 & 0 \\ \mathbf{b}_- c_1 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

that is Hurwitz if and only if $A_1 + b_+ c_1, A_2$ and $A_3$ are Hurwitz.

c) $[n_+ = 1$ and $n_- = 1]$. In this case, $A + \mathbf{b}\mathbf{c}^\top$ is Metzler if and only if $A + \mathbf{b}\mathbf{c}^\top$ takes the form

$$A + \mathbf{b}\mathbf{c}^\top = \begin{bmatrix} \lambda_1 + b_+ c_1 & b_+ c_2 & 0 \\ b_- c_1 & \lambda_2 + b_- c_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

with $b_+ > 0, b_- < 0, c_1 \leq 0$ and $c_2 \geq 0$. This matrix is Hurwitz if and only if both $A_3$ and the $2 \times 2$ matrix

$$M := \begin{bmatrix} \lambda_1 + b_+ c_1 & b_+ c_2 \\ b_- c_1 & \lambda_2 + b_- c_2 \end{bmatrix}$$

are Hurwitz.

*Remark 1:* In case c) of the previous discussion, if the diagonal matrix $A$ is Hurwitz, then every vector $\mathbf{c} \in \mathbb{R}^n_+$ such that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler makes $A + \mathbf{b}\mathbf{c}^\top$ Hurwitz. Indeed, the matrix $A + \mathbf{b}\mathbf{c}^\top$ takes the form

$$A + \mathbf{b}\mathbf{c}^\top = \begin{bmatrix} M & 0 \\ 0 & A_3 \end{bmatrix},$$

where $A_3$ is Hurwitz diagonal by the assumption on $A$. On the other hand, if we constrain $M$ to be Metzler, condition $\lambda_1 < 0$ and $\lambda_2 < 0$ ensures that $\mathrm{tr} M = (\lambda_1 + b_+ c_1) + (\lambda_2 + b_- c_2) \leq \lambda_1 + \lambda_2 < 0$, while $\det M = \lambda_1\lambda_2 + \lambda_1 b_- c_2^{(i)} + \lambda_2 b_+ c_1^{(i)} \geq \lambda_1\lambda_2 > 0$. Consequently, $M$ is Hurwitz.

In order to investigate Cases 2 and 3 of Lemma 1, we first address the case when $\mathbf{b}$ is a strictly positive vector of size $n > 1$.

*Proposition 1:* Given a diagonal matrix $A \in \mathbb{R}^{n \times n}$, $n > 1$, and vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, with $\mathbf{b} \gg 0$, the matrix $A + \mathbf{b}\mathbf{c}^\top$ is Metzler and Hurwitz if and only if

i) $\mathbf{c} \geq 0$;
ii) $A$ is Hurwitz;
iii) the following condition holds:

$$1 + \mathbf{c}^\top A^{-1} \mathbf{b} > 0 \quad (4)$$

*Proof:* $A + \mathbf{b}\mathbf{c}^\top$ is Metzler if and only if $\mathbf{b}\mathbf{c}^\top$ is Metzler and hence, by Lemma 1, if and only if $\mathbf{c} \geq 0$. But this implies that $A \leq A + \mathbf{b}\mathbf{c}^\top$ and hence if $A$ is not Hurwitz then neither $A + \mathbf{b}\mathbf{c}^\top$ can be Hurwitz. Finally, if $A + \mathbf{b}\mathbf{c}^\top$ and $A$ are Hurwitz, both $\det(sI - A - \mathbf{b}\mathbf{c}^\top)$ and $\det(sI - A)$ are positive when $s = 0$, and condition iii) follows from the identity

$$\det(sI - A - \mathbf{b}\mathbf{c}^\top) = \det(sI - A)(1 - \mathbf{c}^\top(sI - A)^{-1}\mathbf{b})$$

