Counting the Number of Fault Patterns in Redundant VLSI Arrays*

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Abstract In VLSI technology, redundancy is a commonly adopted technique to provide reconfiguration capabilities to regular architectures. This paper proves upper and lower bounds on the number of minimal fault patterns (minimal set of faulty processors) which affect a link-redundant linear array in an unrepairable way, for both the cases of bidirectional and unidirectional links.

Keywords Distributed Systems, Fault Tolerance, Parallel Processing, Performance Evaluation.

1 Preliminaries

A standard technique to lower the production costs of VLSI circuits is the provision of *on-chip* redundancy, and accompanying mechanisms to reconfigure the chip components at the occurrence of fabrication faults. Without the presence of reconfiguration capabilities, the yield of very large VLSI chips would be so poor to make their production unacceptable.

In the case of linear arrays of identical *Processing Elements* (PE's), redundant ones (called *spares*) are often placed on the chip to replace faulty PE's and therefore preserve the network connectivity. Besides the regular links between neighboring PE's, extra links (called *bypass* links) are also included to recreate the array topology in the reconfiguration phase [1, 2]. However, regardless of any amount of redundancy and configuration capabilities, there are always sets of faults occurring at strategic positions which affect the chip in an unrepairable way [6]. Such sets of faults, called *Catastrophic Fault Patterns* (CFP's), have been extensively studied in [4, 5, 6] for linear arrays with different varieties of link redundancy.

In this paper we provide upper and lower bounds on the number of CFP's for a special class of redundant arrays, where each PE has exactly one bypass link and where the links can either be unidirectional or bidirectional. As we show in a later section, such bounds can be used in assessing the effectiveness and the cost of the reconfiguration process.

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Formally, let $A = \{p_0, p_1, \ldots, p_{N-1}\}$ denote a linear array of PE's (including both regular and spare elements), which are connected by *regular* links (p_i, p_{i+1}), $0 \le i \le N-2$, and by *bypass* links (p_i, p_{i+g}), $0 \le i \le N-1-g$. This interconnection, referred to as *linear array with link redundancy* g, will be the object of our analysis in the following sections. At the two ends of the array, two special PE's, called I (for Input) and O (for Output), are responsible for the I/O functions of the system. We assume that I is connected to $p_0, p_1, \ldots, p_{g-1}$, while O is connected to $p_{N-g}, p_{N-g+1}, \ldots, p_{N-1}$, so that all PE's in the system have the same degree and reliability bottlenecks at the borders of the array are avoided. Furthermore, we will conduct our analysis under the assumption that only PE's $p_0, p_1, \ldots, p_{N-1}$ can be faulty, while links and I/O nodes always operate correctly.

In [6], faults in the system are characterized as follows.

Definition 1 *For a linear array of size N and any link redundancy, a fault pattern F starting at p_{f_0} is a set of integers $F = \{f_0, f_1, \ldots, f_{m-1}\}$, where $0 \le f_i \le N-1$ and $f_{i-1} < f_i$, $1 \le i \le m \le N$.*

Definition 2 *Given a link-redundant linear array A, a fault pattern $F = \{f_0, f_1, \ldots, f_{m-1}\}$ is catastrophic for A if and only if no path exists between I and O, once the faulty p_i, $i \in F$, and their incident links are removed.*

The occurrence of a CFP implies that there is no way of reconfiguring the system with respect to I/O operations.

It can be easily shown that, for any array A with link redundancy g, a CFP F for A must contain at least g faults. From now on, our analysis will concentrate on such minimal case. In this case, the *width* W_F of a fault pattern $F = \{f_0, f_1, \ldots, f_{g-1}\}$ is defined to be the number of PE's between and including the first and the last fault in F, that is, $W_F = f_{g-1} - f_0 + 1$.

Theorem 1 ([6]) *Let $F = \{f_0, f_1, \ldots, f_{g-1}\}$ be a fault pattern for a linear array A with link redundancy g. Necessary condition for F to be catastrophic is*

$$g \le W_F \le W_F^B = \left(\left\lceil \frac{g}{2} \right\rceil - 1\right) g + \left\lfloor \frac{g}{2} \right\rfloor + 1,$$

in the case of bidirectional links and

$$g \le W_F \le W_F^U = (g-1)^2 + 1,$$

in the case of unidirectional links. □

From the above theorem it easily follows that all CFP's $F = \{f_0, f_1, \ldots, f_{g-1}\}$ starting at p_i, that is, with $f_0 = i$, are such that $f_{g-1} \le i + W_F^x - 1$, where $x \in \{B, U\}$ depends on the link orientation.

