Chapter 2

Recurrence Relations and Divide-and-Conquer Algorithms

Consider the following *recurrence*:

$$\begin{cases} T(n) = & s(n)T(f(n)) + w(n), \quad \text{for } n > n_0, & (2.1.a) \\ T(n) = & T_0, \quad \text{for } n \le n_0. & (2.1.b) \end{cases}$$

In (2.1), n is a nonnegative integer variable, and n_0 and T_0 are nonnegative integer constants. Functions $s(\cdot), f(\cdot)$ and $w(\cdot)$ are nondecreasing, nonnegative integer functions of n (as a consequence, $T(\cdot)$ is also a nondecreasing and nonnegative integer function). Finally, $f(n) < n$ for any $n > n_0$.

Equation (2.1) is often useful in the analysis of *divide-and-conquer* algorithms, where a *problem instance* of size at most n_0 is solved directly, while an instance of size $n > n_0$ is solved by

(i) decomposing the instance into $s(n)$ instances of the same problem of size (at most[1]) $f(n) < n$ each;

(ii) recursively, solving the $s(n)$ smaller instances;

(iii) combining the solutions to the $s(n)$ instances of size (at most) $f(n)$ into a solution to the instance of size n.

[1]Needless to say, whenever the quantities featured in the recurrence are upper bounds, the resulting solution $T(n)$ will be an upper bound to the running time, while exact values yield the exact running time of the resulting algorithm.

Here, $w(n)$ is (an upper bound to) the overall running time of the decomposition and the combination procedures. Also, T_0 is (an upper bound to) the running time of the algorithm on instances of size $n \leq n_0$. With the given interpretation of $n_0, T_0, s(\cdot), f(\cdot)$, and $w(\cdot)$, Equation (2.1) *uniquely* defines a function $T(n)$, which represents (an upper bound to) the running time complexity of the given algorithm for any problem instance of size n.

The following notation is useful to formulate the general solution of Equation (2.1). We let $f^{(0)}(n) = n$, and for $i > 0$, $f^{(i+1)}(n) = f(f^{(i)}(n))$. We also denote by $f^\star(n, n_0)$ the largest k such that $f^{(k)}(n) > n_0$. Note that, if $n \leq n_0$, $f^\star(n, n_0)$ would not be defined. Conventionally, we set $f^\star(n, n_0) = -1$ for $n \leq n_0$.

With the above notation, $f^{(\ell)}(n)$ is the size of a single problem instance at the ℓ-th level of recursion, where $\ell = 0$ corresponds to the initial call. Level $\ell = f^\star(n, n_0)$ is the last for which $f^{(\ell)}(n) > n_0$ and hence it is the last level for which Equation (2.1.a) applies. At level $f^\star(n, n_0) + 1$, Equation (2.1.b) applies instead.

Thus, for $0 \leq \ell \leq f^\star(n, n_0)$, the work spent on a single problem instance at level ℓ is $w(f^{(\ell)}(n))$. For $\ell = f^\star(n, n_0) + 1$, the work per problem instance is T_0.

The instance at level 0 generates $s(n)$ instances at level 1, each of which generates $s(f(n))$ instances at level 2, each of which generates $s(f^{(2)}(n))$ instances at level 3, ..., each of which generates $s(f^{(\ell-1)}(n))$ instances at level ℓ. Therefore, *the total* number of instances at level ℓ is

$$s(n) \cdot s(f(n)) \cdot s(f^{(2)}(n)) \cdot \ldots \cdot s(f^{(\ell-1)}(n)) = \prod_{j=0}^{\ell-1} s(f^{(j)}(n)),$$

where if $\ell - 1 < 0$ the value the above product is conventionally taken to be 1.

By combining the considerations of the last three paragraphs, we obtain the following expression for the general solution of Equation (2.1):

$$T(n) = \sum_{\ell=0}^{f^\star(n,n_0)} \left(\left[\prod_{j=0}^{\ell-1} s(f^{(j)}(n)) \right] w(f^{(\ell)}(n)) \right) + \left[\prod_{j=0}^{f^\star(n,n_0)} s(f^{(j)}(n)) \right] T_0,$$

where, for $f^\star(n, n_0) = -1$, the value of the summation in the above expression is conventionally assumed to be 0.