Viceversa, if i),ii), and iii) hold, condition i) implies that $\mathbf{c}^\top\mathbf{b}$, and hence $A + \mathbf{c}^\top\mathbf{b}$ are Metzler. On the other hand, condition ii) implies that the diagonal elements $\lambda_i$ are negative and, upon setting

$$\alpha_i := -\frac{[\mathbf{c}]_i[\mathbf{b}]_i}{\lambda_i} \geq 0$$

condition iii) implies $0 \leq \sum_{i=1}^n \alpha_i < 1$.
We therefore have

$$\begin{aligned}
& \det(sI - A - \mathbf{b}\mathbf{c}^\top) \\
= \; & \det(sI - A)(1 - \mathbf{c}^\top(sI - A)^{-1}\mathbf{b}) \\
= \; & \prod_{j=1}^n (s - \lambda_j)[1 + \sum_{i=1}^n \frac{\lambda_i\alpha_i}{s - \lambda_i}] \\
= \; & (1 - \sum_{i=1}^n \alpha_i) \prod_{j=1}^n (s - \lambda_j) + (\sum_{i=1}^n \alpha_i) \prod_{j=1}^n (s - \lambda_j) \\
+ \; & \sum_{i=1}^n \lambda_i\alpha_i \prod_{j\neq i} (s - \lambda_j) \\
= \; & (1 - \sum_{i=1}^n \alpha_i) \prod_{j=1}^n (s - \lambda_j) + \sum_{i=1}^n \alpha_i s \prod_{j\neq i} (s - \lambda_j)
\end{aligned}$$

First case: $\mathbf{c}$ is a strictly positive vector.

Assume, first, that $A$ has distinct diagonal entries $\lambda_i, i \in [1, n]$. It entails no loss of generality assuming that

$$0 > \lambda_1 > \lambda_2 > \ldots > \lambda_n,$$

since we can always reduce ourselves to this situation by means of a suitable permutation. We observe that

$$\det(sI_n - A - \mathbf{b}\mathbf{c}^\top) = d(s) - n(s),$$

where

$$\begin{aligned}
d(s) &:= \det(sI_n - A), && (5) \\
n(s) &:= \mathbf{c}^\top \mathrm{adj}(sI_n - A)\mathbf{b}. && (6)
\end{aligned}$$

Moreover, we can easily see that

$$\begin{aligned}
d(s) &= \prod_{i=1}^n (s - \lambda_i), \\
n(s) &= \sum_{i=1}^n [\mathbf{b}]_i[\mathbf{c}]_i \prod_{j\in[1,n], j\neq i} (s - \lambda_j).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
& \det(sI_n - A - \mathbf{b}\mathbf{c}^\top)\big|_{s=\lambda_1} \\
& \quad = 0 - \sum_{i=1}^n [\mathbf{b}]_i[\mathbf{c}]_i \prod_{j\in[1,n], j\neq i} (\lambda_1 - \lambda_j) \\
& \quad = -[\mathbf{b}]_1[\mathbf{c}]_1 \prod_{j\in[1,n], j\neq 1} (\lambda_1 - \lambda_j) < 0, \\
& \det(sI_n - A - \mathbf{b}\mathbf{c}^\top)\big|_{s=\lambda_2} \\
& \quad = 0 - \sum_{i=1}^n [\mathbf{b}]_i[\mathbf{c}]_i \prod_{j\in[1,n], j\neq i} (\lambda_2 - \lambda_j) \\
& \quad = -[\mathbf{b}]_2[\mathbf{c}]_2 \prod_{j\in[1,n], j\neq 2} (\lambda_2 - \lambda_j) > 0, \\
& \det(sI_n - A - \mathbf{b}\mathbf{c}^\top)\big|_{s=\lambda_3} \\
& \quad = 0 - \sum_{i=1}^n [\mathbf{b}]_i[\mathbf{c}]_i \prod_{j\in[1,n], j\neq i} (\lambda_3 - \lambda_j) \\
& \quad = -[\mathbf{b}]_3[\mathbf{c}]_3 \prod_{j\in[1,n], j\neq 1} (\lambda_3 - \lambda_j) < 0, \\
& \qquad \vdots
\end{aligned}$$

By the change of signs of the characteristic polynomial $\det(sI_n - A - \mathbf{b}\mathbf{c}^\top)$ on the negative real half-line, we can deduce that it always has (independently of the specific values of the $[\mathbf{c}]_i$'s) $n - 1$ negative real zeros, say $\tilde{\lambda}_2, \tilde{\lambda}_3, \ldots, \tilde{\lambda}_n$, with $\tilde{\lambda}_i \in (\lambda_i, \lambda_{i-1}), i \in [2, n]$. Therefore $A + \mathbf{b}\mathbf{c}^\top$ is Hurwitz if and only if the remaining (real) zero, $\tilde{\lambda}_1$, is in turn negative. Since the leading coefficient of $\det(sI_n - A - \mathbf{b}\mathbf{c}^\top)$ is positive, and hence this characteristic polynomial eventually takes positive values on the positive real axis, it follows that $A + \mathbf{b}\mathbf{c}^\top$ is Hurwitz if and only if $\tilde{\lambda}_1 \in (\lambda_1, 0)$. But this condition is verified if and only if $\det(sI_n - A - \mathbf{b}\mathbf{c}^\top)\big|_{s=0} > 0$, which means that (4) holds.