Starting from a result in [4], the following section provides very tight upper and lower bounds on the number of minimal CFP's for the bidirectional case. In Section 3 we prove an upper bound on the number of unidirectional CFP's by establishing a correspondence between the unidirectional and bidirectional case. The bounds are finally used in in Section 4 to determine a measure relevant to the cost assessment of the production process of the VLSI chip.

2 The case of bidirectional links

Nayak in [4] develops a technique to count the number of minimal CFP's in the case of bidirectional linear arrays with link redundancy g. Namely, he shows that there is a bijection between minimal CFP's starting at any fixed p_i, with $0 \leq i \leq N - W_F^B$, and the language L of strings of length $g-1$ over the alphabet $\Sigma = \{(,),\star\}$, corresponding to balanced parenthesizations of $\star$'s (e.g., for $g-1=3$, $L = \{\star\star\star, \star(), (\star), ()*\}$). By known combinatorial facts [3], we then have:

$$|L| = \mathrm{F}^B(g) = \sum_{j=0}^{\lfloor \frac{g-1}{2} \rfloor} \binom{g-1}{2j} \binom{2j}{j} \frac{1}{j+1}.$$

The remainder of this section is devoted to prove tight lower and upper bounds on the above quantity. We will make use of the following facts:

Fact 1 *For any $j > 0$,*

$$e^{-\frac{1}{6j}} \frac{2^{2j}}{\sqrt{\pi j}} < \binom{2j}{j} < \frac{2^{2j}}{\sqrt{\pi j}}.$$

Fact 2 *For any $g > 0$,*

$$\sum_{j=o}^{\lfloor \frac{g-1}{2} \rfloor} \binom{g-1}{2j} 2^{2j} = \frac{3^{g-1} + (-1)^{g-1}}{2}.$$

Fact 3 *For any $n > 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{2^{k+1}}{k+1} = \frac{3^{n+1} - 1}{n+1}.$$

Fact 1 can be easily proven by applying Stirling's approximation to $\binom{2j}{j}$. The formulae in Facts 3 and 2 can be derived by means of standard techniques for finite summations [3].

Theorem 2 *For any $g > 1$,*

$$F^B(g) > \frac{e^{-\frac{1}{6}}}{2\sqrt{\pi}} \frac{3^{g-1}+1}{\left\lfloor \frac{g+1}{2} \right\rfloor^{3/2}}.$$

Proof: We have:

$$\mathrm{F}^B(g) = \sum_{j=0}^{\lfloor \frac{g-1}{2} \rfloor} \binom{g-1}{2j} \binom{2j}{j} \frac{1}{j+1} > 1 + \frac{e^{-1/6}}{\sqrt{\pi} \left\lfloor \frac{g+1}{2} \right\rfloor^{3/2}} \sum_{j=1}^{\lfloor \frac{g-1}{2} \rfloor} \binom{g-1}{2j} 2^{2j},$$

by applying Fact 1 and since, for $x \geq 1$, $\mathrm{f}(x) = e^{-\frac{1}{6x}}$ increases and $\mathrm{g}(x) = \frac{1}{(x+1)^{3/2}} \leq \frac{1}{\sqrt{x}(x+1)}$ decreases. For odd values of g we can then establish that

$$\mathrm{F}^B(g) > \frac{e^{-1/6}}{\sqrt{\pi}(\frac{g+1}{2})^{3/2}} \sum_{j=0}^{\frac{g-1}{2}} \binom{g-1}{2j} 2^{2j}.$$

For g even, since $2\frac{e^{-1/6}}{\sqrt{\pi}\lfloor\frac{g+1}{2}\rfloor^{3/2}} < 1$, we can write:

$$\mathrm{F}^B(g) > \frac{e^{-1/6}}{\sqrt{\pi}\left\lfloor\frac{g+1}{2}\right\rfloor^{3/2}} \left(1 + \sum_{j=0}^{\lfloor\frac{g-1}{2}\rfloor} \binom{g-1}{2j} 2^{2j}\right).$$

The theorem then follows from Fact 2 and the two above inequalities. □

Corollary 1 $F^B(g) \in \Omega\left(\frac{3^g}{g^{3/2}}\right)$.