The correctness of the above derivation can be proved by induction on n as follows. Let us start with the base case(s) $n \leq n_0$ and recall that, conventionally, we set $f^\star(n, n_0) = -1$ for $n \leq n_0$. Then, the closed formula correctlt yields T_0, since the summation and the product within evaluate to 0 and 1, respectively.

Assume now that the formula yields the correct value of $T(k)$, for $k < n$ and $n > n_0$.

We have that $T(n) = s(n)T(f(n)) + w(n)$, and, by the inductive hypothesis,

$$T(f(n)) = \sum_{\ell=0}^{f^\star(f(n),n_0)} \left(\left[\prod_{j=0}^{\ell-1} s(f^{(j+1)}(n)) \right] w(f^{(\ell+1)}(n)) \right) + \left[\prod_{j=0}^{f^\star(f(n),n_0)} s(f^{(j+1)}(n)) \right] T_0.$$

Observe that, by the definition of $f^\star$, in case $f(n) \leq n_0$, then $f^\star(f(n), n_0) = -1$, while $f^\star(n, n_0) = 0$. Otherwise, the maximum index k for which $f^{(k)}(f(n)) > n_0$ is clearly one less than the maximum index k for which $f^{(k)}(n) > n_0$, hence, in all cases, $f^\star(f(n), n_0) = f^\star(n, n_0) - 1$. We have:

$$\begin{aligned}
& s(n)T(f(n)) \\
= \;& s(f^{(0)}(n)) \left\{ \sum_{\ell=0}^{f^\star(n,n_0)-1} \left(\left[\prod_{j=0}^{\ell-1} s(f^{(j+1)}(n)) \right] w(f^{(\ell+1)}(n)) \right) + \left[\prod_{j=0}^{f^\star(n,n_0)-1} s(f^{(j+1)}(n)) \right] T_0 \right\} \\
= \;& s(f^{(0)}(n)) \left\{ \sum_{\ell=0}^{f^\star(n,n_0)-1} \left(\left[\prod_{j'=1}^{\ell} s(f^{(j')}(n)) \right] w(f^{(\ell+1)}(n)) \right) + \left[\prod_{j'=1}^{f^\star(n,n_0)} s(f^{(j')}(n)) \right] T_0 \right\}
\end{aligned}$$

(by substituting $j' = j + 1$ in the two products)

$$= \sum_{\ell=0}^{f^\star(n,n_0)-1} \left(\left[\prod_{j'=0}^{\ell} s(f^{(j')}(n)) \right] w(f^{(\ell+1)}(n)) \right) + \left[\prod_{j'=0}^{f^\star(n,n_0)} s(f^{(j')}(n)) \right] T_0$$

(by bringing $s(f^{(0)}(n))$ within the two products)

$$= \sum_{\ell'=1}^{f^\star(n,n_0)} \left(\left[\prod_{j'=0}^{\ell'-1} s(f^{(j')}(n)) \right] w(f^{(\ell')}(n)) \right) + \left[\prod_{j'=0}^{f^\star(n,n_0)} s(f^{(j')}(n)) \right] T_0$$

(by substituting $\ell' = \ell + 1$ in the summation.)

Observe now that $w(n)$ can be rewritten as $\left[\prod_{j'=0}^{0-1} s(f^{(j')}(n))\right] w(f^{(0)}(n))$, which is exactly the term of the summation for $\ell' = 0$. Therefore we obtain

$$T(n) = s(n)T(f(n)) + w(n) = \sum_{\ell'=0}^{f^\star(n,n_0)} \left(\left[\prod_{j'=0}^{\ell'-1} s(f^{(j')}(n)) \right] w(f^{(\ell')}(n)) \right) + \left[\prod_{j'=0}^{f^\star(n,n_0)} s(f^{(j')}(n)) \right] T_0$$

and the inductive thesis follows.