2SAT

- Instance: A 2-CNF formula φ
- Problem: To decide if φ is satisfiable

Example: a 2CNF formula

$$(\neg x \vee y) \wedge (\neg y \vee z) \wedge (x \vee \neg z) \wedge (z \vee y)$$

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2SAT is in P

Theorem: 2SAT is polynomial-time decidable.

Proof: We'll show how to solve this problem efficiently using path searches in graphs…

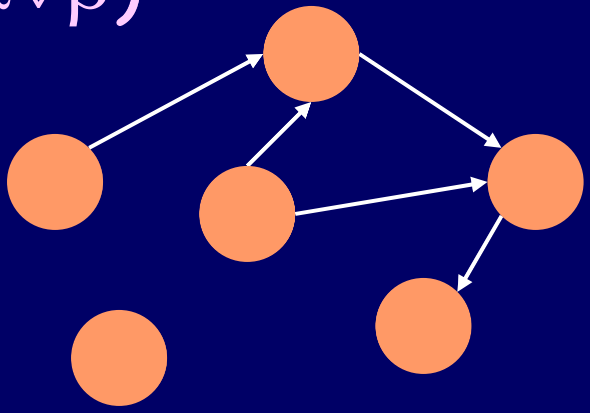
Searching in Graphs

Theorem: Given a graph $G=(V,E)$ and two vertices $s,t \in V$, finding if there is a path from s to t in G is polynomial-time decidable.

Proof: Use some search algorithm (DFS/BFS). ■

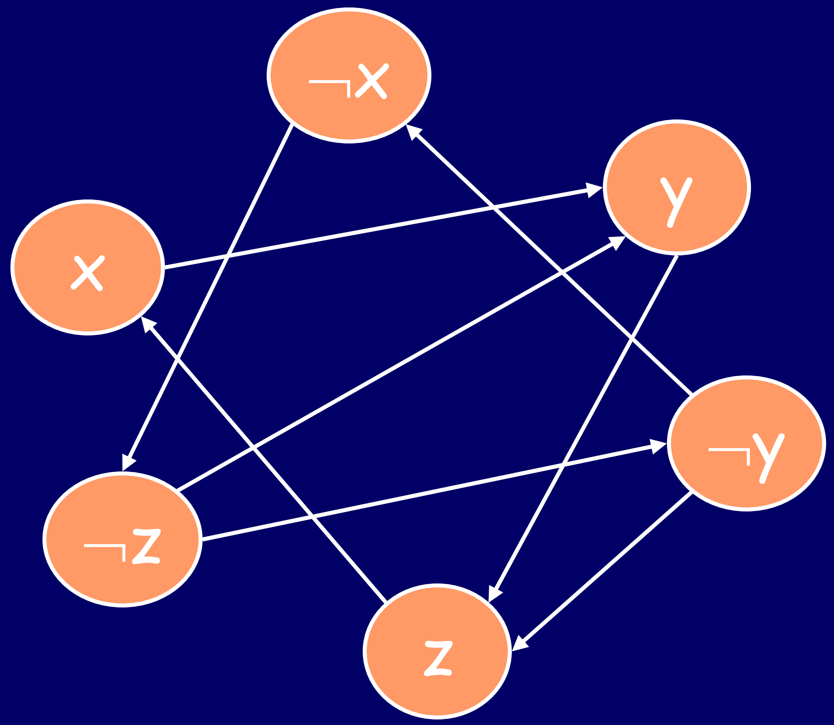
Graph Construction

- Vertex for each variable and a negation of a variable
- Edge (α,β) iff there exists a clause equivalent to $(\neg\alpha\vee\beta)$



Graph Construction: Example

$(\neg x \vee y) \wedge (\neg y \vee z) \wedge (x \vee \neg z) \wedge (z \vee y)$



Observation

Claim: If the graph contains a path from α to β, it also contains a path from $\neg\beta$ to $\neg\alpha$.

Proof: If there's an edge (α,β), then there's also an edge $(\neg\beta,\neg\alpha)$.

