Year 2020-2021
Estimation and filtering

Bayesian estimation using stochastic simulation: theory e applications

Prof. Gianluigi Pillonetto

SUMMARY

- Fisherian vs Bayesian estimation
- Bayesian estimation using Monte Carlo methods
- Bayesian estimation using Markov chain Monte Carlo
- On-line Bayesian estimation (particle filters)

STATE-SPACE MODEL

$$x_k = f\left(x_{k-1}, v_{k-1}\right)$$
$$y_k = h\left(x_k, w_k\right)$$

v, w independent noises

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$x_k \in \mathrm{R}^n$ Markov process

$$p(x_k \mid x_{k-1}, x_{k-2}, \ldots) = p(x_k \mid x_{k-1})$$

$p(x_0)$ given

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FILTERED POSTERIORS

$p(x_{0:k} \mid y_{1:k})$ joint filtered density

$p(x_k \mid y_{1:k})$ filtered density

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PROPAGATION OF THE JOINT FILTERED DENSITY

$$p(x_{0:k} \mid y_{1:k}) = \frac{p(y_{1:k} \mid x_{0:k})\, p(x_{0:k})}{\int p(y_{1:k} \mid x_{0:k})\, p(x_{0:k})\, dx_{0:k}}$$

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Update

$$p(x_{0:k+1} \mid y_{1:k+1}) = p(x_{0:k} \mid y_{1:k}) \,\frac{p(y_{k+1} \mid x_{k+1})\, p(x_{k+1} \mid x_k)}{p(y_{k+1} \mid y_{1:k})}$$

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PROOF

$$p(x_{0:k+1} \mid y_{1:k+1})p(y_{k+1} \mid y_{1:k})p(y_{1:k}) = p(y_{k+1} \mid x_{0:k+1}, y_{1:k})p(x_{k+1} \mid x_{0:k}, y_{1:k})p(x_{0:k} \mid y_{1:k})p(y_{1:k})$$

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$$p(x_{0:k+1} \mid y_{1:k+1}) p(y_{k+1} \mid y_{1:k}) \cancel{p(y_{1:k})} = \underbrace{p(y_{k+1} \mid x_{0:k+1}, y_{1:k})}_{p(y_{k+1} \mid x_{k+1})} \underbrace{p(x_{k+1} \mid x_{0:k}, y_{1:k})}_{p(x_{k+1} \mid x_k)} p(x_{0:k} \mid y_{1:k}) \cancel{p(y_{1:k})}$$

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Time update (Chapman-Kolmogorov)

$$p(x_k \mid y_{1:k-1}) = \int p(x_k \mid x_{k-1})\, p(x_{k-1} \mid y_{1:k-1})\, dx_{k-1}$$

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PROOF

$$p(x_k, x_{k-1} \mid y_{1:k-1}) = \underbrace{\underbrace{p(x_k \mid x_{k-1}, y_{1:k-1})}_{p(x_k \mid x_{k-1})} p(x_{k-1} \mid y_{1:k-1})}_{\text{then integrated over } x_{k-1}}$$

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Measurements update

$$p(x_k \mid y_{1:k}) = \frac{p(y_k \mid x_k) p(x_k \mid y_{1:k-1})}{p(y_k \mid y_{1:k-1})}$$

$$p(y_k \mid y_{1:k-1}) = \int p(y_k \mid x_k) p(x_k \mid y_{1:k-1}) \, dx_k$$

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$$p(y_k \mid y_{1:k-1}) = \int p(y_k \mid x_k) p(x_k \mid y_{1:k-1}) \, dx_k$$

PROOF: $p(y_k \mid x_k, y_{1:k-1}) = p(y_k \mid x_k)$ ....

THE LINEAR GAUSSIAN CASE

$$x_k = Fx_{k-1} + v_{k-1}$$
$$y_k = Hx_k + w_k$$

v and w Gaussian and independent noises

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KALMAN FILTER

$$p(x_{k-1} \mid y_{1:k-1}) = N(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1})$$

Time update

$$p(x_k \mid y_{1:k-1}) = N(x_{k-1}; \hat{x}_{k|k-1}, P_{k|k-1})$$

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Measurements update

$$p(x_k \mid y_{1:k}) = N(x_k; \hat{x}_{k|k}, P_{k|k})$$

$$\hat{x}_{k|k-1} = F \hat{x}_{k-1|k-1}$$
$$P_{k|k-1} = F P_{k-1|k-1} F' + Q$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - H \hat{x}_{k|k-1})$$

$$P_{k|k} = P_{k|k-1} - K_k H P_{k|k-1}$$
$$K_k = P_{k|k-1} H' (H P_{k|k-1} H' + R)^{-1}$$
$$\mathrm{Var}(v_k) = Q, \quad \mathrm{Var}(w_k) = R$$

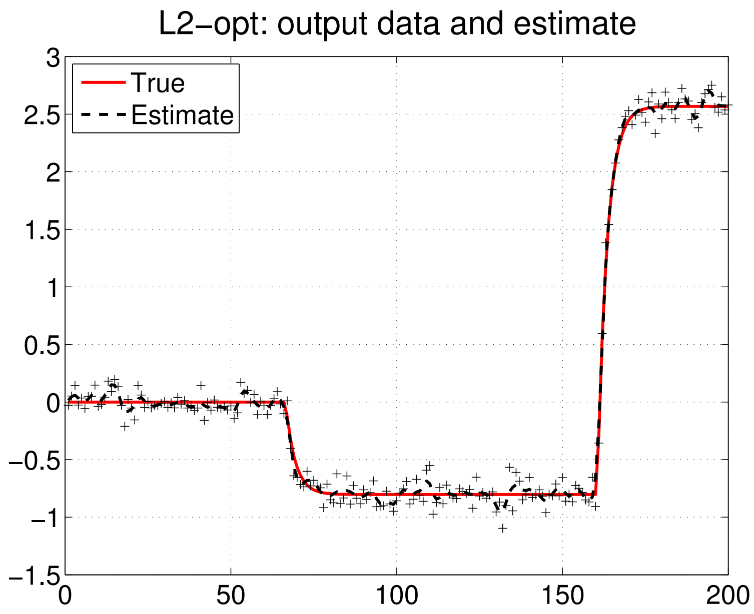
LIMITATIONS OF GAUSSIAN MODELS: LOAD DISTURBANCES AFFECTING A MOTOR

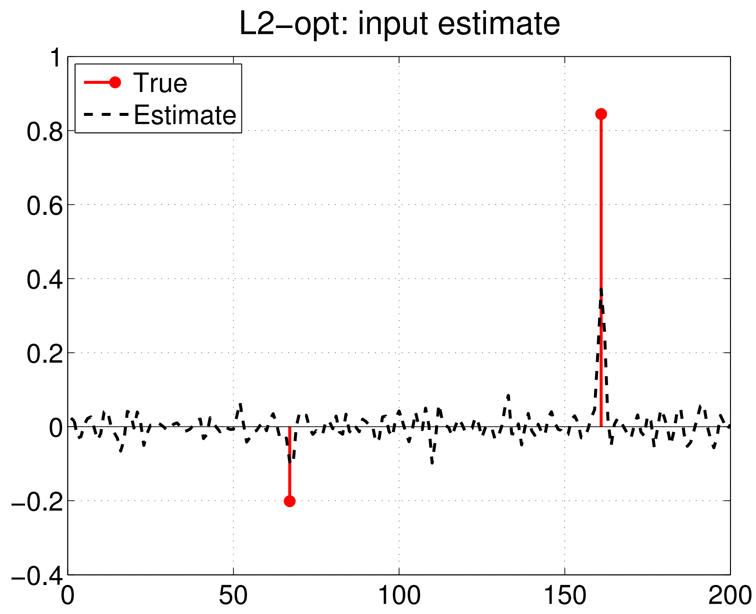
- **State**: angular velocity, angle of motor shaft.
- **Input** u**:** applied torque (known).
- **disturbances** d**:** impulsive, unknown.
- **Dynamics:**

$$x_{t+1} = \begin{pmatrix} 0.7 & 0 \\ 0.08 & 1 \end{pmatrix} x_t + \begin{pmatrix} 11.8 \\ 0.6 \end{pmatrix} (u_t + d_t)$$

$$z_t = \begin{pmatrix} 0 & 1 \end{pmatrix} x_t + e_t$$

- **Right panel:** Best linear estimator of impulsive disturbances d_t is poor. Need a better J to model d_t. (J=model for d_t)





LIMITATIONS OF GAUSSIAN MODELS: OUTLIERS CORRUPTING THE OUTPUTS

- **Dynamics:**

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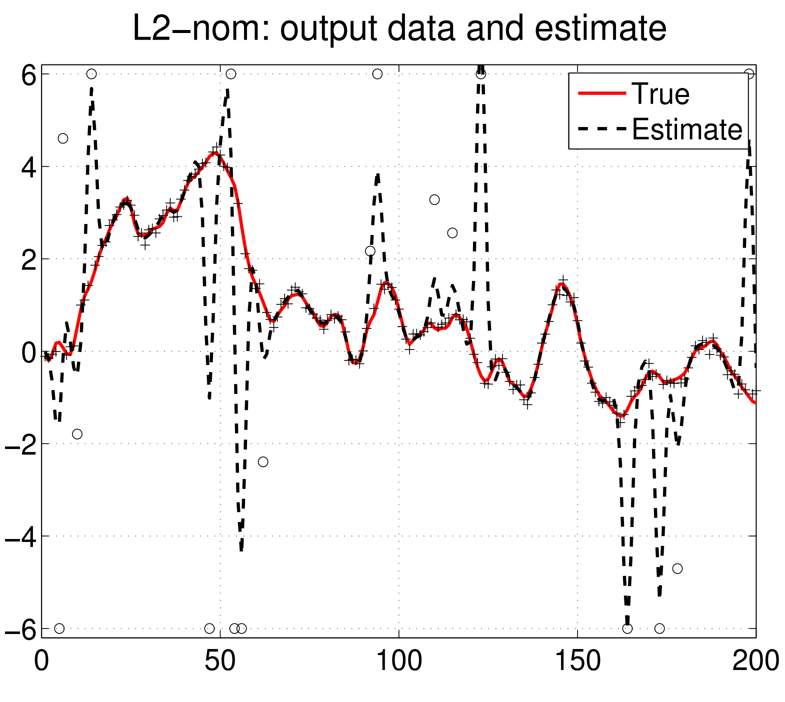
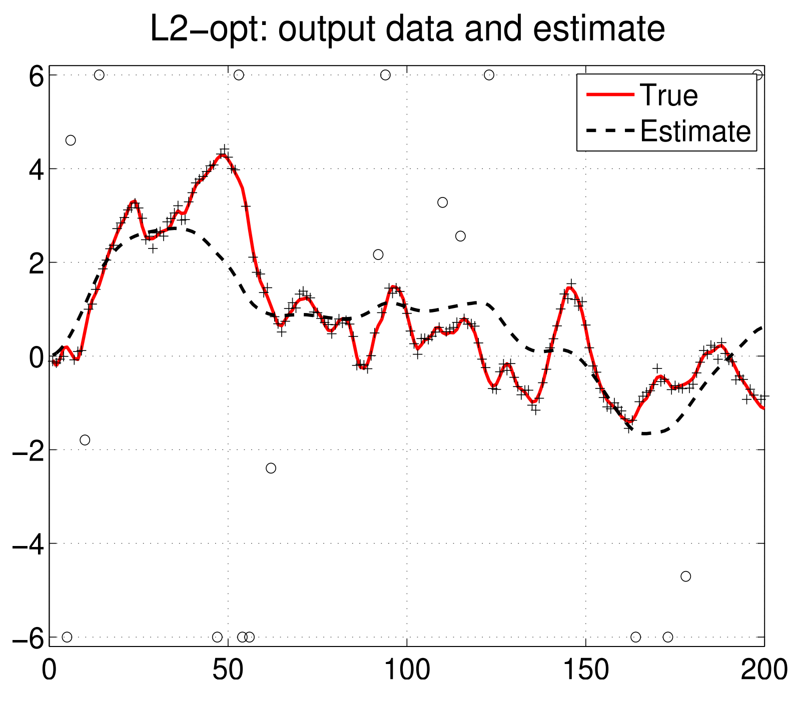
- **Left:** Gaussian model with nominal variance (outliers pull estimate)
- **Right:** Best linear estimate (cannot track signal well)
- **Main point:** . We need a better V to model measurement errors.

(V=model for e_t)





NON LINEAR CASE

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Time update

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$$F_k = \frac{df(\hat{x}_{k-1|k-1})}{dx}$$
$$H_k = \frac{dh(\hat{x}_{k|k-1})}{dx}$$

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1}$$
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$$\mathrm{Var}(v_k) = Q, \quad \mathrm{Var}(w_k) = R$$

EXAMPLE

$$x_k = x_{k-1} + v_{k-1}$$
$$y_{k1} = \left\| x_k - a \right\| + w_{k1}$$
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Localization of an object on a plane with coordinate x_k at instant k

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Localization of an object on a plane with coordinate x_k at instant k

$p(x_1)$ little informative

$\mathrm{Var}(w_{11}) = \mathrm{Var}(w_{12}) = 1$

a=[-5 0], b=[0 5]

$y_{11} = y_{12} = 6$

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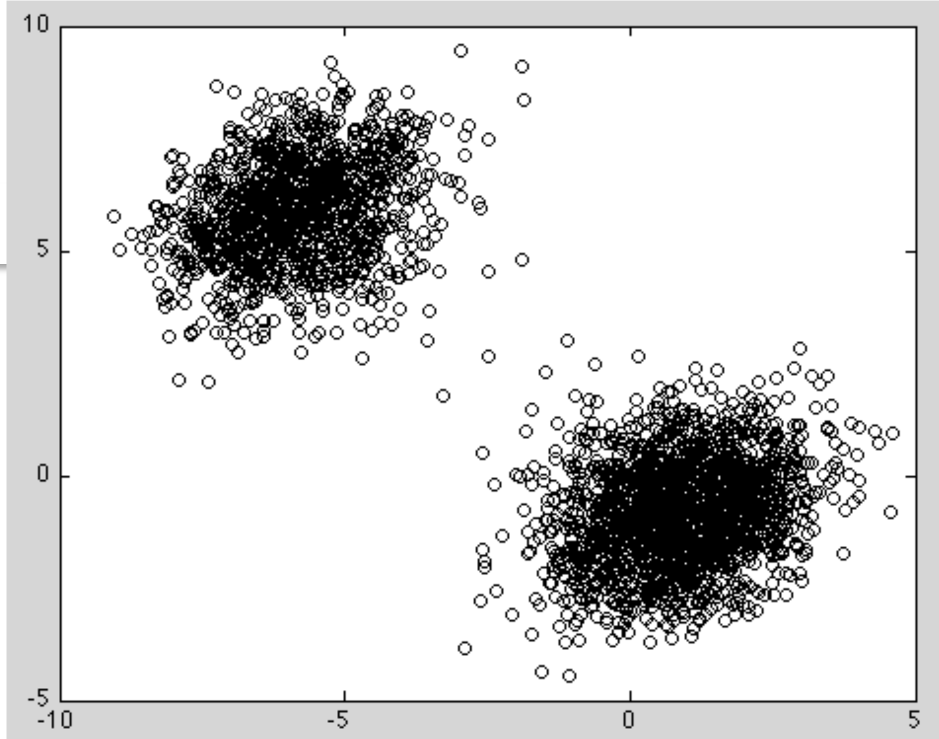
Localization of an object on a plane with coordinate x_k at instant k

$p(x_1 \mid y_1)$



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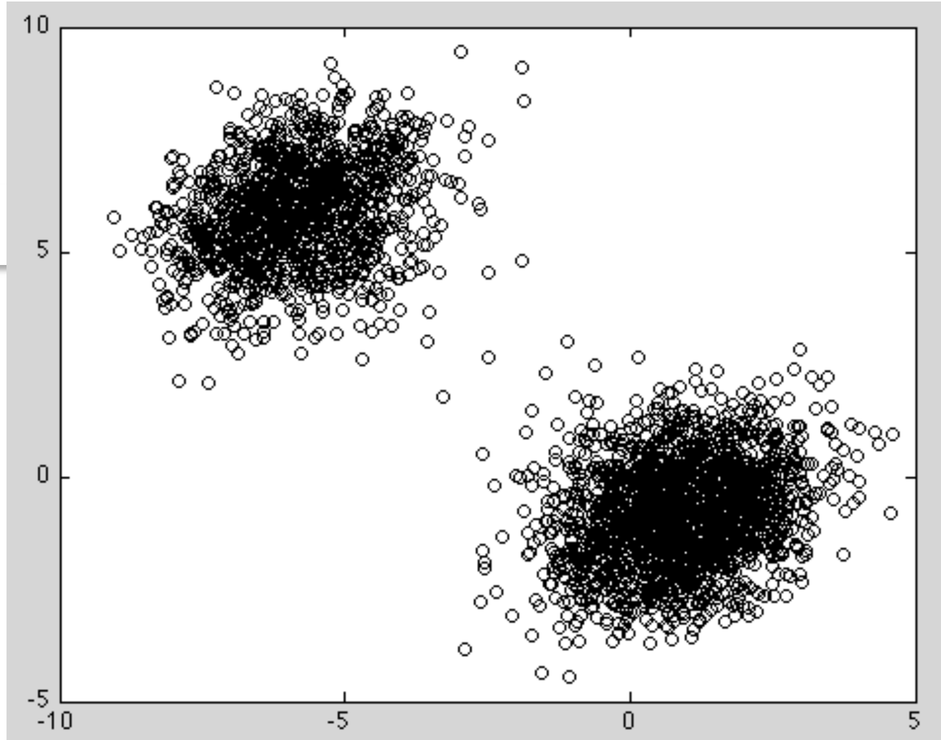

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The Gaussian approximation EKF relies on does not appear reasonable

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$p(x_1 | y_1)$

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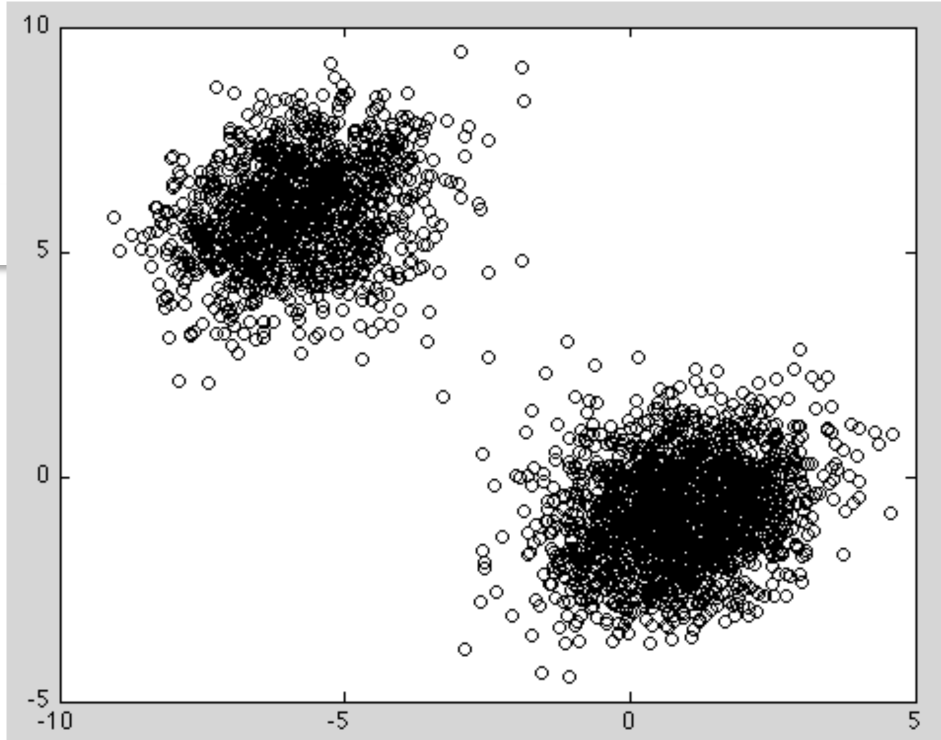

MCMC approach?