Theorem 3 *For any* $g > 1$,

$$F^B(g) < \frac{4(3^g-1)}{\sqrt{\pi}g(\left\lceil\frac{g-1}{2}\right\rceil+2)^{1/2}} + 1.$$

Proof: We will only consider odd values of g, that is, $g = 2i+1$ for $i \geq 1$. The proof for g even follows the same lines. From Fact 1, we have

$$\mathrm{F}^B(g) = \mathrm{F}^B(2i+1) < 1 + \sum_{j=1}^{i} \binom{2i}{2j} \frac{2^{2j}}{\sqrt{\pi j}(j+1)} < 1 + \sqrt{\frac{2}{\pi}} \sum_{j=0}^{i} \binom{2i}{2j} \frac{2^{2j}}{(j+1)^{3/2}}.$$

Note that the function $\mathrm{f}(x) = \frac{2^{2x}}{(x+1)^{3/2}}$ is strictly increasing for $x \geq 1$. Therefore, for $0 \leq j < \left\lceil\frac{i}{2}\right\rceil$, we can bound the term $\binom{2i}{2j} \frac{2^{2j}}{(j+1)^{3/2}}$ with the term $\binom{2i}{2(i-j)} \frac{2^{2(i-j)}}{(i-j+1)^{3/2}}$. Hence,

$$\sum_{j=0}^{i} \binom{2i}{2j} \frac{2^{2j}}{(j+1)^{3/2}} < 2 \sum_{j=\lceil\frac{i}{2}\rceil}^{i} \binom{2i}{2j} \frac{2^{2j}}{(j+1)^{3/2}} < \frac{2\sqrt{2}}{(i+2)^{1/2}} \sum_{j=0}^{i} \binom{2i}{2j} \frac{2^{2j+1}}{2j+1}.$$

The last summation in the above formula adds only and all the even terms of the summation described in Fact 3. Therefore,

$$\mathrm{F}^B(2i+1) < \frac{4(3^{2i+1}-1)}{\sqrt{\pi}(i+2)^{1/2}(2i+1)} + 1.$$

The theorem follows. □

Corollary 2 $F^B(g) \in \Theta\left(\frac{3^g}{g^{3/2}}\right)$.

3 The case of unidirectional links

In order to prove bounds on the number of CFP's in the unidirectional case, we need to recall the following *matrix representation* of any fault pattern F starting a fixed p_i, with $0 \le i \le N - W_F^x$, introduced in [6] for both the bidirectional and unidirectional case. F is represented as a boolean matrix W of size $\bar{W}_F^x \times g$, with $\bar{W}_F^x = \left\lceil \frac{W_F^x}{g} \right\rceil$, defined as follows:

$$W[h,k] = \begin{cases} 1 & \text{if } (i + hg + k) \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Under the above representation, each $f_j \in F$ is mapped into $W[h_j, k_j] = 1$, where $h_j = \left\lfloor \frac{(f_j - i)}{g} \right\rfloor$ and $k_j = (f_j - i) \bmod g$. For instance, the boolean matrix associated to the fault pattern $F = \{0, 4, 6, 8, 12\}$ for a linear array with link redundancy 5 is the following:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{1}$$

Note that F is a CFP for both the bidirectional and the unidirectional case. Note also that, in the matrix representation, regular links "correspond" to either consecutive elements on the same row or to elements $(W[h, g-1], W[h+1, 0])$, while bypass links "correspond" to consecutive elements in the same column.

CFP's can be characterized, with respect to the above matrix representation, as follows:

Theorem 4 ([4]) *Necessary and sufficient condition for a fault pattern F of cardinality g to be catastrophic for a bidirectional linear array with link redundancy g is $W[0,0] = W[0, g-1] = 1$ and, for any $1 \le k \le g-2$,*

- *if $W[h, k-1] = 1$ then exactly one among $W[h-1, k], W[h, k]$ and $W[h+1, k]$ (whenever such elements are defined) is 1.*
- *if $W[h', k+1] = 1$ then exactly one among $W[h'-1, k], W[h', k]$ and $W[h'+1, k]$ (whenever such elements are defined) is 1.* □

Theorem 5 ([4]) *Necessary and sufficient condition for a fault pattern F of cardinality g to be catastrophic for a unidirectional linear array with link redundancy g is $W[0,0] = W[0, g-1] = 1$ and, for any $1 \le k \le g-2$,*

- *if $W[h, k-1] = 1$ then at least one among $W[h-1, k], W[h, k], \ldots, W[\bar{W}_F^U - 1, k]$ (whenever such elements are defined) is 1.*
- *if $W[h', k+1] = 1$ then at least one among $W[0, k], W[1, k], \ldots, W[h'+1, k]$ (whenever such elements are defined) is 1.* □

Note that the matrix representation of any CFP F with $|F| = g$ must have exactly one element $W[h_k, k] = 1$ for each column k, $0 \leq k \leq g-1$ and that any CFP for the bidirectional case is also a CFP for the unidirectional case.