Assume, now, that the diagonal entries of $A$ are not distinct. Then we can denote by

$$0 > \mu_1 > \mu_2 > \ldots > \mu_r$$

the distinct diagonal entries of $A$, and by $k_1, k_2, \ldots, k_r \in \mathbb{Z}_+, k_i > 0$, the corresponding multiplicities. Then, by referring to the same notation used within the first part of the proof, we can observe that

$$\begin{aligned}
d(s) &= \prod_{i=1}^r (s - \mu_i)^{k_i} = \prod_{i=1}^r (s - \mu_i)^{k_i - 1} \cdot \prod_{i=1}^r (s - \mu_i), \\
n(s) &= \prod_{i=1}^r (s - \mu_i)^{k_i - 1} \\
& \quad \cdot \left[ \sum_{i=1}^r \left( \sum_{k\in[1,n]:\lambda_k=\mu_i} [\mathbf{b}]_k[\mathbf{c}]_k \right) \cdot \prod_{j\in[1,r], j\neq i} (s - \mu_j) \right].
\end{aligned}$$

Consequently,

$$\det(sI_n - A - \mathbf{b}\mathbf{c}^\top) = d(s) - n(s) = \prod_{i=1}^{r}(s-\mu_i)^{k_i - 1}$$
$$\cdot \left[\prod_{i=1}^{r}(s-\mu_i) - \sum_{i=1}^{r}\left(\sum_{k\in[1,n]:\lambda_k=\mu_i}[\mathbf{b}]_k[\mathbf{c}]_k\right)\cdot\right.$$
$$\left.\cdot \prod_{j\in[1,r],\mu_j\neq\mu_i}(s-\mu_j)\right].$$

We notice that $\prod_{i=1}^{r}(s-\mu_i)^{k_i-1}$ is surely Hurwitz. On the other hand, we can apply to the polynomial

$$\begin{aligned}\psi(s) \quad := \quad & \prod_{i=1}^{r}(s-\mu_i) - \sum_{i=1}^{r}\left(\sum_{k\in[1,n]:\lambda_k=\mu_i}[\mathbf{b}]_k[\mathbf{c}]_k\right) \\ & \cdot \prod_{j\in[1,r],\mu_j\neq\mu_i}(s-\mu_j)\end{aligned}$$

exactly the same reasoning used in the previous part of the proof, and claim that it surely has $r-1$ negative real roots, and that its $r$th real root is negative if and only if $\psi(0) > 0$. But since

$$\left.\prod_{i=1}^{r}(s-\lambda_i)^{k_i-1}\right|_{s=0} > 0,$$

this is equivalent to requiring that $\det(sI_n - A - \mathbf{b}\mathbf{c}^\top)\big|_{s=0} > 0$, which, in turn, is equivalent to (4).

Second case: $\mathbf{c}$ is a positive but not strictly positive vector. Set $\mathcal{C} := \{i \in [1,n] : [\mathbf{c}]_i > 0\} \subsetneq [1,n]$. It entails no loss of generality assuming that $\mathcal{C} = [1,k], k < n$, since we can always reduce ourselves to this situation, by means of a suitable permutation. This means that $\mathbf{c}^\top = [\,\mathbf{c}_1^\top \quad 0\,], \mathbf{c}_1 \gg 0$. Clearly, $A$ and $\mathbf{b}$ can be accordingly partitioned as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix},$$

with $A_1, A_2$ diagonal Hurwitz matrices, and $\mathbf{b}_1, \mathbf{b}_2$ both strictly positive. Consequently,

$$\begin{aligned} A + \mathbf{b}\mathbf{c}^\top &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} [\,\mathbf{c}_1^\top \quad 0\,] \\ &= \begin{bmatrix} A_1 + \mathbf{b}_1\mathbf{c}_1^\top & 0 \\ \mathbf{b}_2\mathbf{c}_1^\top & A_2 \end{bmatrix}, \end{aligned}$$

and we may apply the same reasoning as before to the submatrix $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$ to say that such a matrix (and hence $A + \mathbf{b}\mathbf{c}^\top$) is Hurwitz if and only if

$$\sum_{i=1}^{k}[\mathbf{b}]_i[\mathbf{c}]_i \prod_{j\in[1,k],j\neq i}(-\lambda_j) < \prod_{i=1}^{k}(-\lambda_i). \tag{7}$$