Not well suited to an on-line context

IMPORTANCE SAMPLING

$$E_\pi[f(x)] = \int f(x)\pi(x)\,dx$$

$$= \int f(x)\pi(x)\frac{q(x)}{q(x)}\,dx \qquad (q(x) > 0)$$

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Proposal density called importance function

IMPORTANCE SAMPLING

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$$= \int f(x)\frac{\pi(x)}{q(x)}q(x)\,dx$$

$$= E_q[\frac{\pi(x)}{q(x)}f(x)]$$

$$\approx \frac{1}{N_s}\sum_{i=1}^{N_s}\frac{\pi(x^i)}{q(x^i)}f(x^i) \qquad x^i \sim \text{q}$$

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unnormalized weights

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unnormalized weights

Normalized weights

$$w^i = \frac{\pi(x^i)}{q(x^i)} \Big/ \sum_{i=1}^{N_s}\frac{\pi(x^i)}{q(x^i)}$$

(it still holds that $E_\pi[f(x)] \approx \sum_{i=1}^{N_s} w^i f(x^i), \quad x^i \sim \mathrm{q}$)

IMPORTANCE SAMPLING

$$E_\pi[f(x)] = \int f(x)\pi(x)\,dx$$

$$= \int f(x)\pi(x)\frac{q(x)}{q(x)}\,dx \qquad (q(x) > 0)$$

$$= \int f(x)\frac{\pi(x)}{q(x)}q(x)\,dx$$

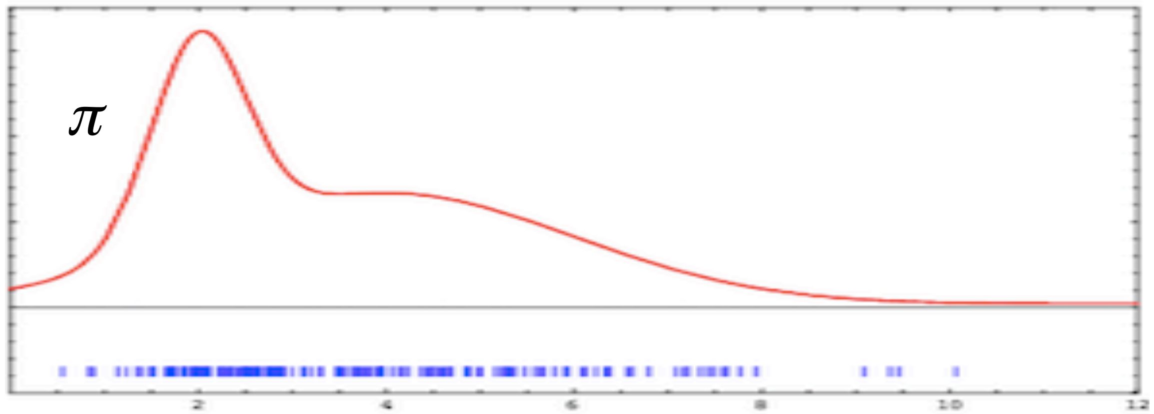
$$= E_q[\frac{\pi(x)}{q(x)}f(x)]$$

$$\approx \frac{1}{N_s}\sum_{i=1}^{N_s}\frac{\pi(x^i)}{q(x^i)}f(x^i) \qquad x^i \sim \text{q}$$

$$\pi(x) \approx \sum_{i=1}^{N_s} w^i \delta(x - x^i)$$

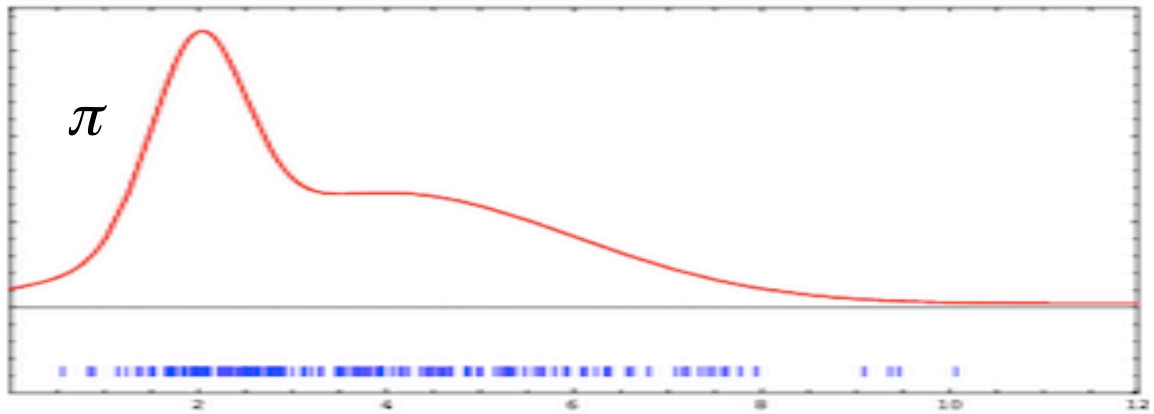
Particle representation of π based on (x^i, w^i)

IMPORTANCE SAMPLING



Target density
and i.i.d. samples
(not obtainable)

IMPORTANCE SAMPLING



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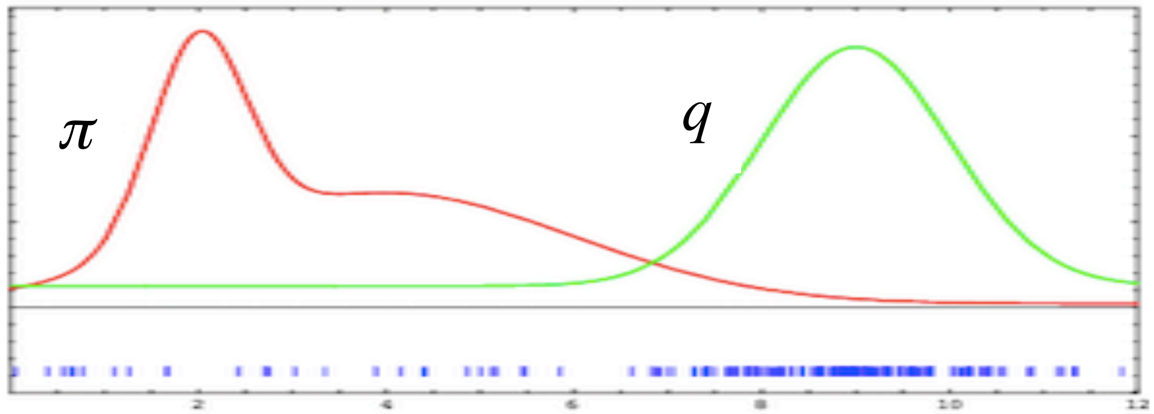
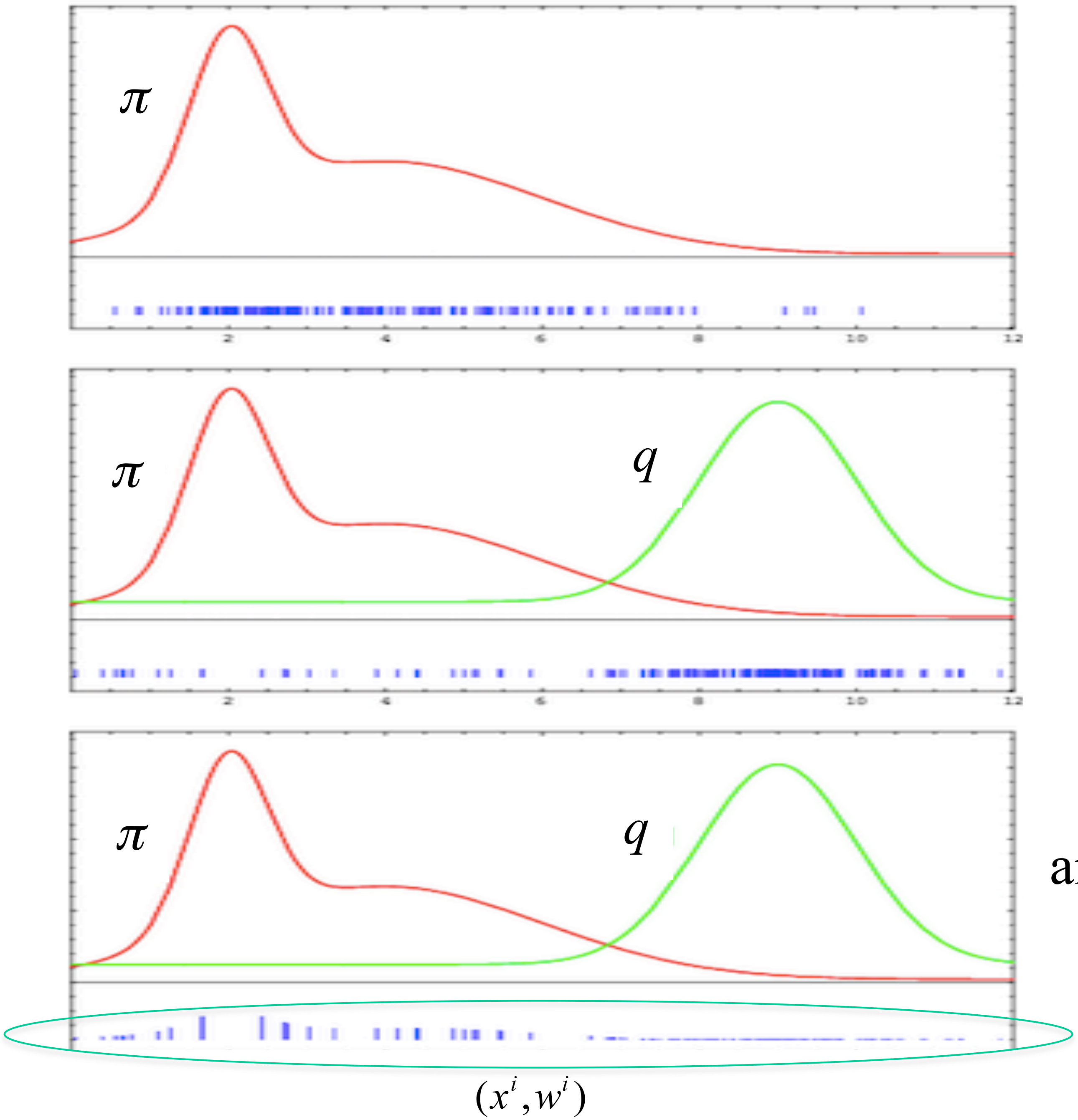


Proposal density
and i.i.d. samples

IMPORTANCE SAMPLING



Target density
and i.i.d. samples
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Proposal density
and i.i.d. samples

Importance weights w^i
and target approximation

$$\pi(x) \approx \sum_{i=1}^{N_s} w^i \delta(x - x^i)$$

SEQUENTIAL IMPORTANCE SAMPLING

Rewriting in state space what we have seen before

$\left\{ x_{0:k}^{i}, w_{k}^{i} \right\}$ probability measure (random)

$$p(x_{0:k} \mid y_{1:k}) \approx \sum_{i=1}^{n} w_k^i \delta(x_{0:k} - x_{0:k}^i) \ , \qquad w_k^i \propto \frac{p(x_{0:k}^i \mid y_{1:k})}{q(x_{0:k}^i \mid y_{1:k})}$$

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Recursive version

We have samples from

$$p(x_{0:k-1} \mid y_{1:k-1})$$

and we want to approximate

$$p(x_{0:k} \mid y_{1:k})$$

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Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

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Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

Here and in what follows we use the equality

$$p(A, B \mid C) = p(A \mid B, C) p(B \mid C)$$

and the assumption

$$q(x_{0:k-1} \mid y_{1:k}) = q(x_{0:k-1} \mid y_{1:k-1})$$

SEQUENTIAL IMPORTANCE SAMPLING

Rewriting in state space what we have seen before

$\left\{x_{0:k}^{i}, w_{k}^{i}\right\}$ probability measure (random)

$$p(x_{0:k} \mid y_{1:k}) \approx \sum_{i=1}^{n} w_k^i \delta(x_{0:k} - x_{0:k}^i) \; , \qquad w_k^i \propto \frac{p(x_{0:k}^i \mid y_{1:k})}{q(x_{0:k}^i \mid y_{1:k})}$$

Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

SEQUENTIAL IMPORTANCE SAMPLING

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Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

$$p(x_{0:k} \mid y_{1:k}) = \frac{p(y_k \mid x_{0:k}, y_{1:k-1}) p(x_{0:k} \mid y_{1:k-1})}{p(y_k \mid y_{1:k-1})}$$

SEQUENTIAL IMPORTANCE SAMPLING

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$p(y_k, x_{0:k} \mid y_{1:k-1})$

SEQUENTIAL IMPORTANCE SAMPLING

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$$p(y_k, x_{0:k} \mid y_{1:k-1}) = p(x_{0:k} \mid y_{1:k}) p(y_k \mid y_{1:k-1})$$

SEQUENTIAL IMPORTANCE SAMPLING

Rewriting in state space what we have seen before

$\{x_{0:k}^i, w_k^i\}$ probability measure (random)

$$p(x_{0:k} \mid y_{1:k}) \approx \sum_{i=1}^{n} w_k^i \delta(x_{0:k} - x_{0:k}^i) \ , \qquad w_k^i \propto \frac{p(x_{0:k}^i \mid y_{1:k})}{q(x_{0:k}^i \mid y_{1:k})}$$

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$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

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$$= \frac{p(y_k \mid x_{0:k}, y_{1:k-1}) p(x_k \mid x_{0:k-1}, y_{1:k-1})}{p(y_k \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

SEQUENTIAL IMPORTANCE SAMPLING

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$\left\{x_{0:k}^{i}, w_{k}^{i}\right\}$ probability measure (random)

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Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

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SEQUENTIAL IMPORTANCE SAMPLING

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$\left\{ x_{0:k}^{i}, w_{k}^{i} \right\}$ probability measure (random)

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Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_{k} \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

$$p(x_{0:k} \mid y_{1:k}) = \frac{p(y_{k} \mid x_{0:k}, y_{1:k-1}) p(x_{0:k} \mid y_{1:k-1})}{p(y_{k} \mid y_{1:k-1})}$$

$$= \frac{p(y_{k} \mid x_{0:k}, y_{1:k-1}) p(x_{k} \mid x_{0:k-1}, y_{1:k-1})}{p(y_{k} \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

$$= \frac{p(y_{k} \mid x_{k}) p(x_{k} \mid x_{k-1})}{p(y_{k} \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

SEQUENTIAL IMPORTANCE SAMPLING

Rewriting in state space what we have seen before

$\left\{x_{0:k}^i, w_k^i\right\}$ probability measure (random)

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Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

$$p(x_{0:k} \mid y_{1:k}) = \frac{p(y_k \mid x_{0:k}, y_{1:k-1}) p(x_{0:k} \mid y_{1:k-1})}{p(y_k \mid y_{1:k-1})}$$

$$= \frac{p(y_k \mid x_{0:k}, y_{1:k-1}) p(x_k \mid x_{0:k-1}, y_{1:k-1})}{p(y_k \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

$$= \frac{p(y_k \mid x_k) p(x_k \mid x_{k-1})}{p(y_k \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

SEQUENTIAL IMPORTANCE SAMPLING

Rewriting in state space what we have seen before

$\left\{x_{0:k}^{i}, w_{k}^{i}\right\}$ probability measure (random)

$$p(x_{0:k} \mid y_{1:k}) \approx \sum_{i=1}^{n} w_k^i \delta(x_{0:k} - x_{0:k}^i) \,, \qquad w_k^i \propto \frac{p(x_{0:k}^i \mid y_{1:k})}{q(x_{0:k}^i \mid y_{1:k})}$$

Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

$$p(x_{0:k} \mid y_{1:k}) = \frac{p(y_k \mid x_{0:k}, y_{1:k-1}) p(x_{0:k} \mid y_{1:k-1})}{p(y_k \mid y_{1:k-1})}$$

$$= \frac{p(y_k \mid x_{0:k}, y_{1:k-1}) p(x_k \mid x_{0:k-1}, y_{1:k-1})}{p(y_k \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

$$= \frac{p(y_k \mid x_k) p(x_k \mid x_{k-1})}{p(y_k \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

SEQUENTIAL IMPORTANCE SAMPLING

Rewriting in state space what we have seen before

$\left\{x_{0:k}^i, w_k^i\right\}$ probability measure (random)

$$p(x_{0:k} \mid y_{1:k}) \approx \sum_{i=1}^{n} w_k^i \delta(x_{0:k} - x_{0:k}^i) \; , \qquad w_k^i \propto \frac{p(x_{0:k}^i \mid y_{1:k})}{q(x_{0:k}^i \mid y_{1:k})}$$

Recursive version

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

$$p(x_{0:k} \mid y_{1:k}) = \frac{p(y_k \mid x_{0:k}, y_{1:k-1}) p(x_{0:k} \mid y_{1:k-1})}{p(y_k \mid y_{1:k-1})}$$

$$= \frac{p(y_k \mid x_{0:k}, y_{1:k-1}) p(x_k \mid x_{0:k-1}, y_{1:k-1})}{p(y_k \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

$$= \frac{p(y_k \mid x_k) p(x_k \mid x_{k-1})}{p(y_k \mid y_{1:k-1})} p(x_{0:k-1} \mid y_{1:k-1})$$

$$\propto p(y_k \mid x_k) p(x_k \mid x_{k-1}) p(x_{0:k-1} \mid y_{1:k-1})$$

SEQUENTIAL IMPORTANCE SAMPLING

Rewriting in state space what we have seen before

$\left\{x_{0:k}^i, w_k^i\right\}$ probability measure (random)

$$p(x_{0:k} \mid y_{1:k}) \approx \sum_{i=1}^{n} w_k^i \delta(x_{0:k} - x_{0:k}^i) \,, \qquad w_k^i \propto \frac{p(x_{0:k}^i \mid y_{1:k})}{q(x_{0:k}^i \mid y_{1:k})}$$

Recursive version

By combining our previous results:

$$q(x_{0:k} \mid y_{1:k}) = q(x_k \mid x_{0:k-1}, y_{1:k}) q(x_{0:k-1} \mid y_{1:k-1})$$

$$p(x_{0:k} \mid y_{1:k}) \propto p(y_k \mid x_k) p(x_k \mid x_{k-1}) p(x_{0:k-1} \mid y_{1:k-1})$$

e

$$w_k^i \propto \frac{p(x_{0:k}^i \mid y_{1:k})}{q(x_{0:k}^i \mid y_{1:k})}$$

SEQUENTIAL IMPORTANCE SAMPLING

We obtain:

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\,p(x_k^i \mid x_{k-1}^i)\,p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\,q(x_{0:k-1}^i \mid y_{1:k-1})}$$