Correctness

Claim:

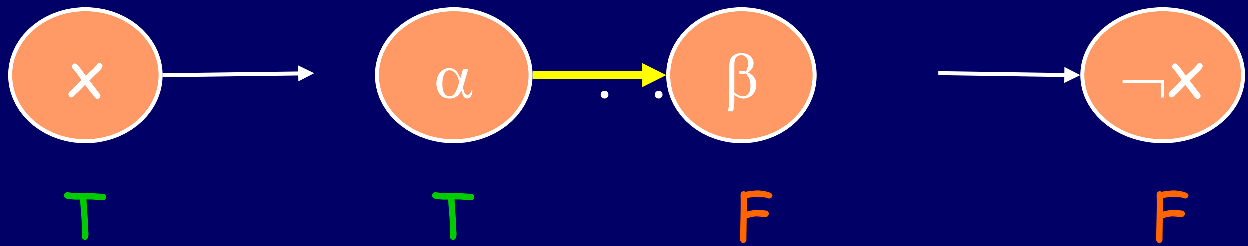
a 2-CNF formula φ is unsatisfiable iff there exists a variable x, such that:

1. there is a path from x to $\neg x$ in the graph
2. there is a path from $\neg x$ to x in the graph

Correctness (1)

- Suppose there are paths $x..\neg x$ and $\neg x..x$ for some variable x, but there's also a satisfying assignment ρ.
- If $\rho(x)=T$ (similarly for $\rho(x)=F$):

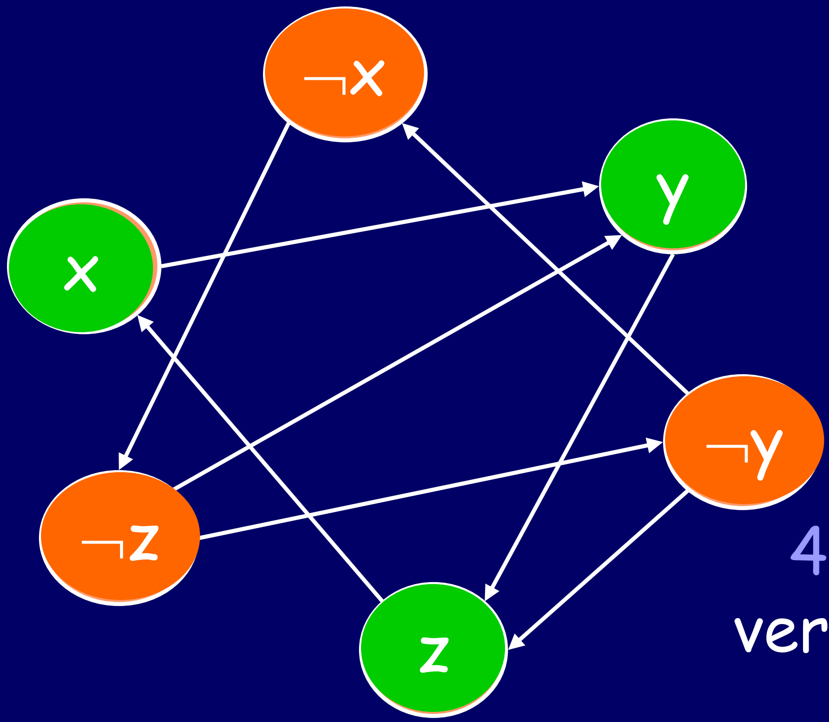
$(\neg\alpha\vee\beta)$ is false!



Correctness (2)

- Suppose there are no such paths.
- Construct an assignment as follows:

1. pick an unassigned vertex and assign it T



2. assign T to all reachable vertices
3. assign F to their negations
4. Repeat until all vertices are assigned

Correctness (2)

Claim: The algorithm is well defined.

Proof: If there were a path from x to both y and $\neg y$,

then there would have been a path from x to $\neg y$ and from $\neg y$ to $\neg x$.

Correctness

A formula is unsatisfiable iff there are no paths of the form $x..\neg x$ and $\neg x..x$. ■

2SAT is in P

We get the following efficient algorithm for 2SAT:

- For each variable x find if there is a path from x to $\neg$x and vice-versa.
- Reject if any of these tests succeeded.
- Accept otherwise

$\Rightarrow$ 2SAT$\in$P. ■