But if we multiply both sides by $\prod_{i=k+1}^{n}(-\lambda_i)$ and we keep in mind that $[\mathbf{b}]_i[\mathbf{c}]_i = 0$ for every $i \in [k+1,n]$, we note that (7) is just equivalent to (4). ∎

*Remark 2:* As clarified within the proof, condition (4) is equivalent to the fact that $\det(sI_n - A - \mathbf{b}\mathbf{c}^\top)\big|_{s=0} > 0$. On the other hand, since $A$ is Hurwitz, and hence $\det(-A) > 0$, from the identity

$$\begin{aligned} \det(-A - \mathbf{b}\mathbf{c}^\top) &= \det(-A)\det(I_n + A^{-1}\mathbf{b}\mathbf{c}^\top) \\ &= \det(-A)(1 + \mathbf{c}^\top A^{-1}\mathbf{b}), \end{aligned}$$

one deduces that condition (4) may also be written as

$$1 + \mathbf{c}^\top A^{-1}\mathbf{b} > 0. \tag{8}$$

*Remark 3:* From the proof of Proposition 1 it is clear that if $A$ is diagonal Hurwitz, $\mathbf{b}$ is strictly positive and $\mathbf{c}$ is a positive vector such that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler Hurwitz, then $A + \mathbf{b}\mathbf{c}^\top$ has all negative real eigenvalues. Even more, if the eigenvalues of $A$ have multiplicities at most 2, then $A + \mathbf{b}\mathbf{c}^\top$ has all distinct negative real eigenvalues (so, it is diagonalizable).

We complete our analysis by putting together the cases $\mathbf{b} > 0$ and $\mathbf{b} < 0$. The derivation of these characterizations is straightforward from the previous proposition and Lemma 1, and hence we omit the proof.

*Corollary 1:* Given a diagonal matrix $A \in \mathbb{R}^{n\times n}$, $n > 1$, and vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, assume w.l.o.g. that they are described as in (2) and (3), and that either $n_+$ or $n_-$ is zero (namely, all nonzero entries of $\mathbf{b}$ have the same sign). The matrix $A + \mathbf{b}\mathbf{c}^\top$ is Metzler and Hurwitz if and only if

i) if either $n_+ > 1$ or $n_- > 1$, then the nonzero entries of $\mathbf{c}$ have the same sign as the nonzero entries of $\mathbf{b}$ (namely $\mathbf{b}\mathbf{c}^\top \geq 0$), while if $\mathbf{b}$ has a unique nonzero entry then the nonzero entries of $\mathbf{c}_3$ have the same sign as the nonzero entry of $\mathbf{b}$;

ii) $A$ is Hurwitz;

iii) condition (8) holds.

## III. Continuous-time positive switched systems with rank one difference: stability analysis

In the rest of the paper we consider continuous-time positive switched systems (CPSSs) described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{9}$$

where $\mathbf{x}(t)$ is the $n$-dimensional state variable and $\sigma(t)$ the switching sequence at time $t$. At every time $t \in \mathbb{R}_+$,

$$A_{\sigma(t)} \in \{A + \mathbf{b}\mathbf{c}_1^\top, A + \mathbf{b}\mathbf{c}_2^\top, \dots, A + \mathbf{b}\mathbf{c}_p^\top\}, \tag{10}$$

with $A \in \mathbb{R}^{n\times n}$, while $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n$, for every $i \in [1,p]$. We assume that $A$ is a diagonal matrix, and that for every index $i \in [1,p]$, the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler. This latter condition ensures that the switched system (9) is positive, by this meaning that if the initial state $\mathbf{x}(0)$ is positive then the whole state trajectory remains in the positive orthant $\mathbb{R}^n_+$ for every choice of the switching sequence. For this class of systems, we want to define and characterize asymptotic stability.