SEQUENTIAL IMPORTANCE SAMPLING

We obtain:

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})} \quad w_{k-1}^i$$

SEQUENTIAL IMPORTANCE SAMPLING

We obtain:

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})}$$

$$= w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})}$$

SEQUENTIAL IMPORTANCE SAMPLING

We obtain:

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})}$$

$$= w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})}$$

Markovian proposal assumption

$$q(x_k^i \mid x_{0:k-1}^i, y_{1:k}) = q(x_k^i \mid x_{k-1}^i, y_k)$$

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})} = w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, y_k)}$$

SEQUENTIAL IMPORTANCE SAMPLING

We obtain:

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})}$$

$$= w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})}$$

Markovian proposal assumption

$$q(x_k^i \mid x_{0:k-1}^i, y_{1:k}) = p(x_k^i \mid x_{k-1}^i)$$

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})} = w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, y_k)}$$

SEQUENTIAL IMPORTANCE SAMPLING

We obtain:

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})}$$

$$= w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})}$$

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$$q(x_k^i \mid x_{0:k-1}^i, y_{1:k}) = p(x_k^i \mid x_{k-1}^i)$$

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})} = w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, y_k)}$$

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$$= w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})}$$

Markovian proposal assumption

$$q(x_k^i \mid x_{0:k-1}^i, y_{1:k}) = p(x_k^i \mid x_{k-1}^i)$$

$$w_k^i \propto w_{k-1}^i\, p(y_k \mid x_k^i)$$

SEQUENTIAL IMPORTANCE SAMPLING

We obtain:

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})}$$

$$= w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})}$$

Markovian proposal assumption

$$q(x_k^i \mid x_{0:k-1}^i, y_{1:k}) = q(x_k^i \mid x_{k-1}^i, y_k)$$

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})} = w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, y_k)}$$

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$$= w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})}$$

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$$q(x_k^i \mid x_{0:k-1}^i, y_{1:k}) = q(x_k^i \mid x_{k-1}^i, y_k)$$

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SIS (PARTICLE FILTER)

Particle representation and propagation of the filtered density

$$p(x_k \mid y_{1:k}) \approx \sum_{i=1}^{N_s} w_k^i \delta(x_k - x_k^i)$$

$$w_k^i \propto w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, y_k)}$$

SEQUENTIAL IMPORTANCE SAMPLING

We obtain:

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$$= w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})}$$

Markovian proposal assumption

$$q(x_k^i \mid x_{0:k-1}^i, y_{1:k}) = q(x_k^i \mid x_{k-1}^i, y_k)$$

$$w_k^i \propto \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)\, p(x_{0:k-1}^i \mid y_{1:k-1})}{q(x_k^i \mid x_{0:k-1}^i, y_{1:k})\, q(x_{0:k-1}^i \mid y_{1:k-1})} = w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, y_k)}$$

SIS (PARTICLE FILTER)

Particle representation and propagation of the filtered density

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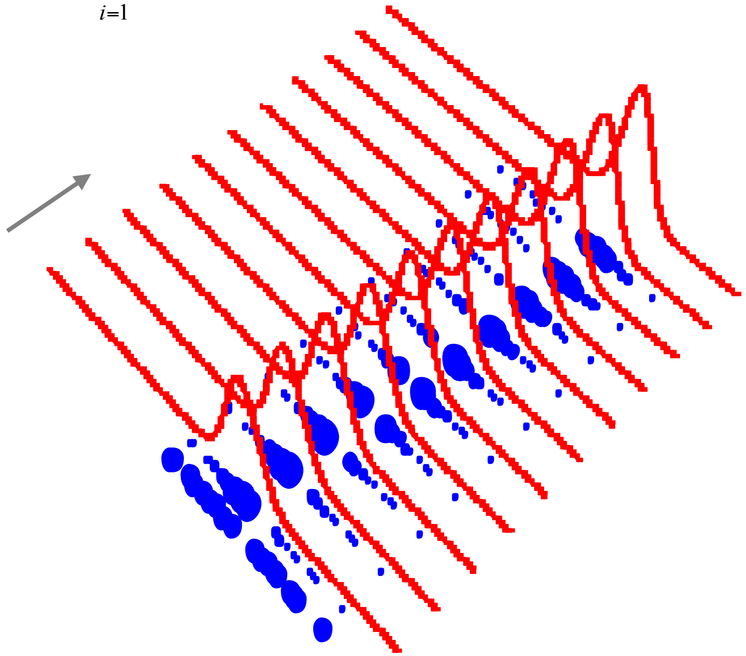
Algorithm 1: SIS Particle Filter

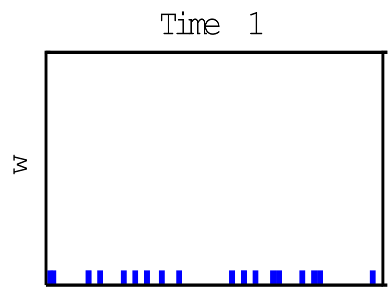
$[\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}] = \mathrm{SIS}[\{\mathbf{x}_{k-1}^i, w_{k-1}^i\}_{i=1}^{N_s}, y_k]$

- FOR $i = 1 : N_s$
  - Draw $\mathbf{x}_k^i \sim q(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i, y_k)$
  - Assign the particle a weight, w_k^i, according to $w_k^i \propto w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, y_k)}$
- END FOR

SIS IN ACTION

$$p(x_k \mid y_{1:k}) \approx \sum_{i=1}^{N_s} w_k^i \delta(x_k - x_k^i)$$





Algorithm 1: SIS Particle Filter

$[\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}] = \text{SIS}[\{\mathbf{x}_{k-1}^i, w_{k-1}^i\}_{i=1}^{N_s}, y_k]$

- FOR $i = 1$: N_s
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- END FOR

SIS IN ACTION

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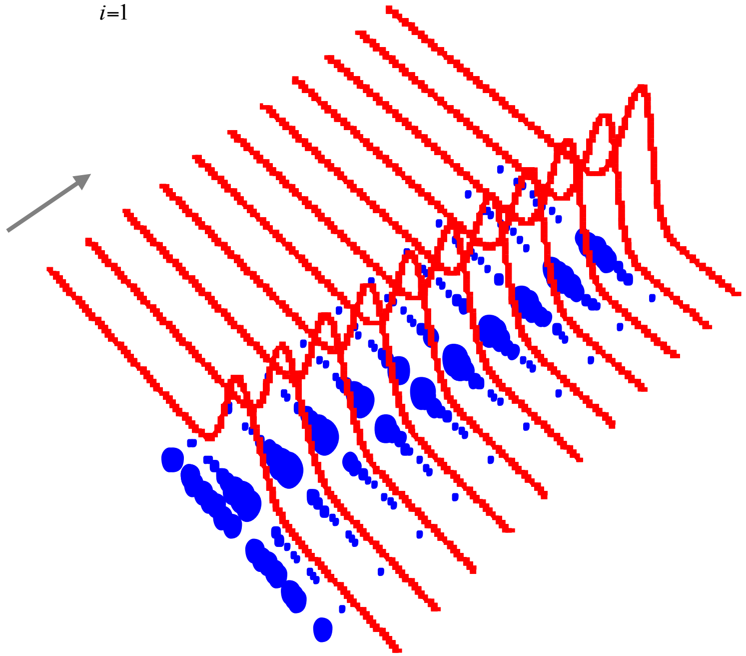

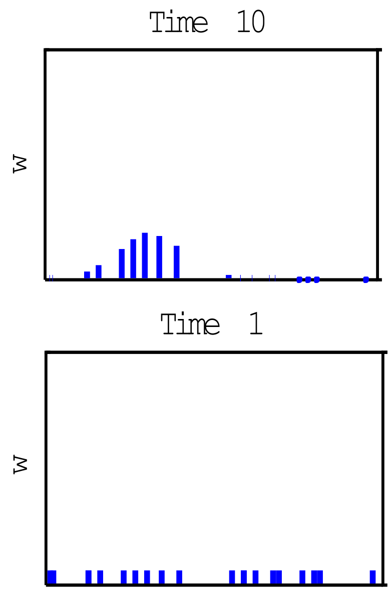

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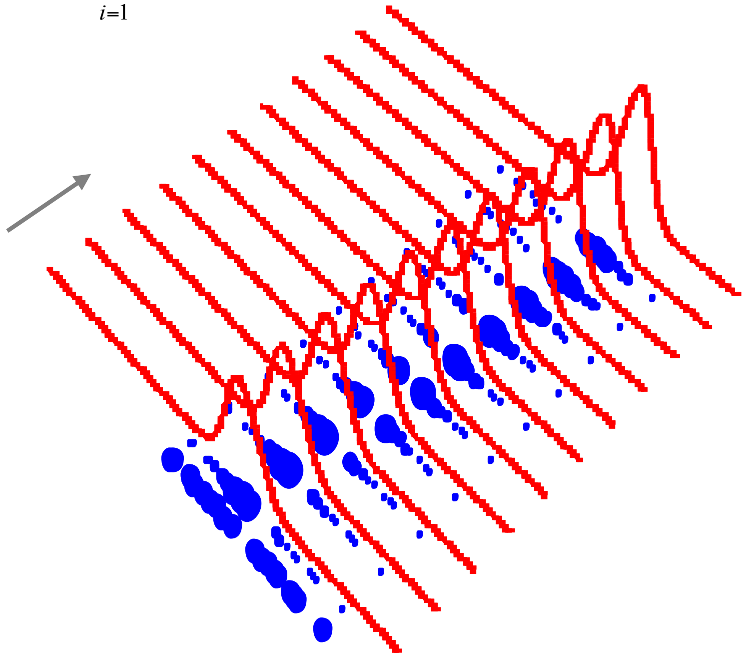

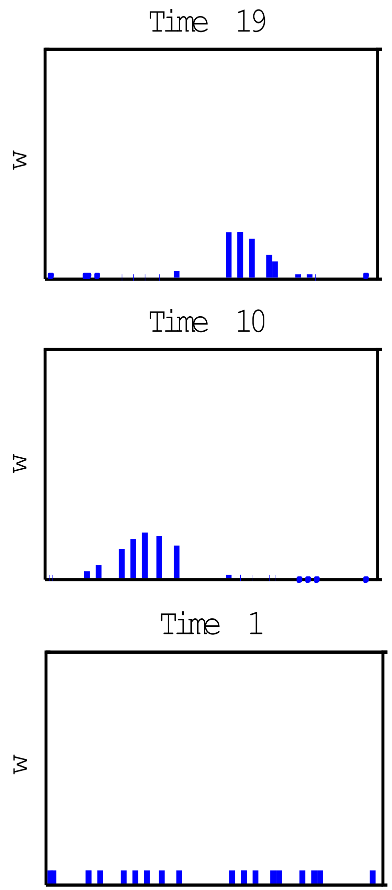

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SIS LIMITATIONS: DEGENERACY

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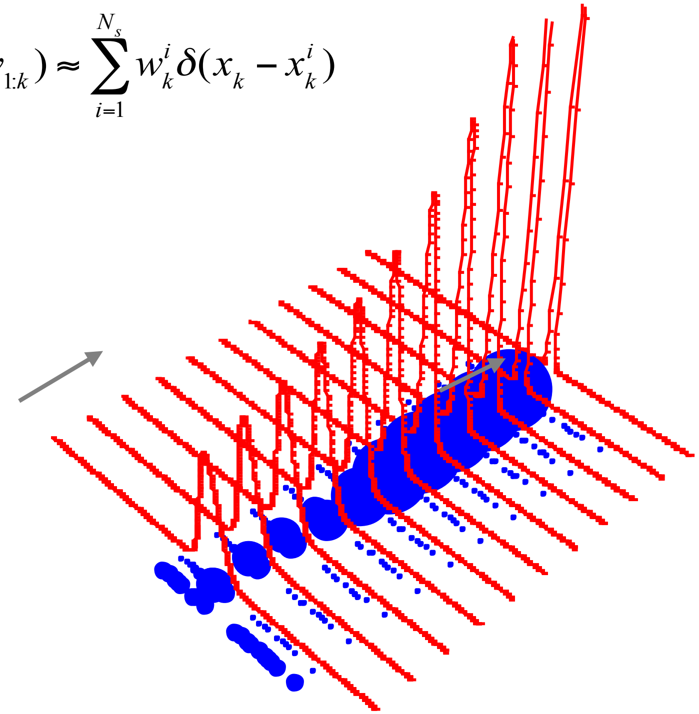

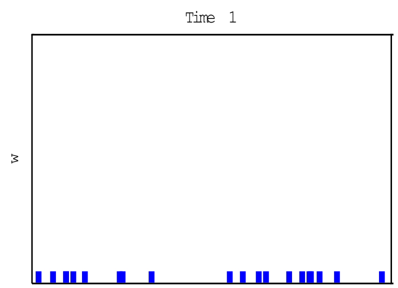

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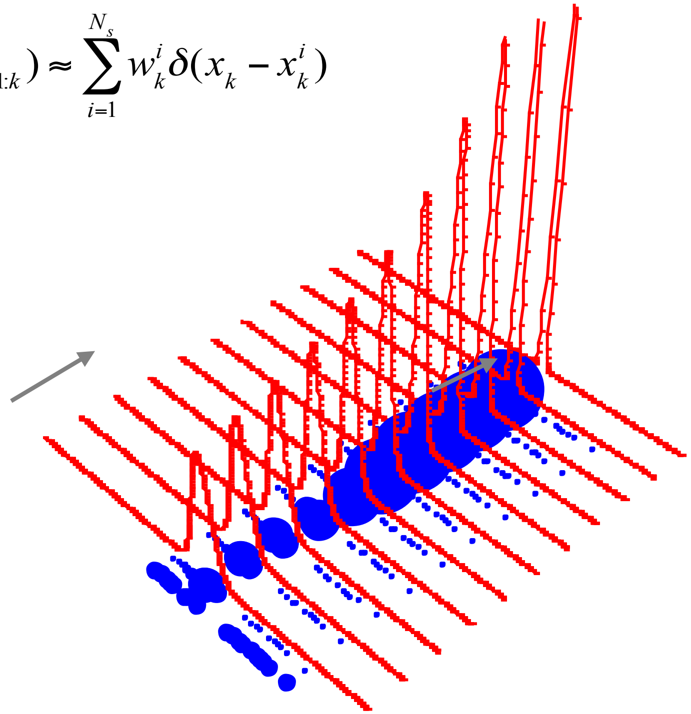

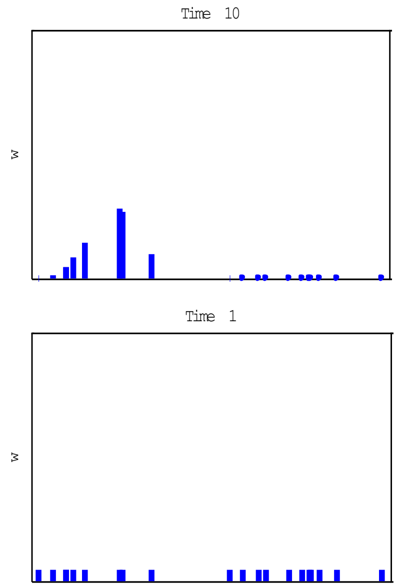

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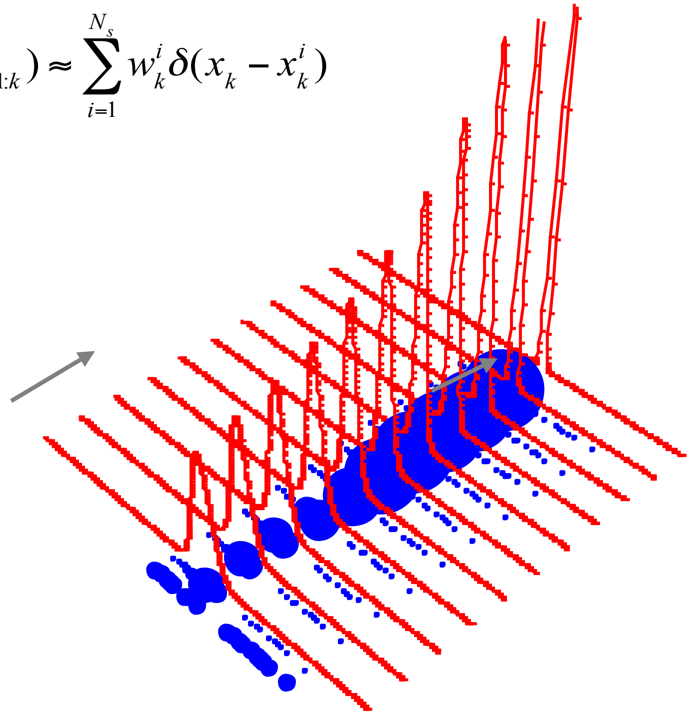

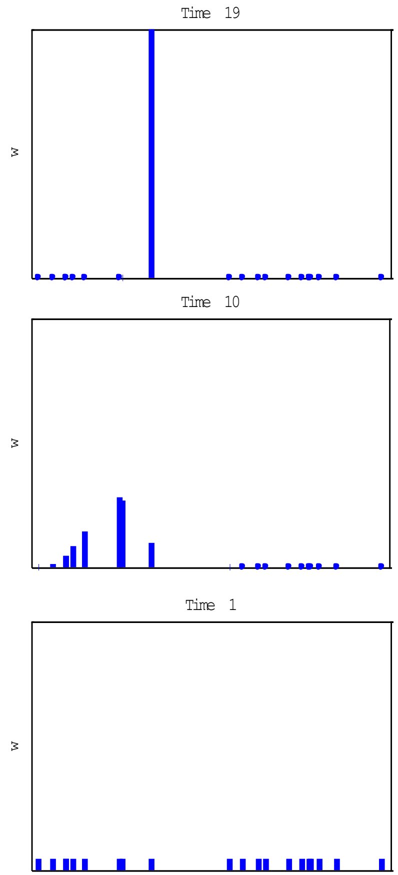

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- END FOR

DEGENERACY: THEORY

Best proposal: $q(x_{0:k}^{i} \mid y_{1:k}) = p(x_{0:k}^{i} \mid y_{1:k})$

Thinking of the weights as random variables it is obtained using:

$E\left[w(x_{0:k})\right] = 1$ (normalized with N_s=1)

$Var\left[w(x_{0:k})\right] = 0$

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Note: **conditional on** $x^{i}_{0:k-1}$ **and** $y_{1:k}$ **the best proposal (not usable in practice) is**

$$p(x_k \mid x_{k-1}^{i}, y_k)$$

DEGENERACY: THEORY

Best proposal: $q(x^i_{0:k} \mid y_{1:k}) = p(x^i_{0:k} \mid y_{1:k})$

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Theorem: for proposals (importance functions) of the type

$$q(x_{0:n} \mid y_{1:n}) = q(x_0)\prod_{k=1}^{n} q(x_k \mid x_{0:k-1}, y_{1:k})$$