Using the above two theorems, we can represent each minimal CFP starting at any fixed p_i, with $i \leq N - W_F^x$, as a sequence of $g-1$ integer "moves" m_j, with $1 \leq j \leq g-1$, where each m_j indicates the increment of the row index from the (only) element set to 1 in column $j-1$ to the (only) one in column j in the associated matrix W. More formally, let $W[h_{i-1}, i-1]$ and $W[h_i, i]$ both be 1. Then $m_i = h_{i-1} - h_i$. As an example, the sequence associated to the matrix in (1) is $\langle -1, -1, 1, 1\rangle$.

We will refer to sequences representing CFP's as *catastrophic sequences*. For minimal CFP's starting at any fixed position $i \leq N - W_F^x$, catastrophic sequences can be characterized as follows:

Proposition 1 *Let $\langle m_1, m_2, \ldots, m_{g-1}\rangle$ be a sequence of moves such that:*

- $-1 \leq m_i \leq 1$ *for* $1 \leq i \leq g-1$*;*
- $\sum_{i=1}^{k} m_i \leq 0$ *for any* $1 \leq k \leq g-2$*;*
- $\sum_{i=1}^{g-1} m_i = 0$.

Then any such sequence corresponds to a minimal CFP for the bidirectional case and vice versa.

Proposition 2 *Let $\langle m_1, m_2, \ldots, m_{g-1}\rangle$ be a sequence of moves such that:*

- $m_i \leq 1$ *for* $1 \leq i \leq g-1$*;*
- $\sum_{i=1}^{k} m_i \leq 0$ *for any* $1 \leq k \leq g-2$*;*
- $\sum_{i=1}^{g-1} m_i = 0$.

Then any such sequence corresponds to a minimal CFP for the unidirectional case and vice versa.

The above propositions follow by observing the immediate bijection between catastrophic sequences and matrix representations of minimal CFP's

Let us now come to bounding the number $\mathrm{F}^U(g)$ of catastrophic fault patterns starting at any p_i, $0 \leq i \leq N - W_F^U$ under the assumption of unidirectional links. Clearly, $\mathrm{F}^U(g)$ is lower-bounded by $\mathrm{F}^B(g)$. In order to determine an upper bound to $\mathrm{F}^U(g)$, we establish a mapping from unidirectional catastrophic sequences to bidirectional ones. Let S_g^B (resp., S_g^U) be the set of catastrophic sequences (of length $g-1$) for bidirectional (resp., unidirectional) linear arrays with link redundancy g. Moreover, let $\bar{S}_g^U \subset S_g^U$ be the subset of unidirectional catastrophic sequences $\langle m_1, m_2, \ldots, m_{g-1}\rangle$ such that $m_i \neq 0$, for $1 \leq i \leq g-1$. We have:

Lemma 1 $\left|\bar{S}_g^U\right| < F^B(2g-1)$.

Proof: Any sequence $\langle m_1, m_2, \ldots, m_{g-1}\rangle$ belonging to $\bar{S}_g^U$ can be transformed into one for the bidirectional case by substituting each $m_i \leq -2$ with a string $m_{i_1}, m_{i_2}, \ldots, m_{i_l}$ with $l = |m_i| + 2$, $m_{i_1} = m_{i_l} = 0$ and $m_{i_j} = -1$ for $2 \leq j \leq l-1$. It is straightforward to see that the new sequence satisfies the properties given in Proposition 1 for a given $g' \geq g$. We are then left with bounding g'. Note that $g' - 1 = g - 1 + \left|\sum_{m_i \leq -2} m_i\right| + n$, where n is the number of terms in the summation. Given that

$$0 = -\left|\sum_{m_i \leq -2} m_i\right| + \sum_{m_i > -2} m_i \leq -\left|\sum_{m_i \leq -2} m_i\right| + g - 1 - n,$$

we have that $n + \left|\sum_{m_i \leq -2} m_i\right| \leq g - 1$. Therefore $g' \leq 2g - 1$. In order to obtain the desired mapping, we simply "pad" each transformed sequence of length $< 2g - 1$ with zeroes. It is immediate to see that the obtained mapping is injective. The lemma follows. $\square$

Theorem 6 $F^U(g) \in O\left(\frac{10^g}{g^{3/2}}\right)$.