*Definition 1:* The CPSS (9) is *asymptotically stable* if for every initial state $\mathbf{x}(0) \geq 0$ and every switching sequence $\sigma(t), t \in \mathbb{R}_+$, the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

Clearly, if the stability problem is solvable, then all the system matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, must be (Metzler and) Hurwitz. In this section we want to prove that, when $A$ is a diagonal matrix, the fact that all matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are Hurwitz is, in fact, also sufficient for asymptotic stability.

As a first case, we address the situation when the vector $\mathbf{b}$ has nonzero entries of different signs. We have the following result.

*Proposition 2:* Given a diagonal matrix $A \in \mathbb{R}^{n\times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, assume w.lo.g. that $A$ and $\mathbf{b}$ are described as in (2) and each $\mathbf{c}_i$ can be accordingly partitioned as

$$\mathbf{c}_i^\top = [\, \mathbf{c}_1^{(i)\top} \quad \mathbf{c}_2^{(i)\top} \quad \mathbf{c}_3^{(i)\top} \,], \qquad i \in [1,p]. \tag{11}$$

If $n_+ \geq 1$, $n_- \geq 1$, and the matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler and Hurwitz, then the CPSS (9) is asymptotically stable.

*Proof:* By referring to Lemma 1 and the subsequent discussion, we distinguish the following subcases:

a) $[n_+ > 1$ and $n_- > 1]$. If so, we have already seen that each matrix $A + \mathbf{b}\mathbf{c}_i^\top$ can be Metzler Hurwitz if and only if it coincides with $A$ and $A$ is Hurwitz. Therefore it is clear that in this case the Metzler Hurwitz property of the matrices guarantees the asymptotic stability of the CPSS (9).

b) $[n_+ > 1$ and $n_- = 1]$ or $[n_+ = 1$ and $n_- > 1]$. We have already noticed that, when the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler Hurwitz, it is also upper triangular (in the first case) or lower triangular (in the second case). But a CPSS whose matrices are all Hurwitz and in the same kind of triangular form is necessarily asymptotically stable [10].

c) $[n_+ = 1$ and $n_- = 1]$. In this case, $A+\mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $A + \mathbf{b}\mathbf{c}_i^\top$ takes the form

$$A + \mathbf{b}\mathbf{c}_i^\top = \begin{bmatrix} M_i & 0 \\ 0 & A_3 \end{bmatrix}, \tag{12}$$

where

$$M_i := \begin{bmatrix} \lambda_1 + b_+ c_1^{(i)} & b_+ c_2^{(i)} \\ b_- c_1^{(i)} & \lambda_2 + b_- c_2^{(i)} \end{bmatrix}, \; i \in [1,p], \tag{13}$$

and $A_3$ are all Hurwitz matrices, $b_+ > 0, b_- < 0, c_1 \leq 0$ and $c_2 \geq 0$. If the matrices $M_i, i \in [1,p]$, are all Metzler Hurwitz, then (see Proposition 3 in [5]) all their convex combinations are Metzler Hurwitz, in turn, and this ensures [7] that the two-dimensional CPSS

$$\dot{\mathbf{z}}(t) = M_{\sigma(t)}\mathbf{z}(t), \;\; M_{\sigma(t)} \in \{M_1, M_2, \dots, M_p\}, \tag{14}$$

is asymptotically stable. Therefore, as the two-dimensional system (14) is asymptotically stable and $A_3$ is Hurwitz, the CPSS system (9) described by the matrices (12) is asymptotically stable.

■

We now address the case when all the nonzero entries of the vector $\mathbf{b}$ have the same sign.

*Proposition 3:* Given a diagonal matrix $A \in \mathbb{R}^{n\times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, assume w.lo.g. that $A$ and $\mathbf{b}$ are described as in (2) and each $\mathbf{c}_i$ is accordingly partitioned as in (11). If either $n_+ = 0$ or $n_- = 0$, and the matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler and Hurwitz, then the CPSS (9) is asymptotically stable.