$Var\left[w(x_{0:k})\right]$ **always increases as *k* increases**

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NON DEGENERACY MEASURE

$$N_{eff} = \frac{N_s}{1 + \mathrm{Var}(w^{*i}_k)}$$

$$w^{*i}_k = \frac{p(x^i_k \mid y_{1:k-1})}{q(x^i_k \mid x^i_{k-1}, y_k)}$$

true weight

DEGENERACY: THEORY

Best proposal: $q(x_{0:k}^i \mid y_{1:k}) = p(x_{0:k}^i \mid y_{1:k})$

Thinking of the weights as random variables it is obtained using:

$$E\left[w(x_{0:k})\right] = 1 \quad \text{(normalized with } N_s\text{=1)}$$

$$Var\left[w(x_{0:k})\right] = 0$$

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$$q(x_{0:n} \mid y_{1:n}) = q(x_0)\prod_{k=1}^{n} q(x_k \mid x_{0:k-1}, y_{1:k})$$

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NON DEGENERACY MEASURE

$$N_{eff} = \frac{N_s}{1 + \mathrm{Var}(w_k^{*i})}$$

$$w_k^{*i} = \frac{p(x_k^i \mid y_{1:k-1})}{q(x_k^i \mid x_{k-1}^i, y_k)}$$

true weight

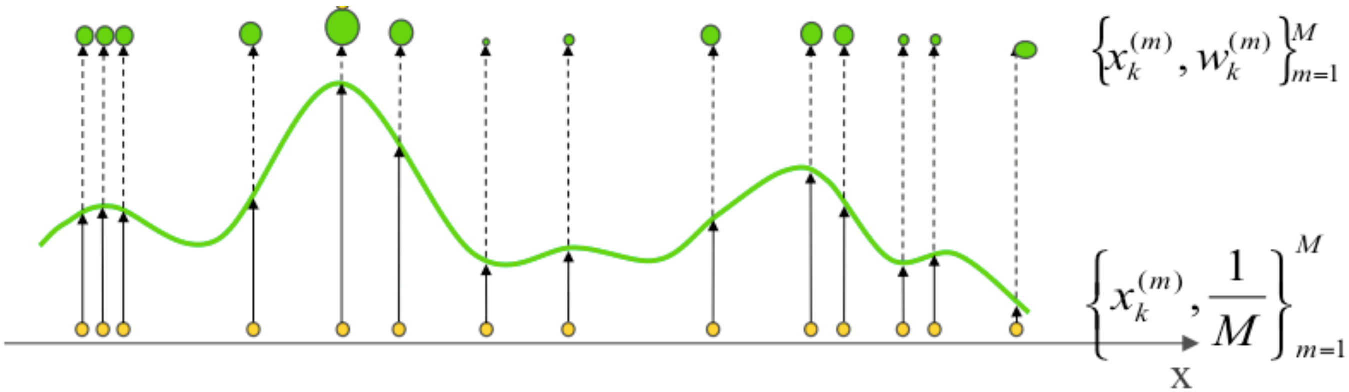
$$N_{eff} = \frac{1}{\sum_{i=1}^{N_s} (w_k^i)^2}$$

Degeneration estimate using normalized weights

REGENERATION (RESAMPLING)

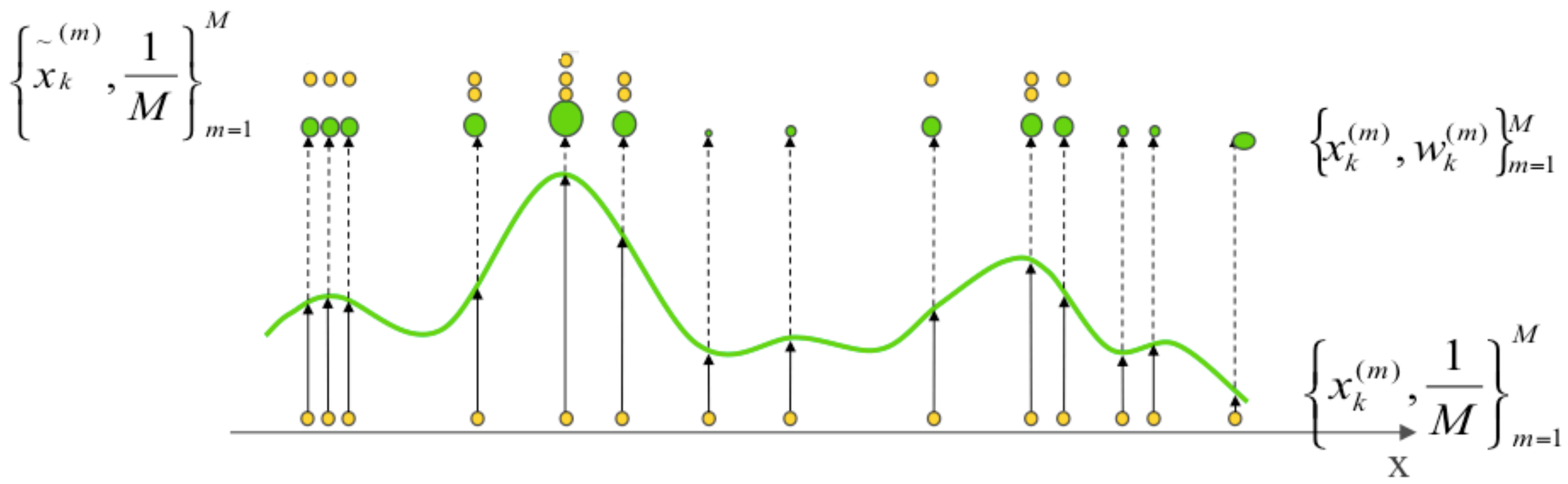
Starting point is a particle density with uniform weights

REGENERATION (RESAMPLING)



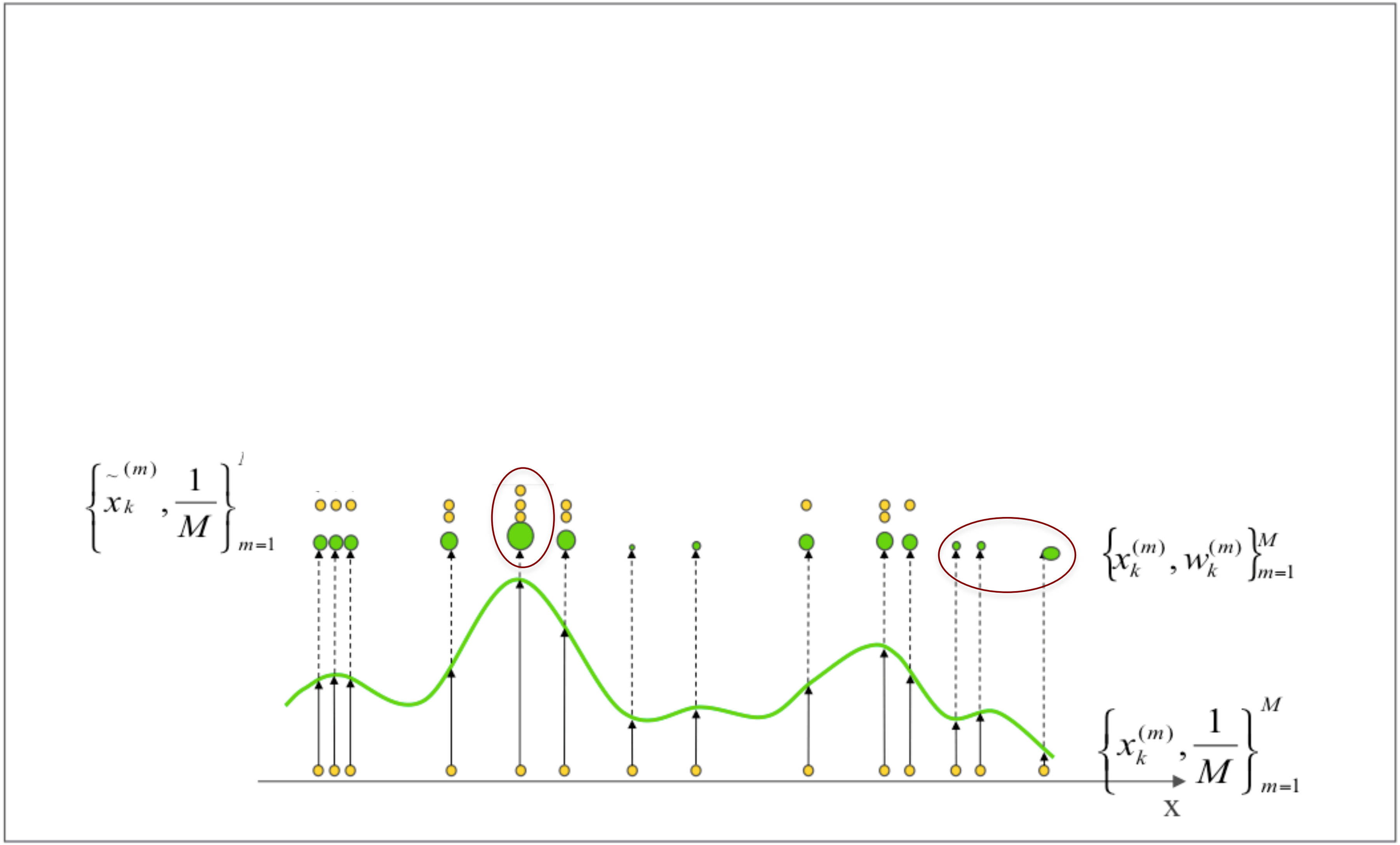
New data arrive and we propagate the pdf obtaining non uniform weights

REGENERATION (RESAMPLING)



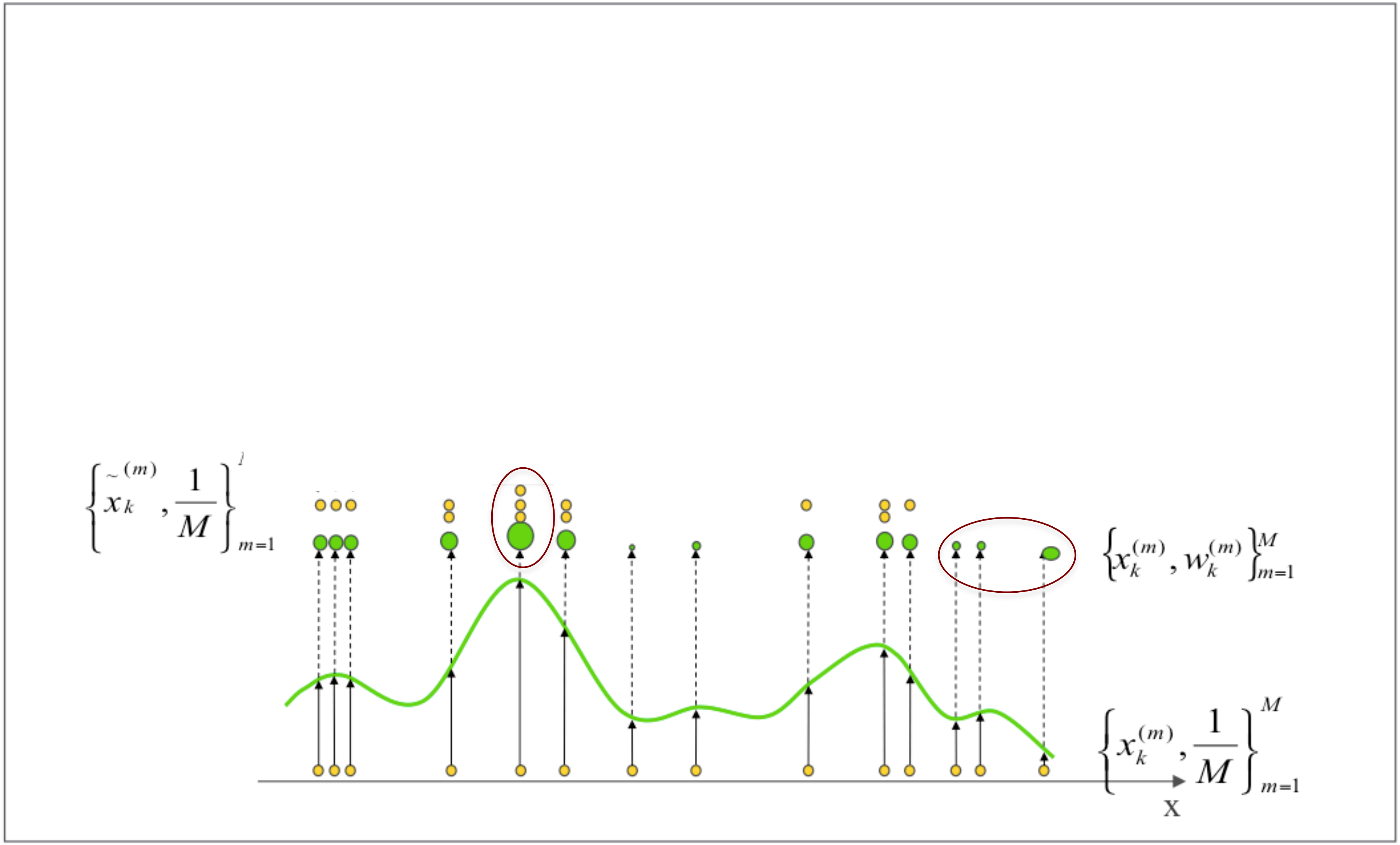
Resampling of the particle density
leads to uniform weights and possible repetitions of x

REGENERATION (RESAMPLING)



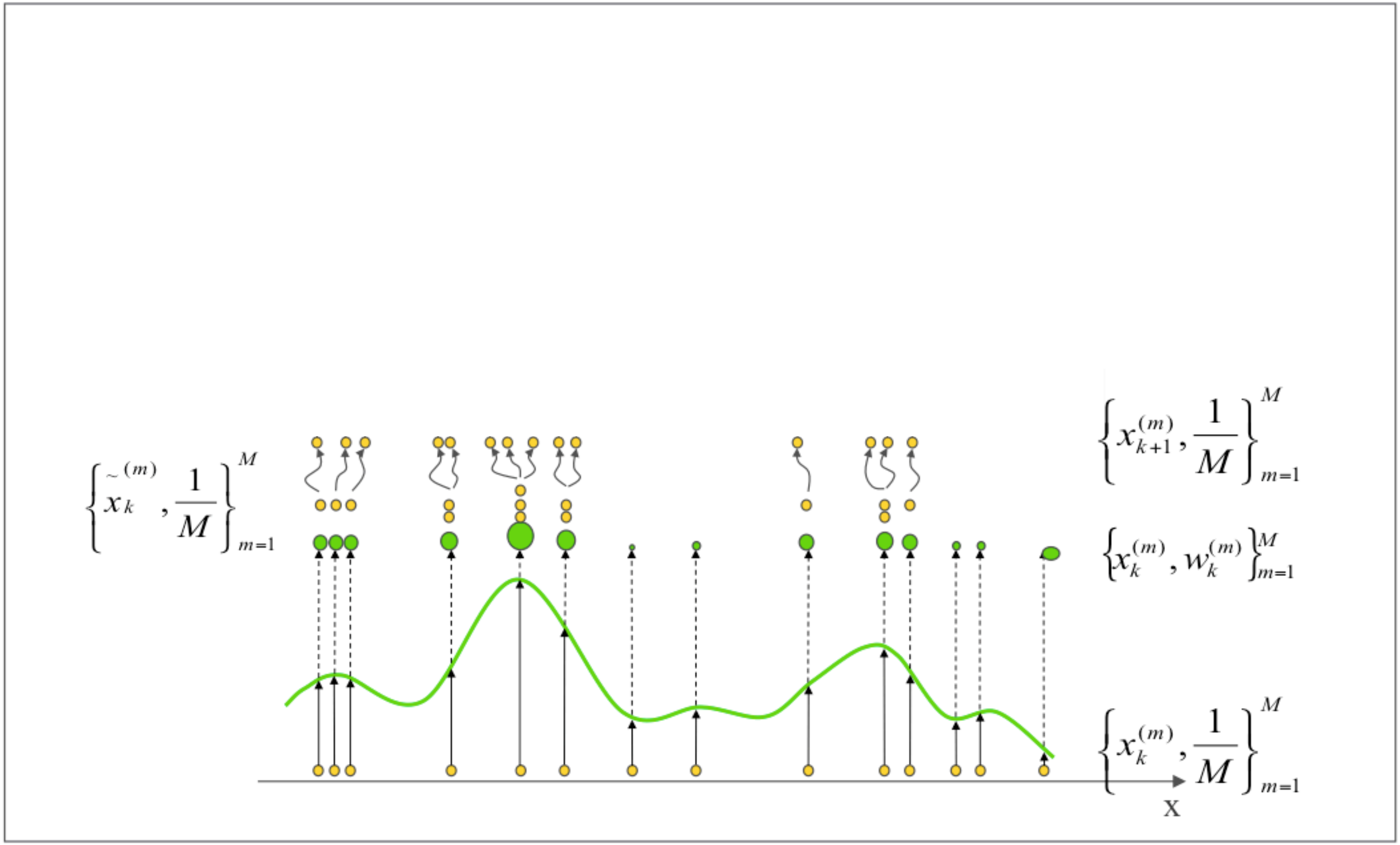
Note that high-probability regions are emphasized
and unimportant regions are lost

REGENERATION (RESAMPLING)



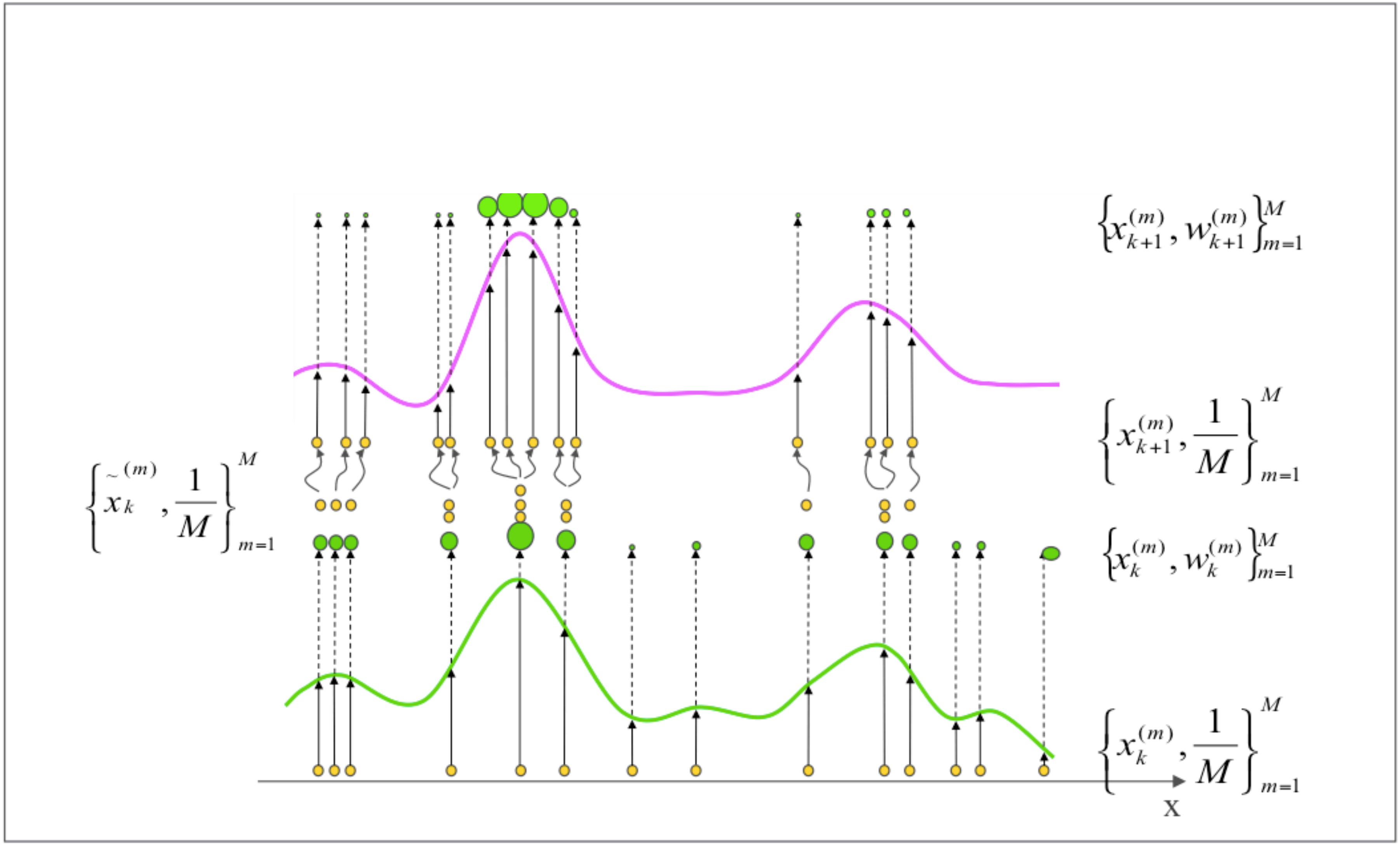
Similar to MCMC philosophy: “to move around regions found to have large probability” (Resample and Move)