Proof: Any catastrophic sequence in S_g^U (except for the sequence of $g-1$ zeroes) can be univocally obtained by interleaving a catastrophic sequence in $\bar{S}_j^U$, $3 \leq j \leq g$, with $g - j$ zeroes. Vice versa, any interleaving of a given catastrophic sequence in $\bar{S}_j^U$, $3 \leq j \leq g$, with $g - j$ zeroes yields a catastrophic sequence in S_g^U different from the sequence of $g - 1$ zeroes. Therefore, from Lemma 1:

$$\begin{aligned} \mathrm{F}^U(g) &= 1 + \sum_{j=3}^{g} \left|\bar{S}_j^U\right| \binom{g-1}{g-j} \\ &\leq 1 + \sum_{j=3}^{g} \mathrm{F}^B(2g-1) \binom{g-1}{g-j}. \end{aligned}$$

By applying theorem 3, after some trivial manipulations we can determine a small constant $c > 0$ such that

$$\mathrm{F}^U(g) \leq 1 + c\sum_{j=2}^{g-1} \frac{3^{2j}}{j^{3/2}} \binom{g-1}{j} \in O\left(\frac{10^g}{g^{3/2}}\right).$$

$\square$

4 Application

In the previous sections, we have proved bounds on the number of minimal Catastrophic Fault Patterns (CFP's) for both bidirectional and unidirectional arrays with link redundancy g. Our bounds are tight for the bidirectional case.

The study of minimal CFP's for such architectures has received vast attention in the open literature. In particular, efficient testing algorithms have been devised [5] to detect the presence of such patterns. Restricting attention to minimal CFP's is justified by the fact that, in VLSI manufacturing, expensive on-chip reconfiguration is attempted only when the number of faults does not exceed a certain threshold. Otherwise, the defective chip is simply discarded [2].

The study of minimal CFP's relates to the case when reconfiguration is attempted only if at most g faults are detected. A measure of interest is then the fraction f of chips for which no reconfiguration strategy is successful: bounds on f are useful when assessing the *cost* of the manufacturing process. Note that f is the probability of the following event:

$\mathbf{C}$ = "There are exactly g faulty PE's in the array which disconnect node I from node O".

We evaluate f under a well established probabilistic framework, where each PE has an independent probability p of being faulty and $(1-p)$ of being operating correctly [2]. Let **CFP** be the event: "g faulty PE's in the array form a CFP" and **G** the event: "There are exactly g faulty PE's". Then

$$f = \Pr(\mathbf{C}) = \Pr(\mathbf{CFP} \cap \mathbf{G}) = \Pr(\mathbf{CFP}|\mathbf{G}) \binom{N}{g} p^g (1-p)^{N-g}.$$

We are then left with bounding $\Pr(\mathbf{CFP}|\mathbf{G})$ for both the bidirectional and unidirectional case. Let $\mathrm{F}_j^x(g)$, $x \in \{B, U\}$, be the number of CFP's starting at any p_j with $0 \leq j \leq N-g$. Clearly,

$$\Pr(\mathbf{CFP}|\mathbf{G}) = \frac{\sum_{j=0}^{N-g} \mathrm{F}_j^x(g)}{\binom{N}{g}} \leq \frac{(N-g)\mathrm{F}^x(g)}{\binom{N}{g}}, \tag{2}$$

since any $\mathrm{F}_j^x(g)$, $0 \leq j \leq N-g$, can be upper bounded by $\mathrm{F}^x(g)$ (note that the above bound is particularly accurate for $N \gg g$). By expanding (2) on the basis of Theorems 3 and 6 we then have, for the bidirectional case,

$$\Pr(\mathbf{C}) \in O\left(\frac{N(3p)^g(1-p)^{N-g}}{g^{3/2}}\right).$$

Finally, for the unidirectional case, we have

$$\Pr(\mathbf{C}) \in O\left(\frac{N(10p)^g(1-p)^{N-g}}{g^{3/2}}\right).$$

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