*Proof:* Consider, first, the case when $\mathbf{b} \gg 0$ (namely $n = n_+ > 1$ and $n_- = 0$). By Proposition 1, if $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler Hurwitz and $\mathbf{b} \gg 0$, then $\mathbf{c}_i \geq 0$, $A$ is Hurwitz and condition (8) holds, namely $1+\mathbf{c}_i^\top A^{-1}\mathbf{b} > 0$. Set $\mathbf{w} := -A^{-1}\mathbf{b}$. It is easy to see that $\mathbf{w} \gg 0$ and that

$$(A + \mathbf{b}\mathbf{c}_i^\top)\mathbf{w} \quad = \quad -(1 + \mathbf{c}_i^\top A^{-1}\mathbf{b})\mathbf{b} \ll 0, \quad \forall\, i \in [1,p].$$

This ensures that the positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}^\top \mathbf{x}(t), \qquad A_\sigma(t) \in \{A + \mathbf{b}\mathbf{c}_1^\top, \dots, A + \mathbf{b}\mathbf{c}_p^\top\},$$

is asymptotically stable. But then, for each choice of the switching sequence, the product of the matrix exponentials converges to the zero matrix and so does its transposed, thus ensuring that also the positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad A_\sigma(t) \in \{A + \mathbf{b}\mathbf{c}_1^\top, \dots, A + \mathbf{b}\mathbf{c}_p^\top\},$$

is asymptotically stable.

Consider now the case when $\mathbf{b} > 0$ has some zero entries. In this case $A, \mathbf{b}$ and the vectors $\mathbf{c}_i$ become

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}, \; \begin{bmatrix} \mathbf{b}_+ \\ 0 \end{bmatrix}, \; \mathbf{c}_i = \begin{bmatrix} \mathbf{c}_1^{(i)} \\ \mathbf{c}_3^{(i)} \end{bmatrix}, \qquad \text{with } \mathbf{b}_+ \gg 0,$$

where $\dim\, \mathbf{b}_+ = \dim\, A_1 = \dim\, \mathbf{c}_1^{(i)} = n_+$. Consequently,

$$A + \mathbf{b}\mathbf{c}_i^\top = \begin{bmatrix} A_1 + \mathbf{b}_+\mathbf{c}_1^{(i)\top} & \mathbf{b}_+\mathbf{c}_3^{(i)\top} \\ 0 & A_3 \end{bmatrix}.$$

If $n_+ > 1$, set $\mathbf{w}_1 := -A_1^{-1}\mathbf{b}_+$. By the same reasoning as in the first part of the proof, we can claim that $\mathbf{w}_1 \gg 0$. Also, for every $\epsilon > 0$, $\mathbf{w}_2 := \epsilon\mathbf{1}_{n-n_+} \gg 0$. It is easy to see that, for a suitably small value of $\epsilon > 0$, the quantity $\alpha := -(1 + \mathbf{c}_1^{(i)\top} A_1^{-1}\mathbf{b}_+) + \epsilon(\mathbf{c}_2^{(i)\top}\mathbf{1}_{n-n_+})$ is negative, and hence the vector

$$(A + \mathbf{b}\mathbf{c}_i^\top) \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \alpha\mathbf{b}_+ \\ \epsilon(A_2\mathbf{1}_{n-n_+}) \end{bmatrix}$$

is strictly negative. Again, this proves that system (9) is asymptotically stable. If $n_+ = 1$, we can come to the same conclusion by using $\mathbf{w}^\top = [\,1 \quad \epsilon\mathbf{1}_{n-1}^\top\,]$.

The cases $\mathbf{b} \ll 0$ and $\mathbf{b} < 0$ can be addressed along the same lines. ■

By putting together Propositions 2 and 3, we finally get the following result.

*Theorem 1:* Let $A \in \mathbb{R}^{n\times n}$ be a diagonal matrix, and let $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, be vectors such that $A+\mathbf{b}\mathbf{c}_i^\top$ is Metzler for every index $i \in [1,p]$. The following facts are equivalent:

i) $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz for every index $i \in [1,p]$;
ii) the CPSS (9) is asymptotically stable.

## IV. CPPSs with rank one difference: stabilizability analysis

*Definition 2:* The CPSS (9) is *stabilizable* if for every positive initial state $\mathbf{x}(0)$ there exists a switching sequence $\sigma(t), t \in \mathbb{R}_+$, such that the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

In the general case, the stabilization problem is solvable if at least one of the system matrices is Hurwitz. More generally, if there exist $\alpha_i, i \in [1,p]$, $0 \le \alpha_i \le 1$, such that the convex combination $\sum_{i=1}^p \alpha_i(A+\mathbf{b}\mathbf{c}_i^\top)$ is (Metzler and) Hurwitz, then the system is stabilizable [18]. For matrices that different by a rank one matrix, these two sufficient conditions for stabilizability are in fact equivalent.