REGENERATION (RESAMPLING)



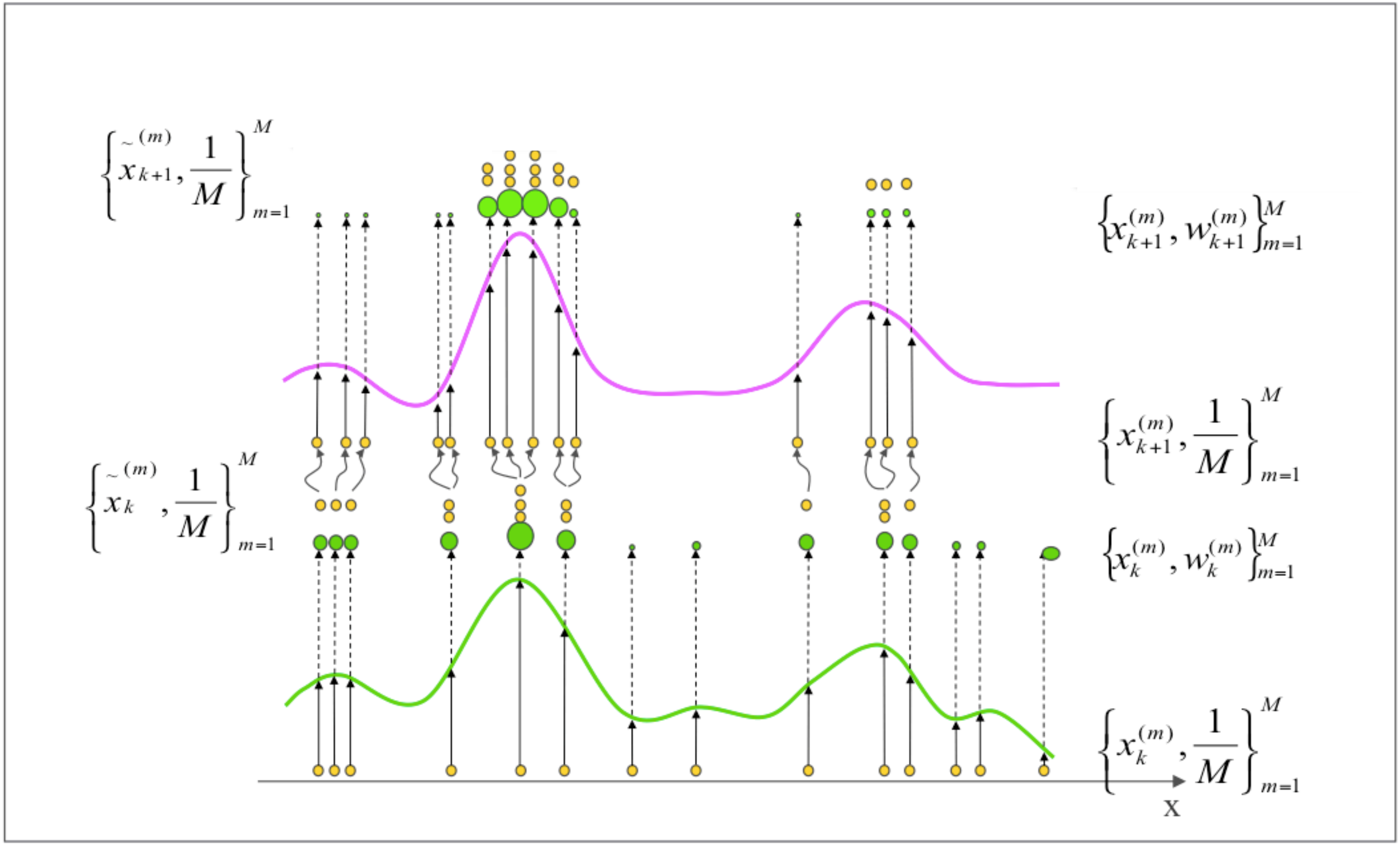
The new density then evolves by adding the transition noise
(weights remain uniform)

REGENERATION (RESAMPLING)



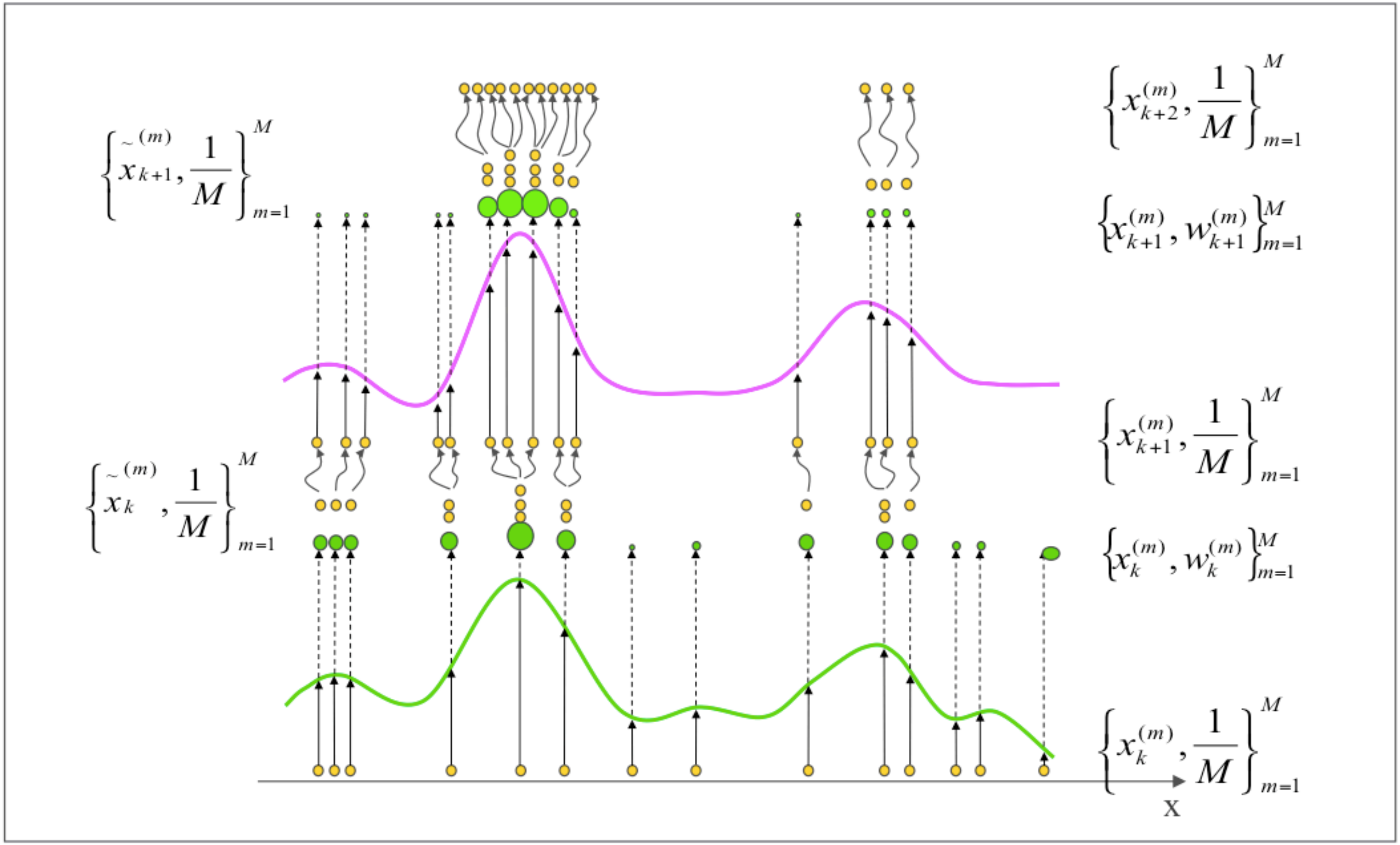
New measurements update

REGENERATION (RESAMPLING)



Regeneration

REGENERATION (RESAMPLING)



We add transition noise

PARTICLE FILTER

Algorithm 1: SIS Particle Filter

$[\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}] = \text{SIS}[\{\mathbf{x}_{k-1}^i, w_{k-1}^i\}_{i=1}^{N_s}, y_k]$

- FOR $i = 1: N_s$
  - Draw $\mathbf{x}_k^i \sim q(\mathbf{x}_k | \mathbf{x}_{k-1}^i, y_k)$
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- END FOR

Algorithm 2: Resampling Algorithm

$[\{\mathbf{x}_k^{j*}, w_k^j, i^j\}_{j=1}^{N_s}] = \text{RESAMPLE}\ [\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}]$

- Initialize the CDF: $c_1 = 0$
- FOR $i = 2: N_s$
  - Construct CDF: $c_i = c_{i-1} + w_k^i$
- END FOR
- Start at the bottom of the CDF: $i = 1$
- Draw a starting point: $u_1 \sim \mathbb{U}[0, N_s^{-1}]$
- FOR $j = 1: N_s$
  - Move along the CDF: $u_j = u_1 + N_s^{-1}(j-1)$
  - WHILE $u_j > c_i$
  - * $i = i + 1$
  - END WHILE
  - Assign sample: $\mathbf{x}_k^{j*} = \mathbf{x}_k^i$
  - Assign weight: $w_k^j = N_s^{-1}$
  - Assign parent: $i^j = i$
- END FOR

Algorithm 3: Generic Particle Filter

$[\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}] = \text{PF}[\{\mathbf{x}_{k-1}^i, w_{k-1}^i\}_{i=1}^{N_s}, y_k]$

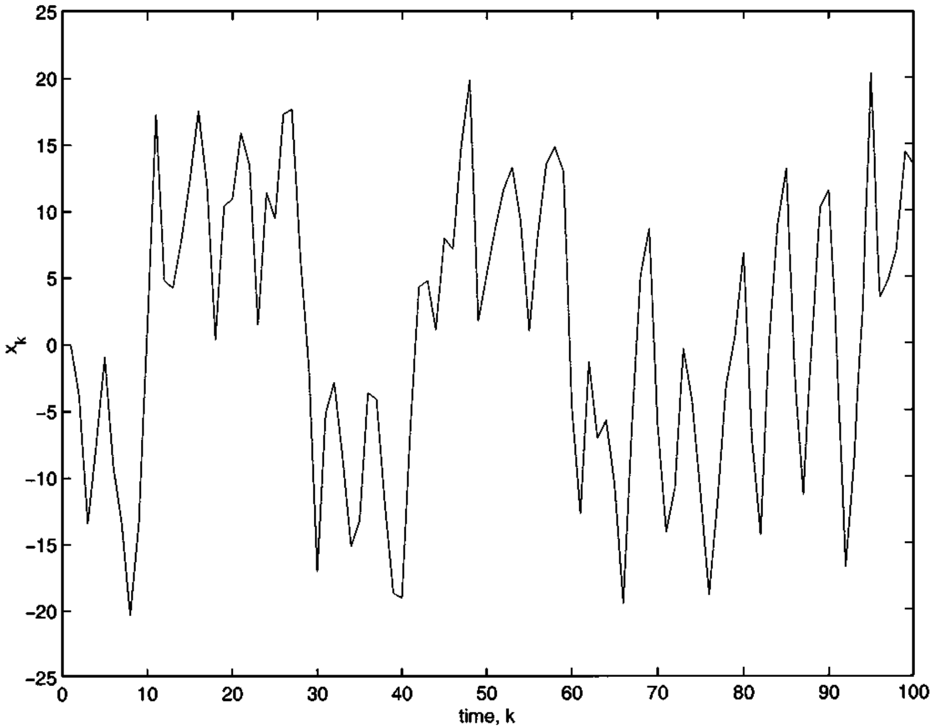
- FOR $i = 1: N_s$
  - Draw $\mathbf{x}_k^i \sim q(\mathbf{x}_k | \mathbf{x}_{k-1}^i, y_k)$
  - Assign the particle a weight, w_k^i, according to $w_k^i \propto w_{k-1}^i \frac{p(y_k | x_k^i) p(x_k^i | x_{k-1}^i)}{q(x_k^i | x_{k-1}^i, y_k)}$
- END FOR
- Calculate total weight: $t = \text{SUM}[\{w_k^i\}_{i=1}^{N_s}]$
- FOR $i = 1: N_s$
  - Normalize: $w_k^i = t^{-1} w_k^i$
- END FOR
- Calculate $\widehat{N_{eff}}$
- IF $\widehat{N_{eff}} < N_T$
  - Resample using algorithm 2:
  - * $[\{\mathbf{x}_k^i, w_k^i, -\}_{i=1}^{N_s}] = \text{RESAMPLE}[\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}]$
- END IF

EXAMPLE

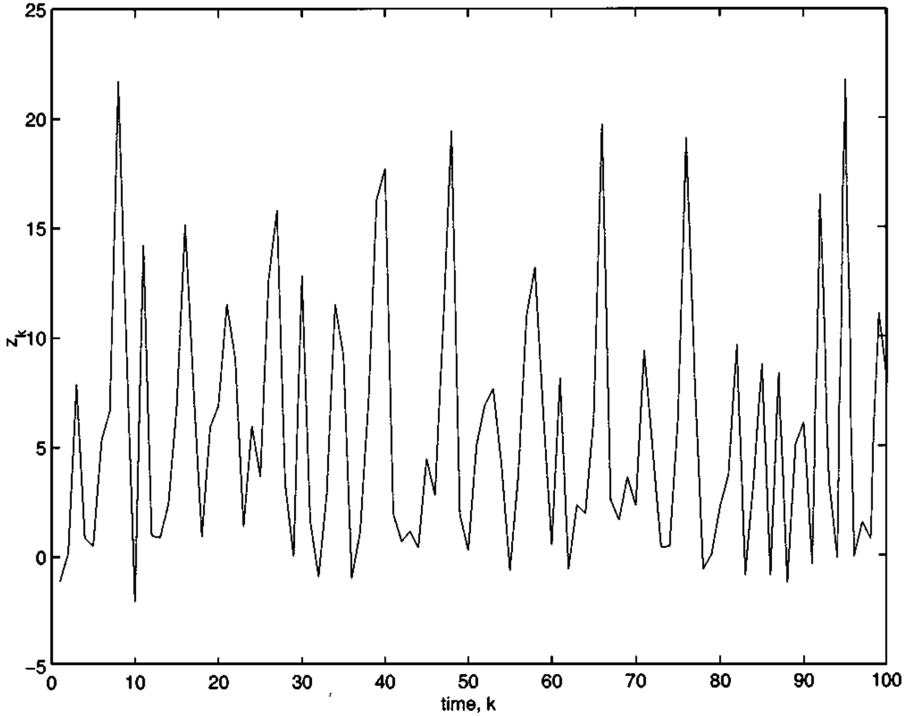
$$x_k = \frac{x_{k-1}}{2} + \frac{25x_{k-1}}{1+x_{k-1}^2} + 8\cos(1.2k) + v_k, \quad Q = 10$$

$$y_k = \frac{x_k^2}{20} + w_k, \quad R = 1$$

True x_k



y_k



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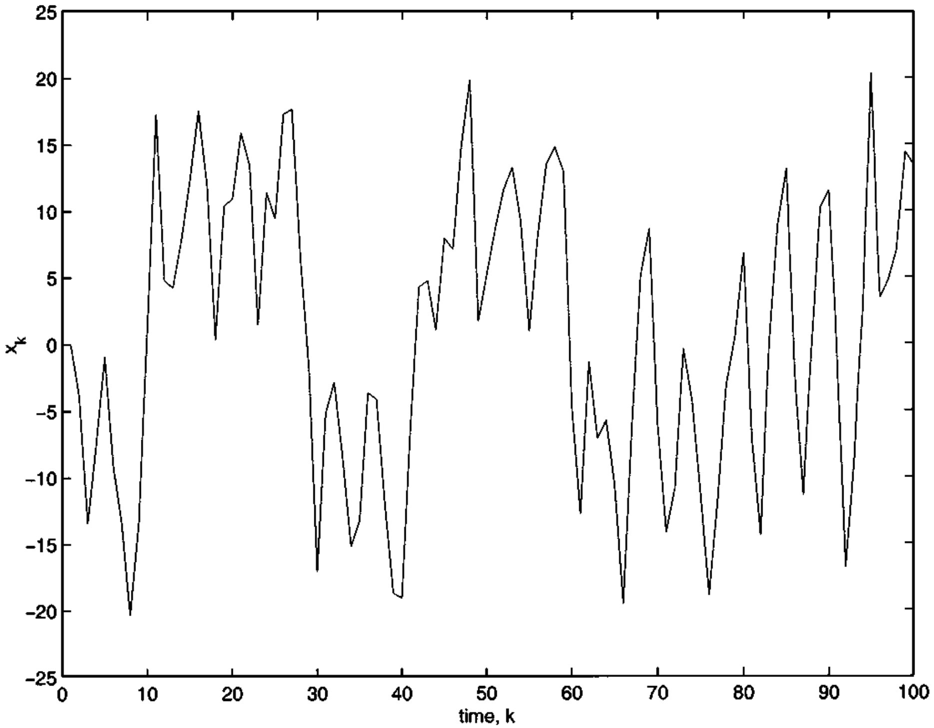
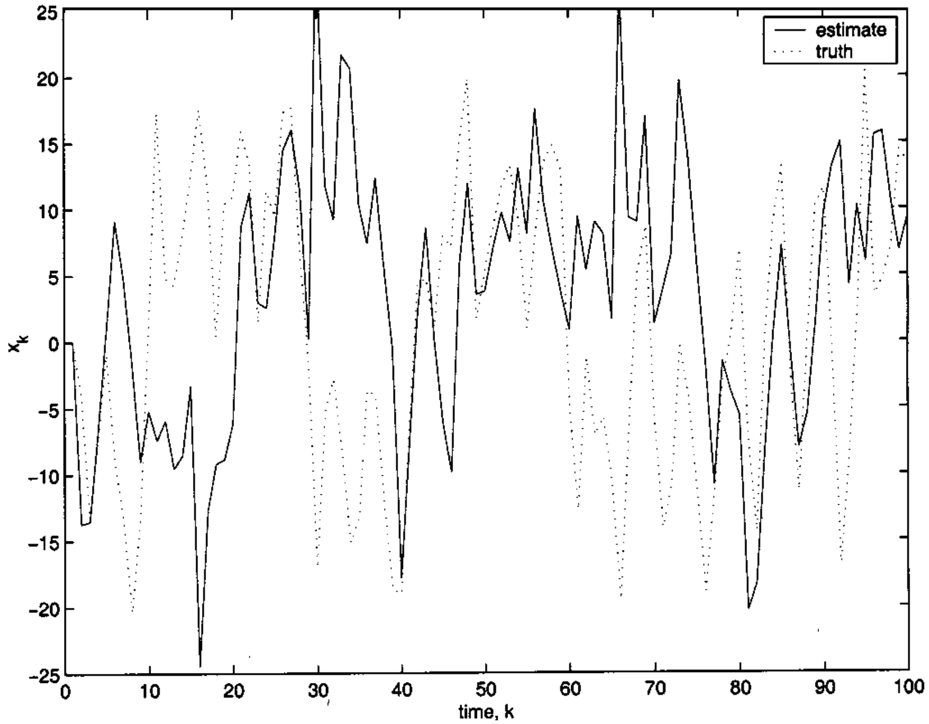


EKF estimates of x_k



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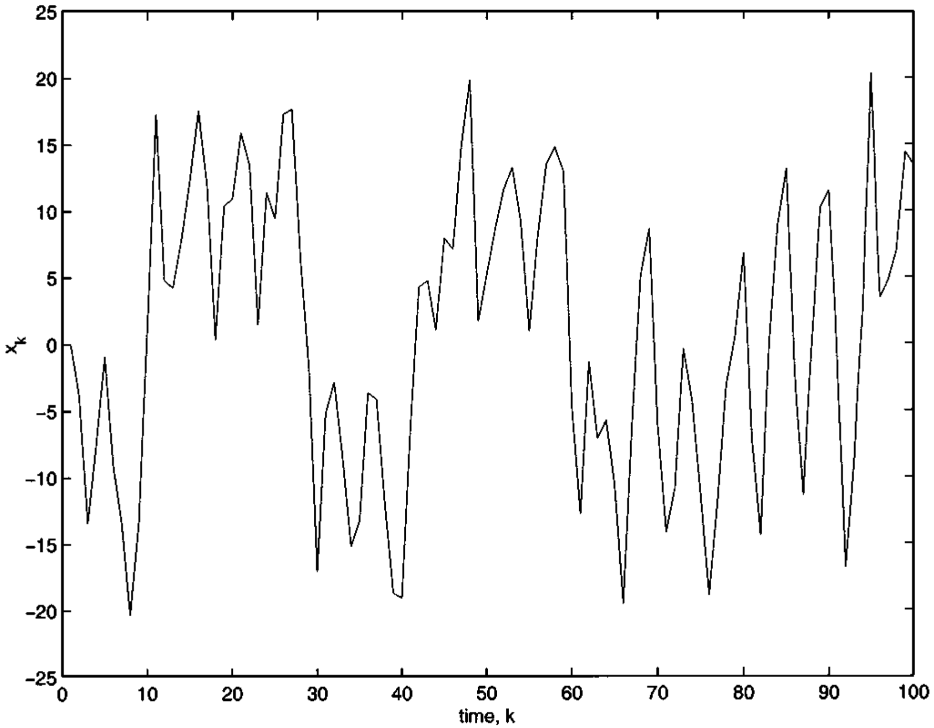

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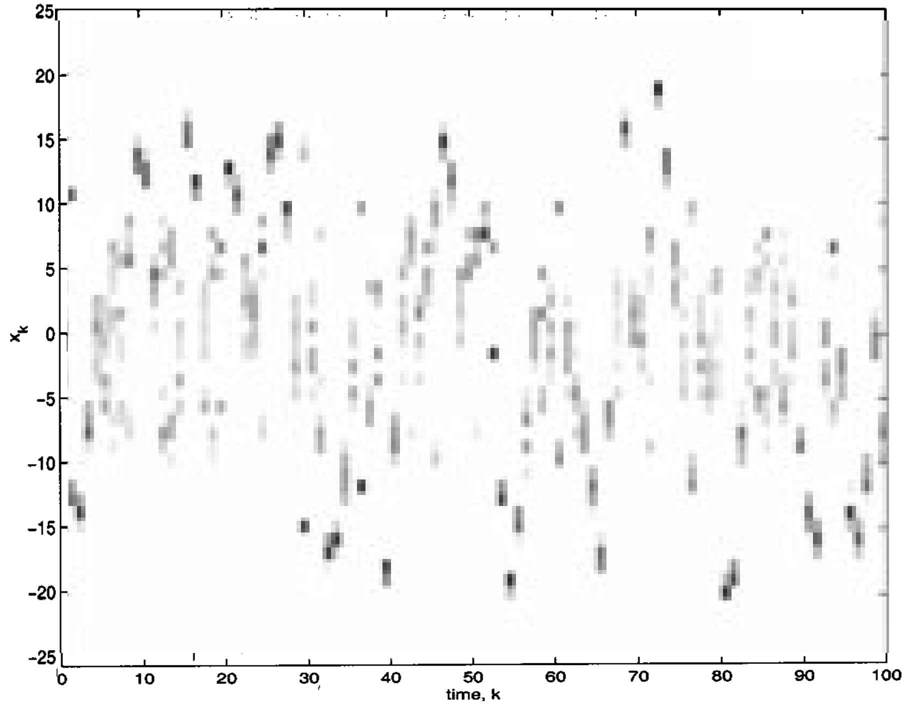
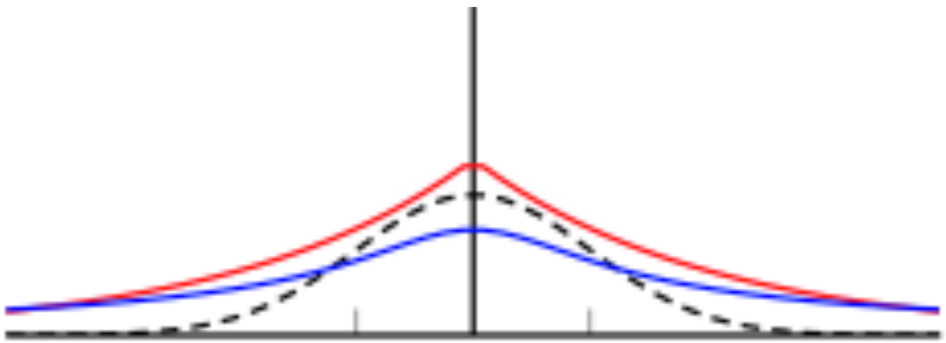


GAUSSIAN LIMITATIONS:
LOAD DISTURBANCES AND OUTLIERS

- **State**: angular velocity, angle of motor shaft.
- **Input** u**:** applied torque (known).
- **disturbances** d**:** impulsive, unknown.
- **Dynamics:**

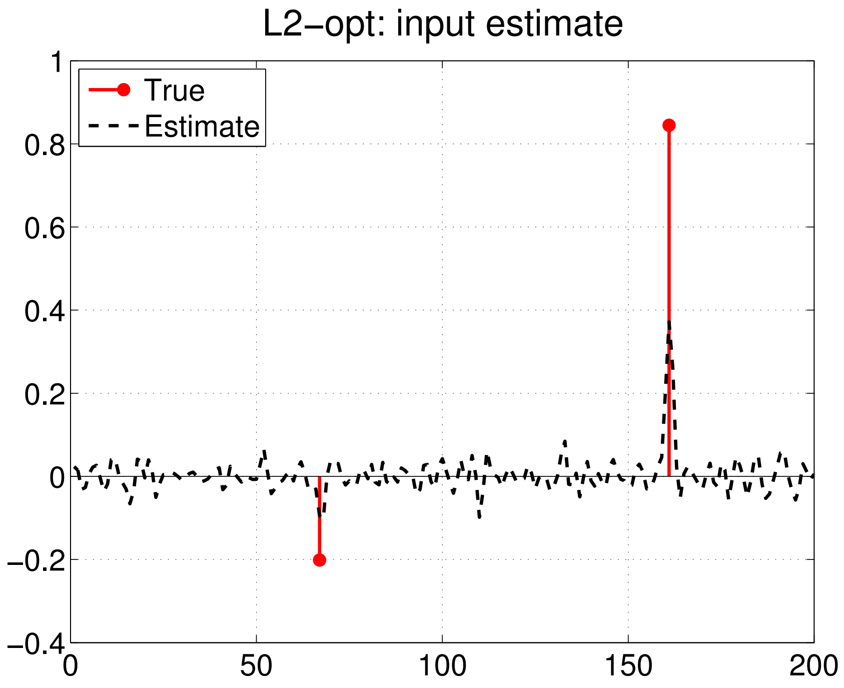
$$x_{t+1} = \begin{pmatrix} 0.7 & 0 \\ 0.08 & 1 \end{pmatrix} x_t + \begin{pmatrix} 11.8 \\ 0.6 \end{pmatrix} (u_t + d_t)$$

$$z_t = \begin{pmatrix} 0 & 1 \end{pmatrix} x_t + e_t$$



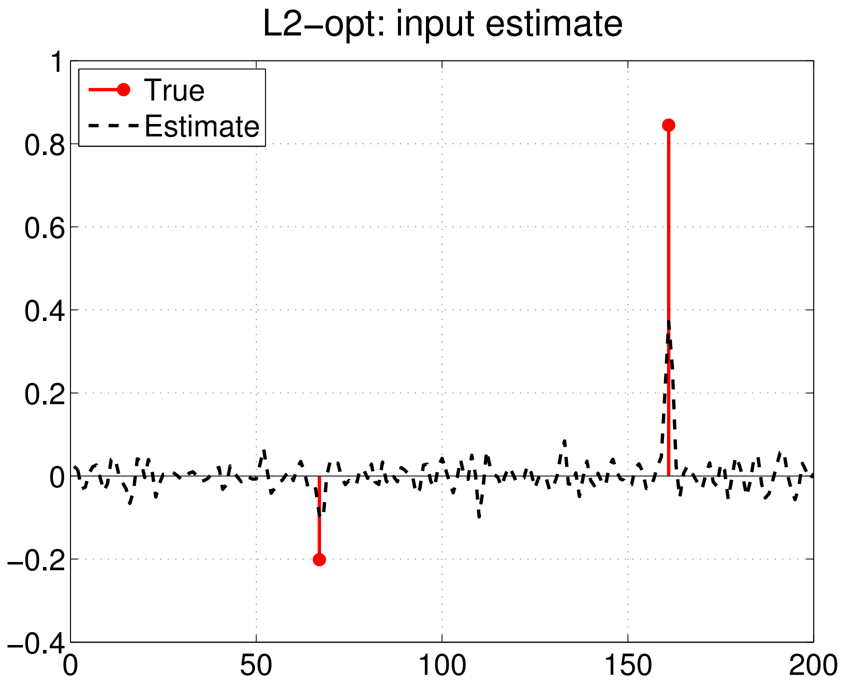
We use a Laplacian density to model d_t and e_t

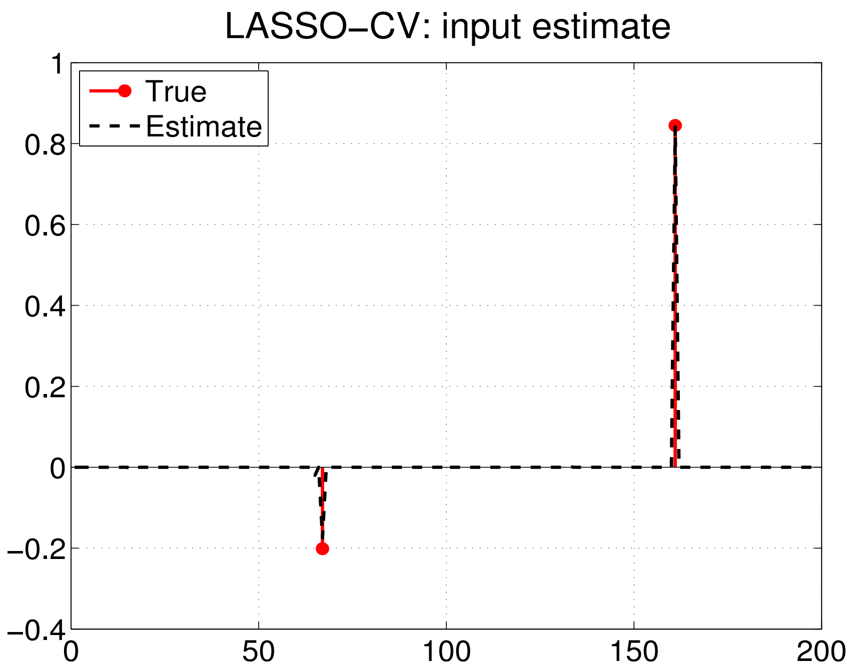
RESULTS



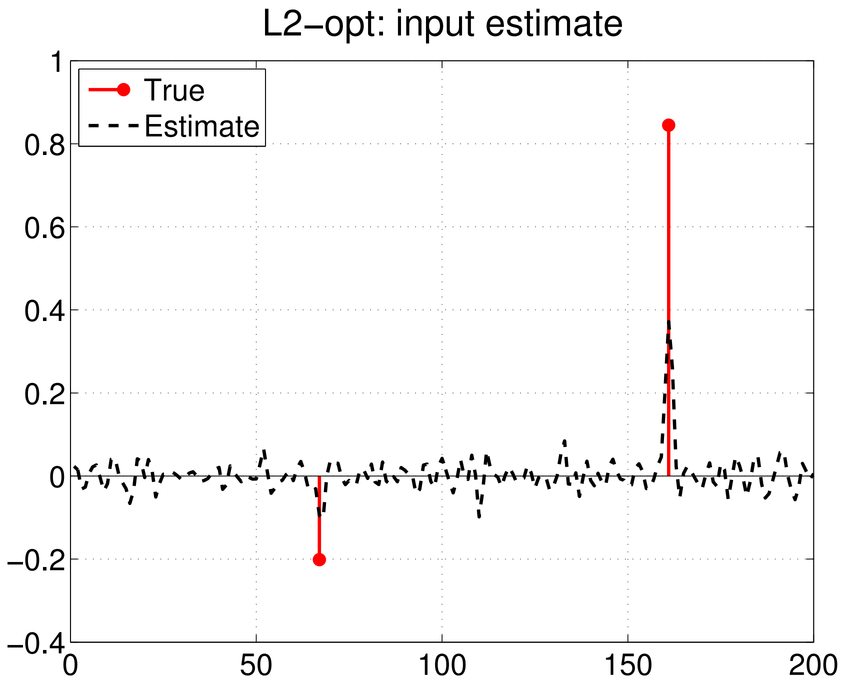
L2−opt: input estimate
True
Estimate
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
50
100
150
200

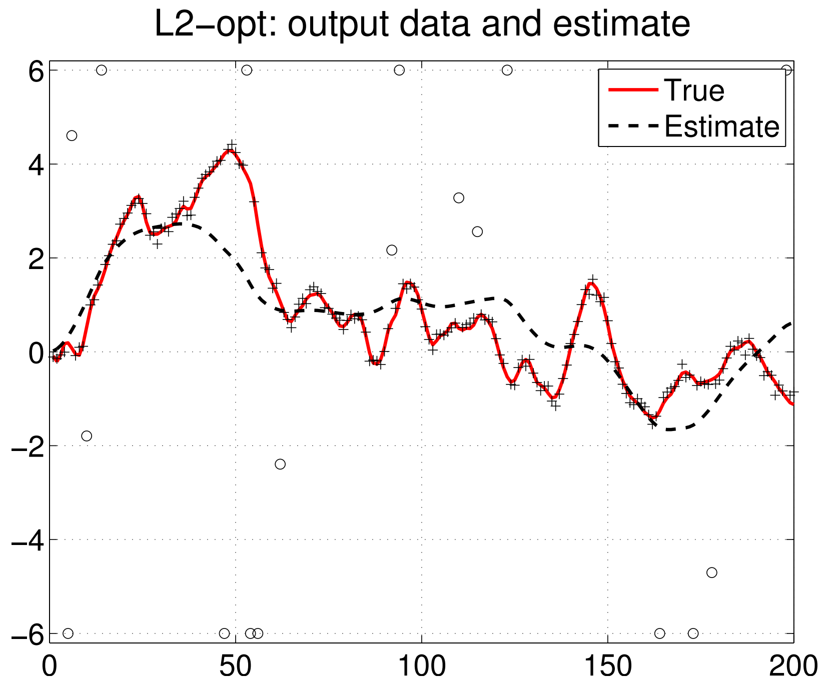
RESULTS

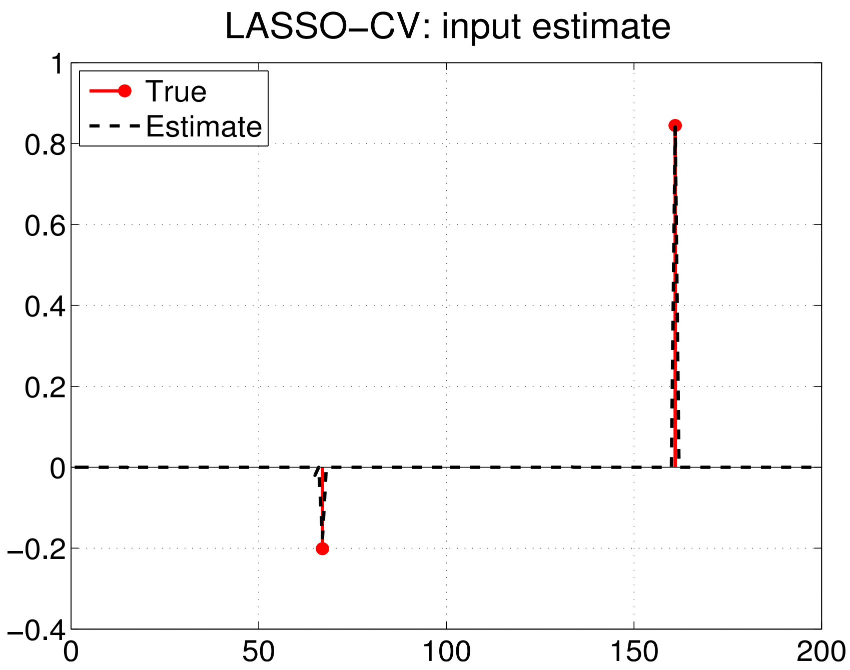

L2−opt: input estimate
True
Estimate
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
50
100
150
200


LASSO−CV: input estimate
True
Estimate
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
50
100
150
200