*Proposition 4:* [6] Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix, and assume that $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, are column vectors such that the matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are Metzler. There exist $\alpha_i, i \in [1,p], 0 \le \alpha_i \le 1$, with $\sum_{i=1}^p \alpha_i = 1$, such that $\sum_{i=1}^p \alpha_i(A + \mathbf{b}\mathbf{c}_i^\top)$ is Hurwitz if and only if there exists $i \in [1,p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

By making use of the previous result, we can provide an important characterization of the stabilizability property for the class of CPSSs described as in (9), under the additional assumption that $A$ is a diagonal matrix. As in the previous section, we have to address separately the case when $\mathbf{b}$ has entries of opposite signs and the case when the nonzero entries have all the same sign.

*Proposition 5:* Given a diagonal matrix $A \in \mathbb{R}^{n\times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, assume w.lo.g. that $A$ and $\mathbf{b}$ are described as in (2) and each $\mathbf{c}_i$ is accordingly partitioned as in (11). If $n_+ \ge 1$, $n_- \ge 1$, and the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler, then the CPSS (9) is stabilizable if and only if there exists an index $i \in [1,p]$ such that the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

*Proof:* Sufficiency is obvious, so we only prove necessity. As in the proof of Proposition 2, we proceed by considering all possible subcases of Case 1 enlightened in Lemma 1 and following discussion (see Section II):

a) $[n_+ > 1$ and $n_- > 1]$. If so, $A+\mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $\mathbf{c}_i = 0$, and hence all matrices $A+\mathbf{b}\mathbf{c}_i^\top$ coincide with $A$. So, stabilizability requires that all matrices $A+\mathbf{b}\mathbf{c}_i^\top = A$ are Metzler Hurwitz.

b) $[n_+ > 1$ and $n_- = 1]$ or $[n_+ = 1$ and $n_- > 1]$. In the first case, $\mathbf{c}_1^{(i)} = 0$, $\mathbf{c}_3^{(i)} = 0$ and $c_2^{(i)}$ is a nonnegative scalar. Consequently, all matrices
$$A + \mathbf{b}\mathbf{c}_i^\top = \begin{bmatrix} A_1 & \mathbf{b}_+ c_2^{(i)} & 0 \\ 0 & A_2 + b_- c_2^{(i)} & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$
are upper triangular, and the system is stabilizable if and only if $A_1$ and $A_3$ are Hurwitz and there exists at least one index $i \in [1,p]$ such that $A_2 + b_- c_2^{(i)} < 0$. But this means that there exists an index $i \in [1,p]$ such that $A+\mathbf{b}\mathbf{c}_i^\top$ is Hurwitz. The second case is symmetric (see the proof of Proposition 2).

c) $[n_+ = 1$ and $n_- = 1]$. In this case, $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $A+\mathbf{b}\mathbf{c}_i^\top$ takes the form (12), with the matrix $M_i$ described as in (13). Clearly, the CPSS system (9) with such matrices is stabilizable if and only if $A_3$ is Hurwitz and the two-dimensional CPSS
$$\dot{\mathbf{z}}(t) = M_{\sigma(t)}\mathbf{z}(t), \qquad M_{\sigma(t)} \in \{M_1, M_2, \dots, M_p\},$$
is stabilizable. But for a two-dimensional CPSS, stabilizability is equivalent [1], [2] to the existence of a Hurwitz convex combination of the matrices. By Proposition 4, this implies that at least one of $M_i$'s is Hurwitz. Therefore there exists an index $i \in [1,p]$ such that the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

■

*Proposition 6:* Given a diagonal matrix $A \in \mathbb{R}^{n\times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, assume w.l.o.g. that $A$ and $\mathbf{b}$ are described as in (2) and each $\mathbf{c}_i$ is accordingly partitioned as in (11). If either $n_+ = 0$ or $n_- = 0$, and the matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler, then the CPSS (9) is asymptotically stable if and only if there exists an index $i \in [1,p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