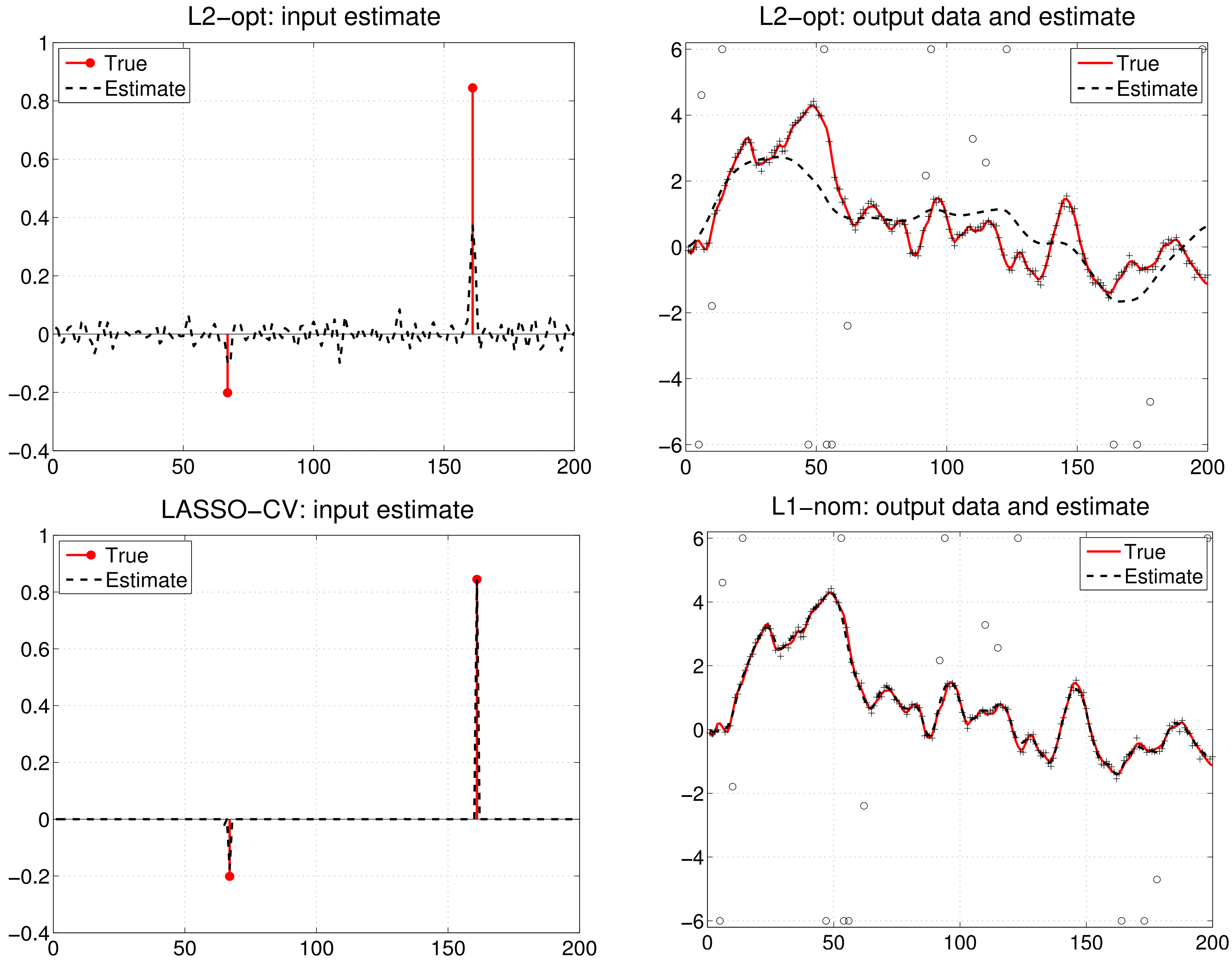
RESULTS


L2−opt: input estimate
True
Estimate
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
50
100
150
200


L2−opt: output data and estimate
True
Estimate
6
4
2
0
−2
−4
−6
0
50
100
150
200

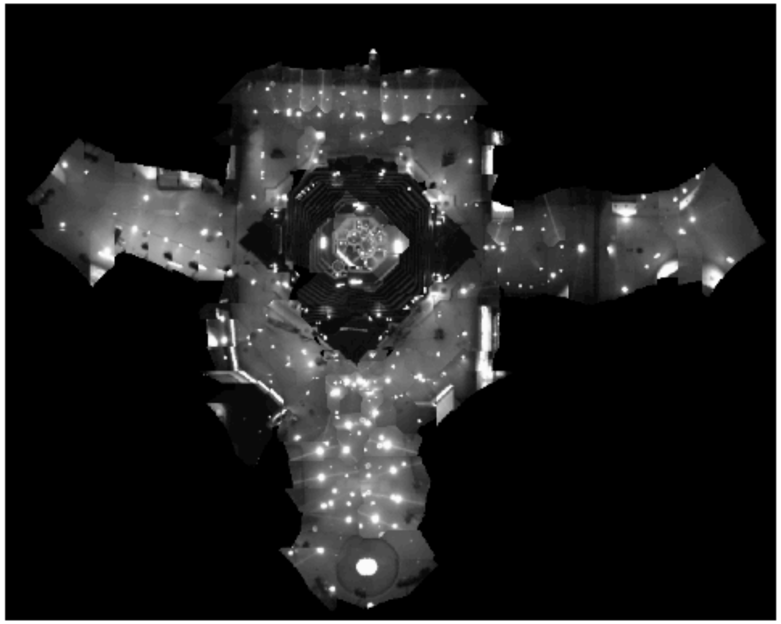

LASSO−CV: input estimate
True
Estimate
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
50
100
150
200

RESULTS

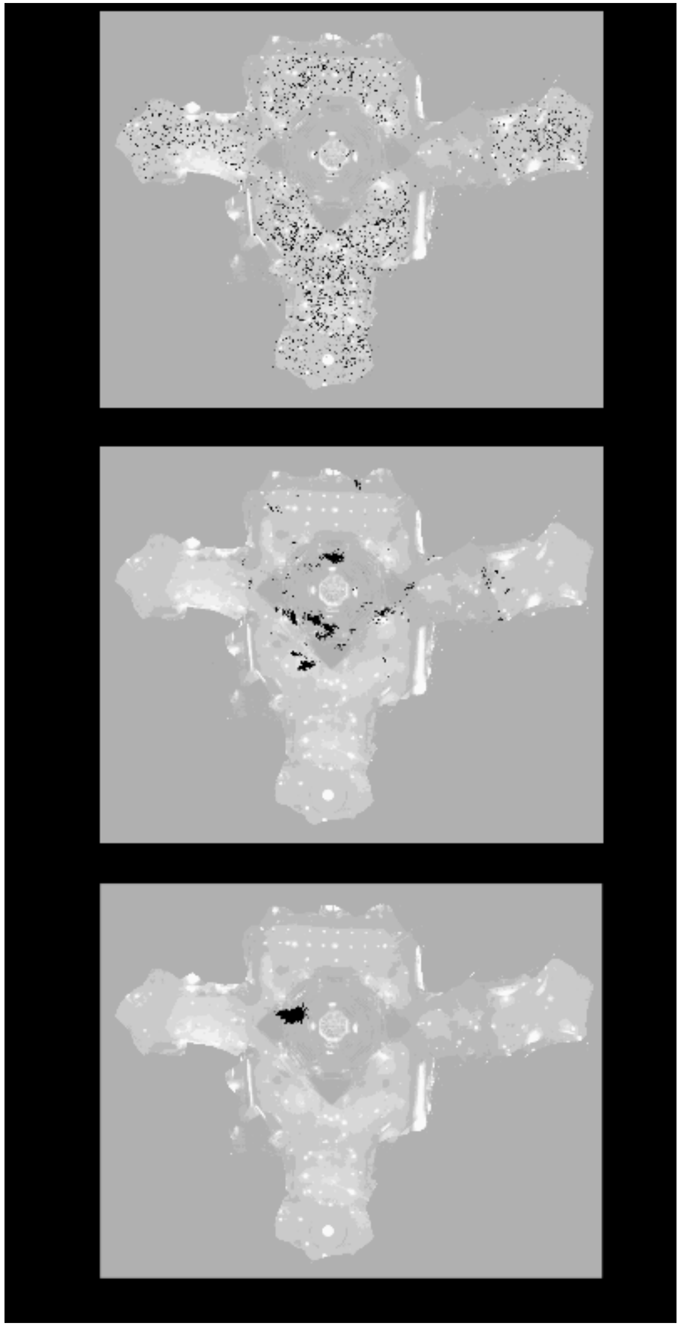


L2−opt: input estimate
True
Estimate
L2−opt: output data and estimate
True
Estimate
LASSO−CV: input estimate
True
Estimate
L1−nom: output data and estimate
True
Estimate

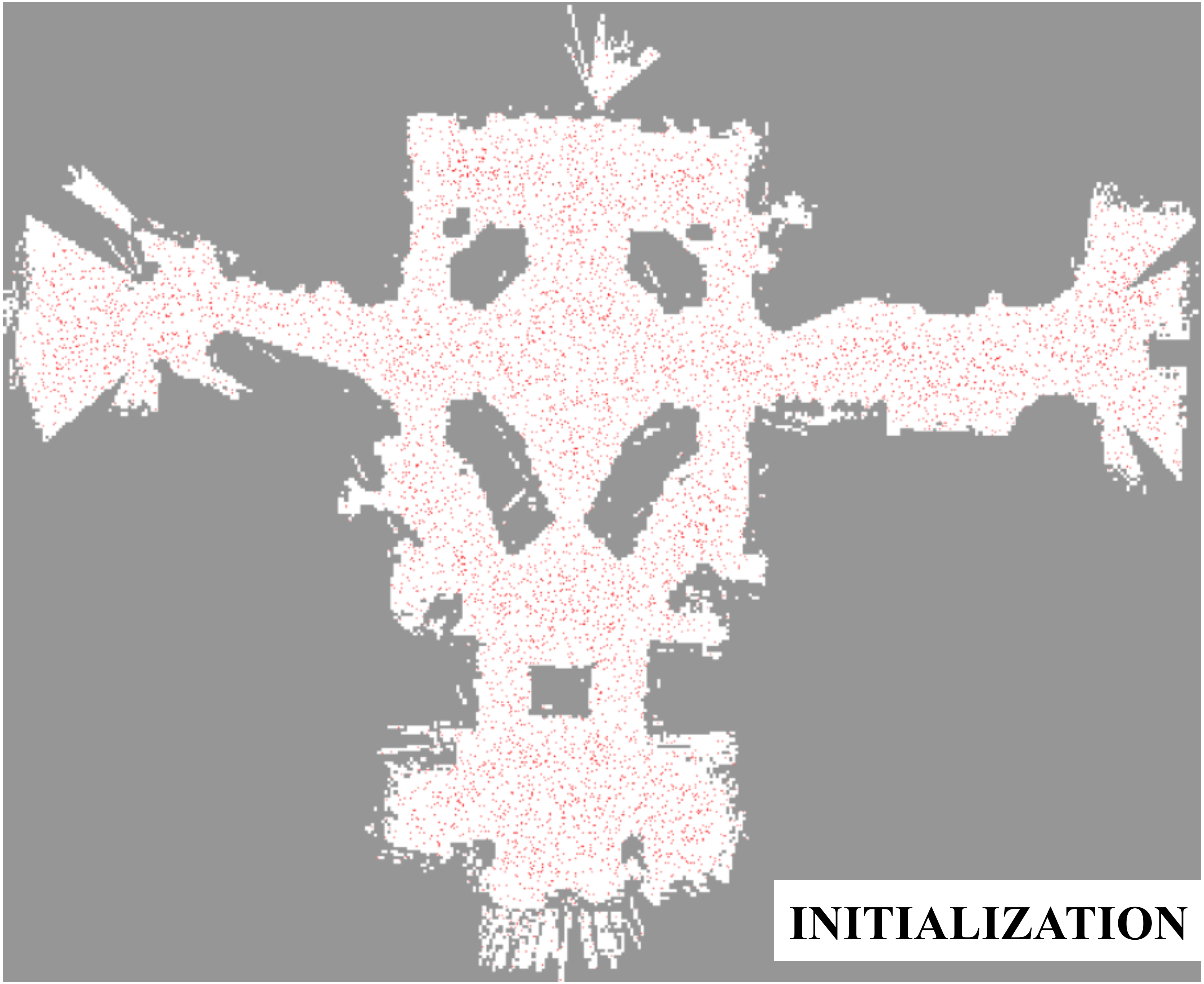
EXAMPLE: LOCALIZATION



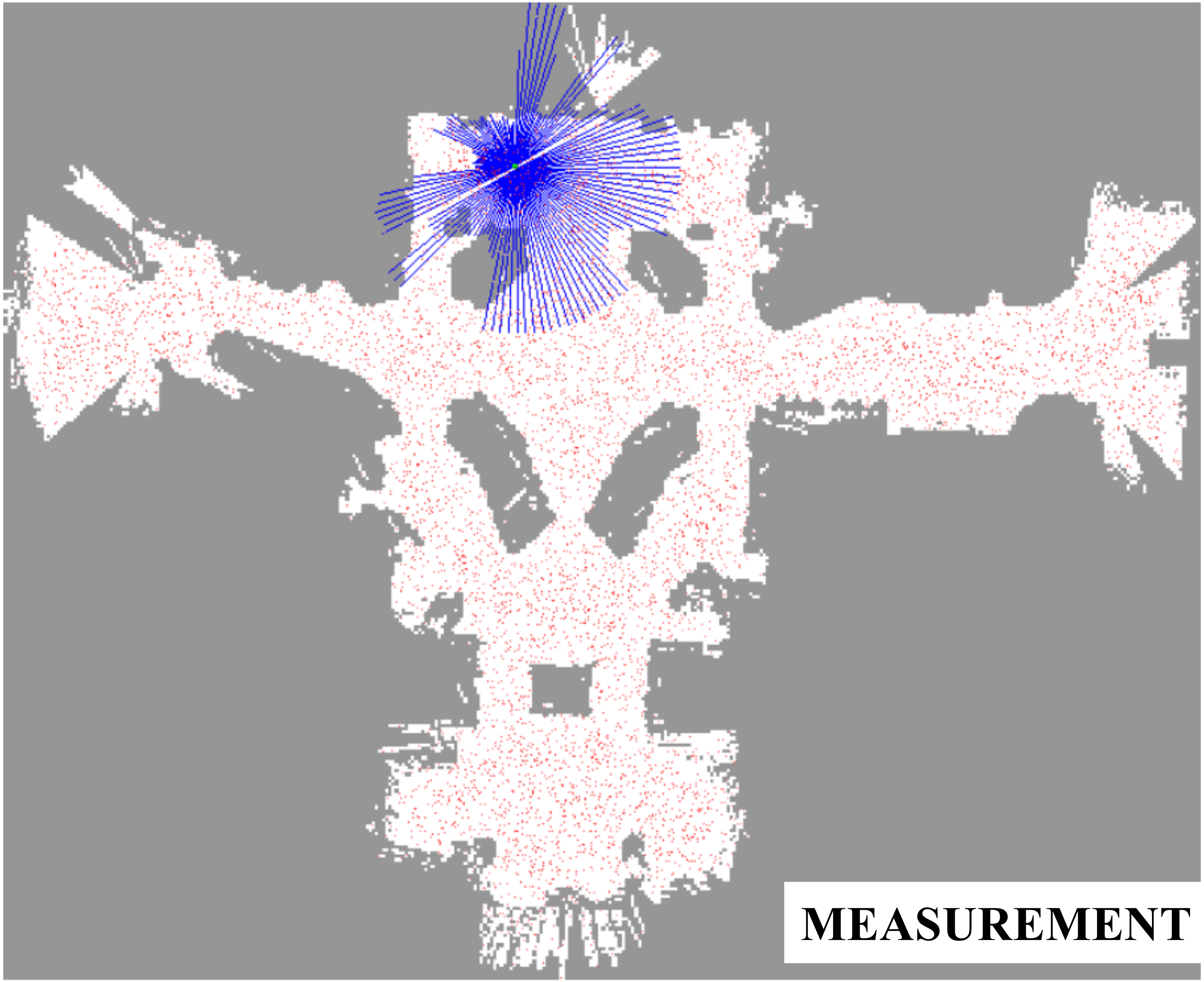
Ceiling map of the National Museum of American History, which was used as the perceptual model in navigating with a vision sensor.