*Proof:* Again, we only need to prove the necessity. Consider, first, the case when $\mathbf{b} \gg 0$. We preliminary notice that, by Lemma 1, if $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler and $\mathbf{b} \gg 0$, then $\mathbf{c}_i \ge 0$, and this ensures that $A + \mathbf{b}\mathbf{c}_i^\top \ge A$. If the system is stabilizable then [2] it is consistently stabilizable, by this meaning that there exists a switching sequence $\bar{\sigma} : \mathbb{R}_+ \to [1,p]$ that asymptotically drives to zero every initial state. This implies [17] that there exists a periodic switching signal that asymptotically stabilizes the switched system, namely that there exist $r \in \mathbb{Z}_+, i_1, i_2, \dots, i_r \in [1,p], \tau_1, \tau_2, \dots, \tau_r \in \mathbb{R}_+$, such that
$$Z := e^{(A+\mathbf{b}\mathbf{c}_{i_1}^\top)\tau_1} e^{(A+\mathbf{b}\mathbf{c}_{i_2}^\top)\tau_2} \dots e^{(A+\mathbf{b}\mathbf{c}_{i_r}^\top)\tau_r}$$
is Schur. But since
$$Z \ge e^{A\tau_1} e^{A\tau_2} \dots e^{A\tau_r},$$
this latter matrix must be Schur, too, and hence the diagonal matrix $A$ must be a Hurwitz. Since $A$ is diagonal Hurwitz and $\mathbf{b} \gg 0$, by Proposition 1, if each $A + \mathbf{b}\mathbf{c}_i^\top$ were not Hurwitz then it should be $1 + \mathbf{c}_i^\top A^{-1}\mathbf{b} \le 0, \forall\, i \in [1,p]$. Set $\mathbf{w} := -A^{-1}\mathbf{b} \gg 0$ and note, again, that
$$(A + \mathbf{b}\mathbf{c}_i^\top)\mathbf{w} = -(1 + \mathbf{c}_i^\top A^{-1}\mathbf{b})\mathbf{b}.$$
So, if none of the system matrices were Hurwitz, there would be a (strictly) positive vector $\mathbf{w}$ such that
$$(A + \mathbf{b}\mathbf{c}_i^\top)\mathbf{w} \ge 0, \qquad \forall\, i \in [1,p],$$

thus preventing stabilizability [6]. So, there must be an index $i \in [1,p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

Consider now the case when $\mathbf{b} > 0$ has some zero entries, and hence

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}, \begin{bmatrix} \mathbf{b}_+ \\ 0 \end{bmatrix}, \mathbf{c}_i = \begin{bmatrix} \mathbf{c}_1^{(i)} \\ \mathbf{c}_3^{(i)} \end{bmatrix}, \qquad \text{with } \mathbf{b}_+ \gg 0,$$

where $\dim A_1 = \dim \mathbf{b}_+ = \dim \mathbf{c}_1^{(i)} = n_+$. Consequently,

$$A + \mathbf{b}\mathbf{c}_i^\top = \begin{bmatrix} A_1 + \mathbf{b}_+\mathbf{c}_1^{(i)\top} & \mathbf{b}_+\mathbf{c}_3^{(i)\top} \\ 0 & A_3 \end{bmatrix}.$$

If $A_3$ is not Hurwitz, clearly the system cannot be stabilizable. So, suppose that $A_3$ is Hurwitz but all the matrices $A_1 + \mathbf{b}_+\mathbf{c}_1^{(i)\top}, i \in [1,p]$, are not. If $n_+ > 1$, set $\mathbf{w}_1 := -A_1^{-1}\mathbf{b}_+$. By the same reasoning as in the first part of the proof, we can claim that in this case it would be

$$(A + \mathbf{b}\mathbf{c}_i^\top) \begin{bmatrix} \mathbf{w}_1 \\ 0 \end{bmatrix} \geq 0, \qquad \forall\ i \in [1,p],$$

thus preventing stabilizability. If $n_- = 1$, we can come to the same conclusion by using $\mathbf{w}^\top = \mathbf{e}_1$. ■

By putting together Propositions 5 and 6, we finally get the following result.

*Theorem 2:* Let $A \in \mathbb{R}^{n\times n}$ be a diagonal matrix, and let $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, be vectors such that $A+\mathbf{b}\mathbf{c}_i^\top$ is Metzler for every index $i \in [1,p]$. The following facts are equivalent:

i) there exists $i \in [1,p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz;
ii) the CPSS (9) is stabilizable.

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