EXAMPLE: LOCALIZATION



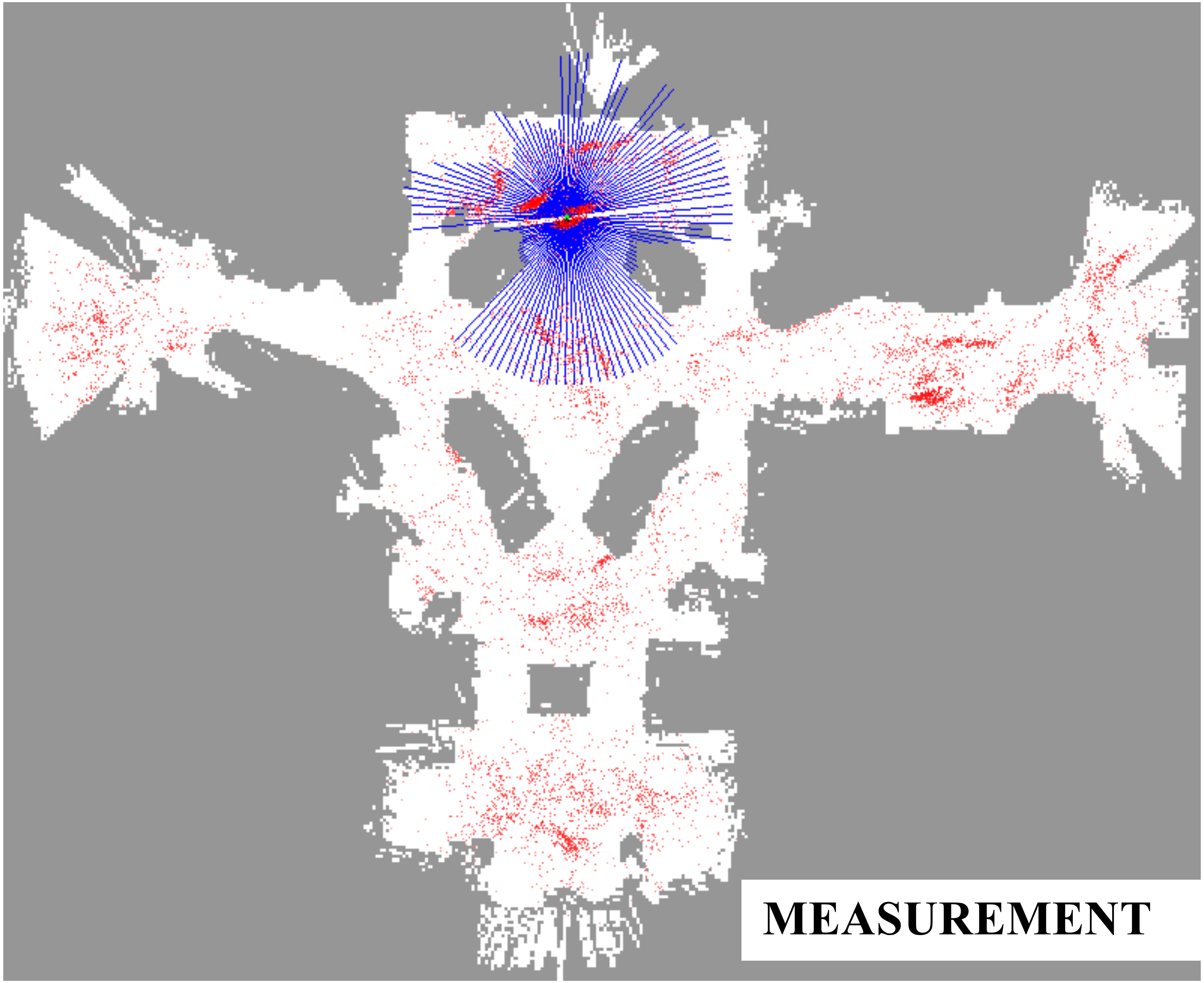
EXAMPLE: LOCALIZATION



EXAMPLE: LOCALIZATION

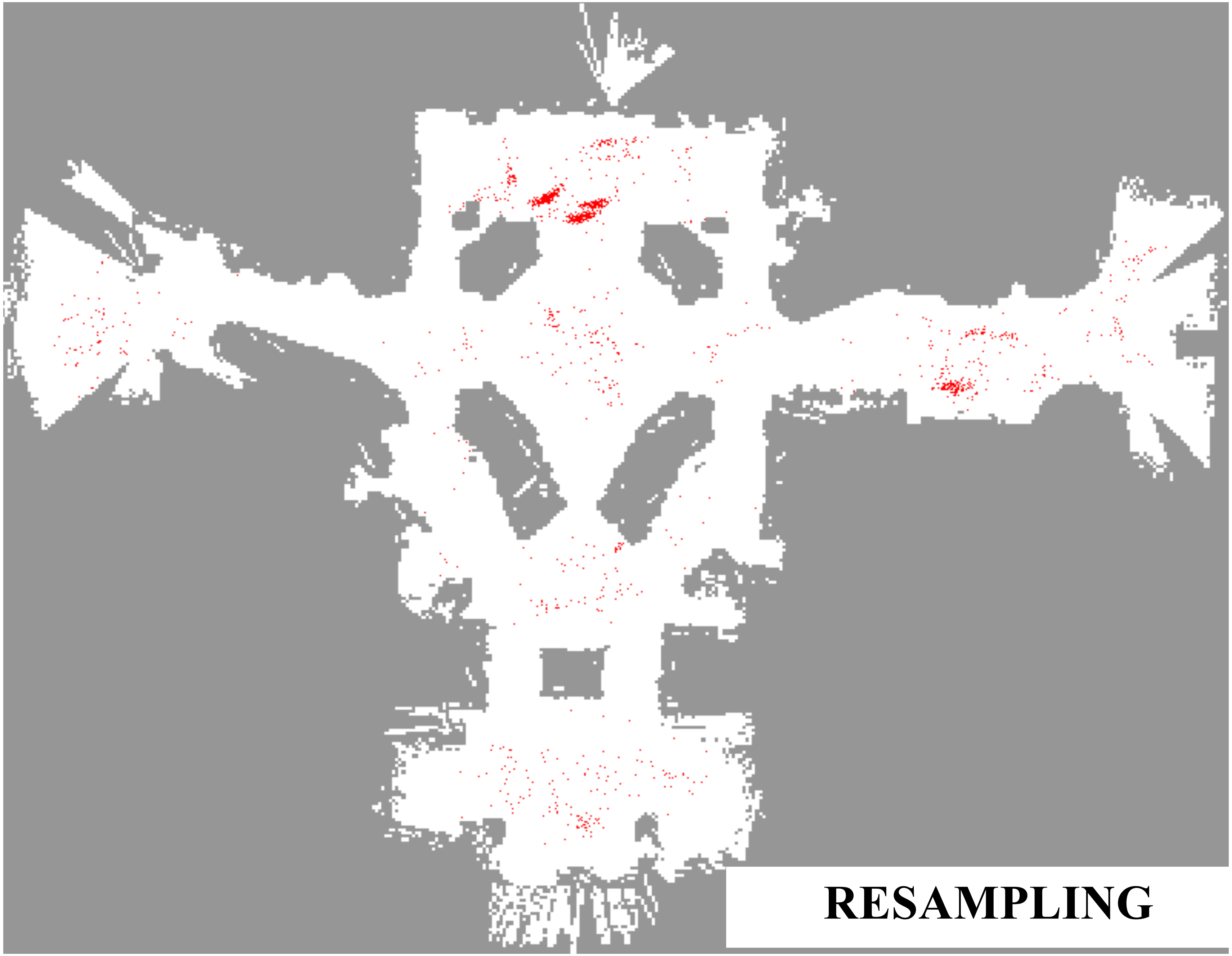
WEIGHTS UPDATE

EXAMPLE: LOCALIZATION

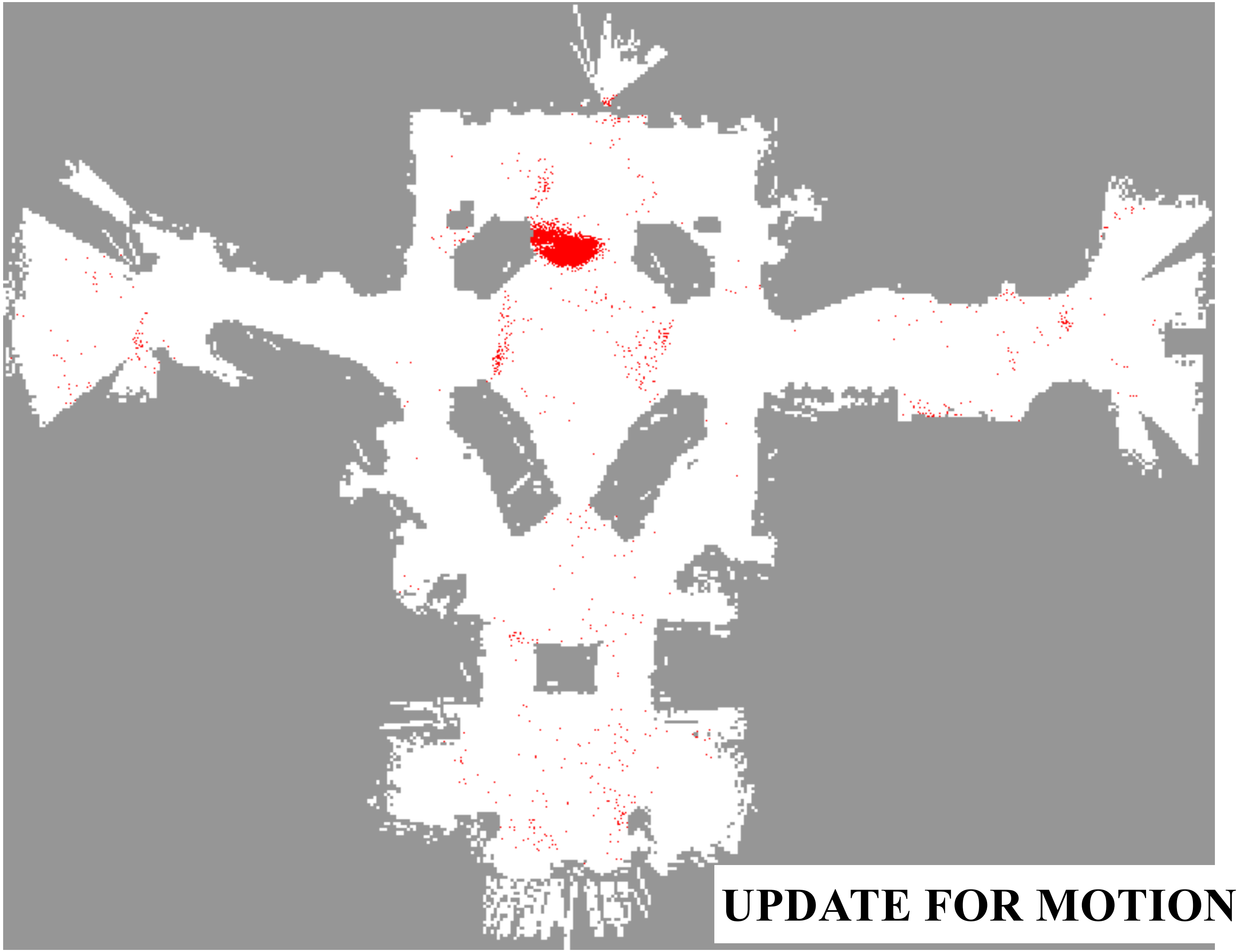


EXAMPLE: LOCALIZATION

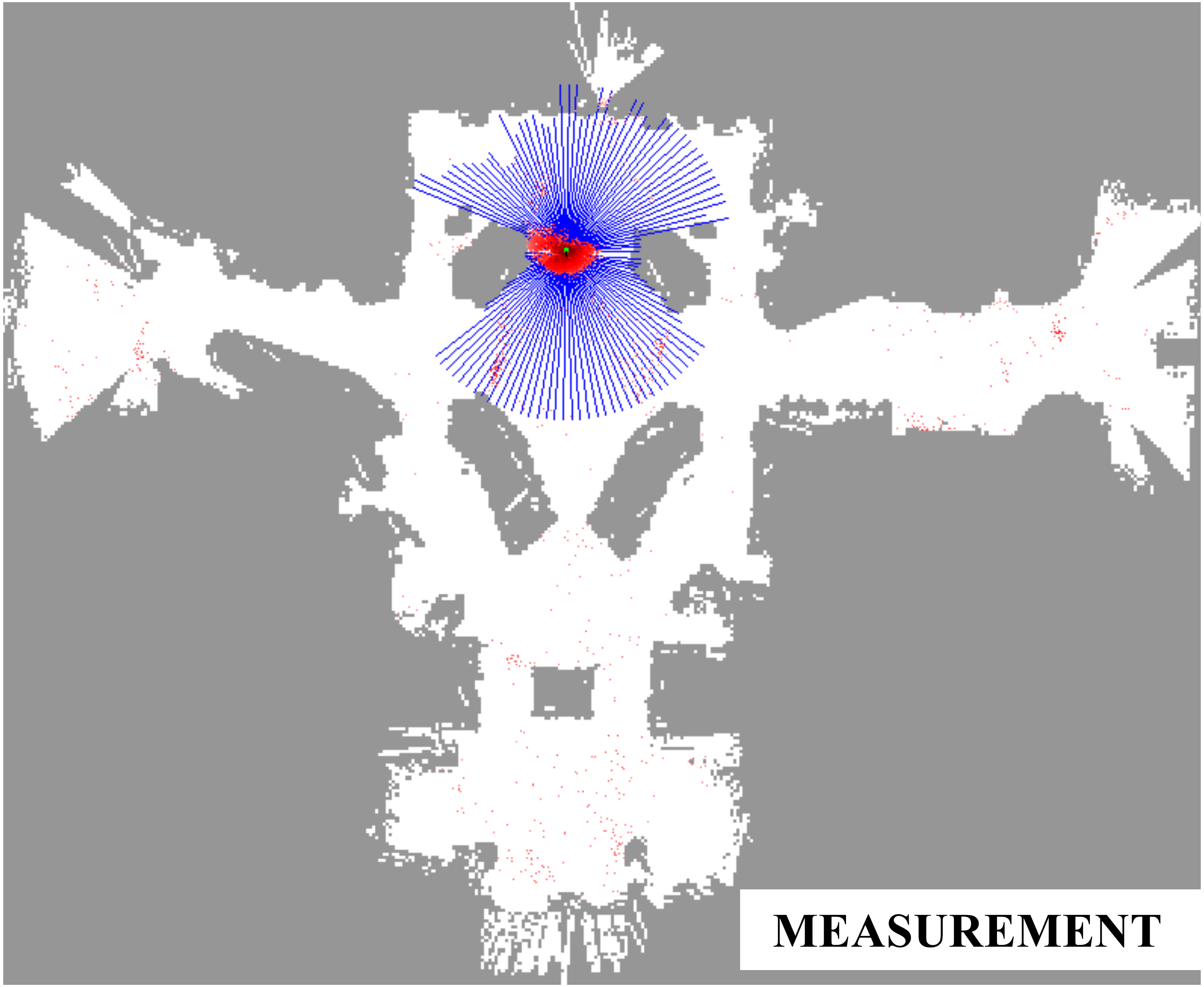
EXAMPLE: LOCALIZATION



EXAMPLE: LOCALIZATION



EXAMPLE: LOCALIZATION



EXAMPLE: LOCALIZATION

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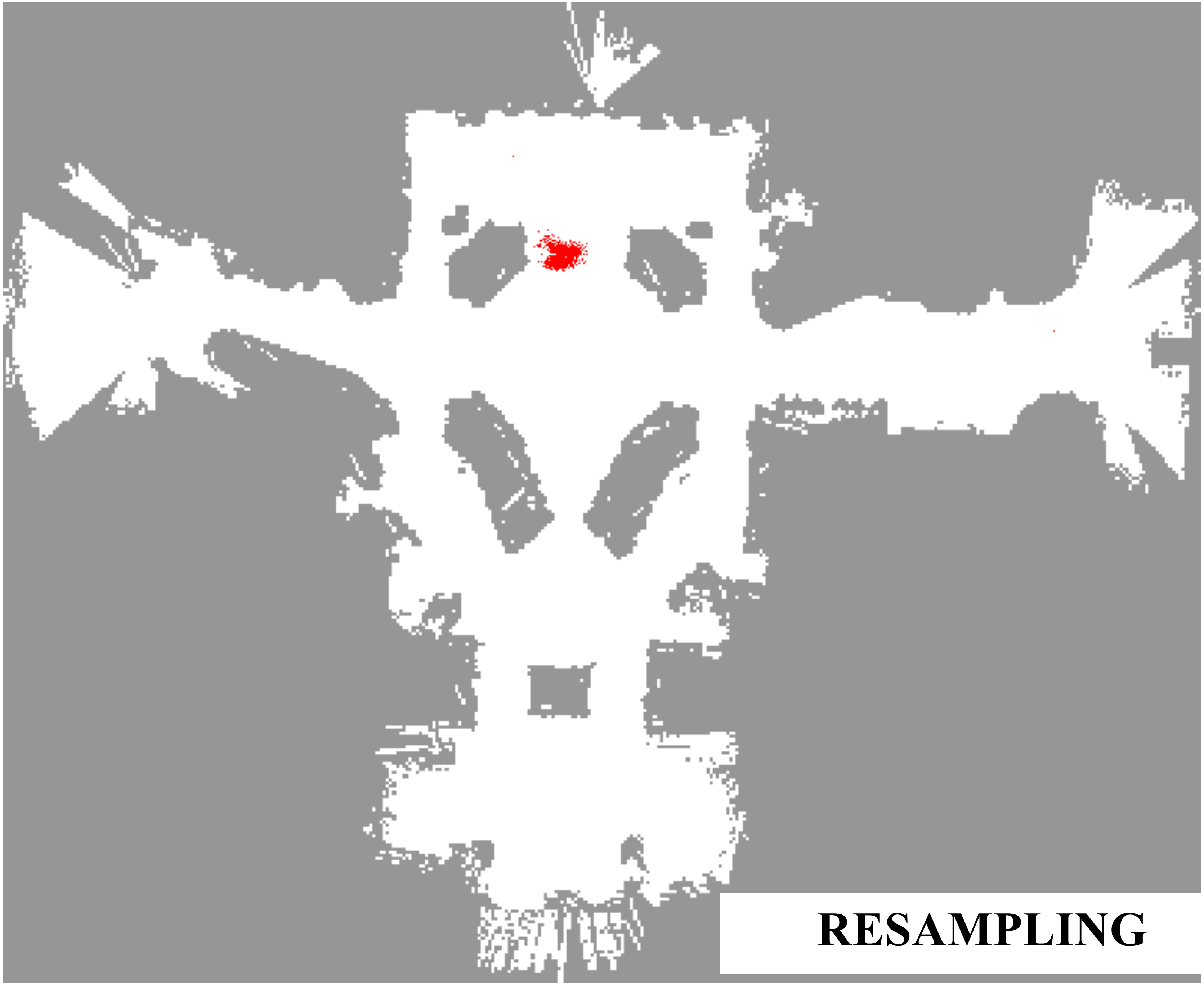
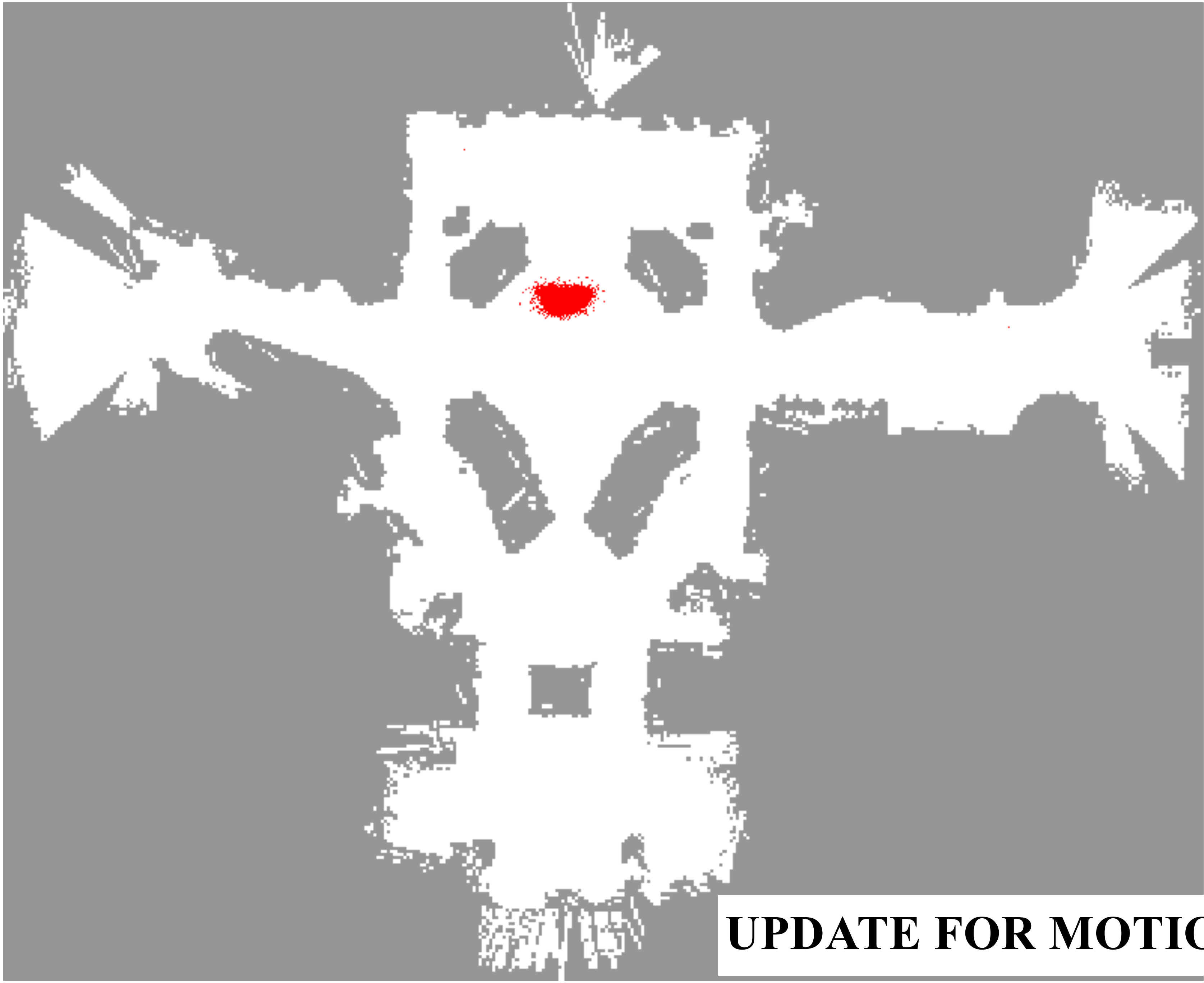
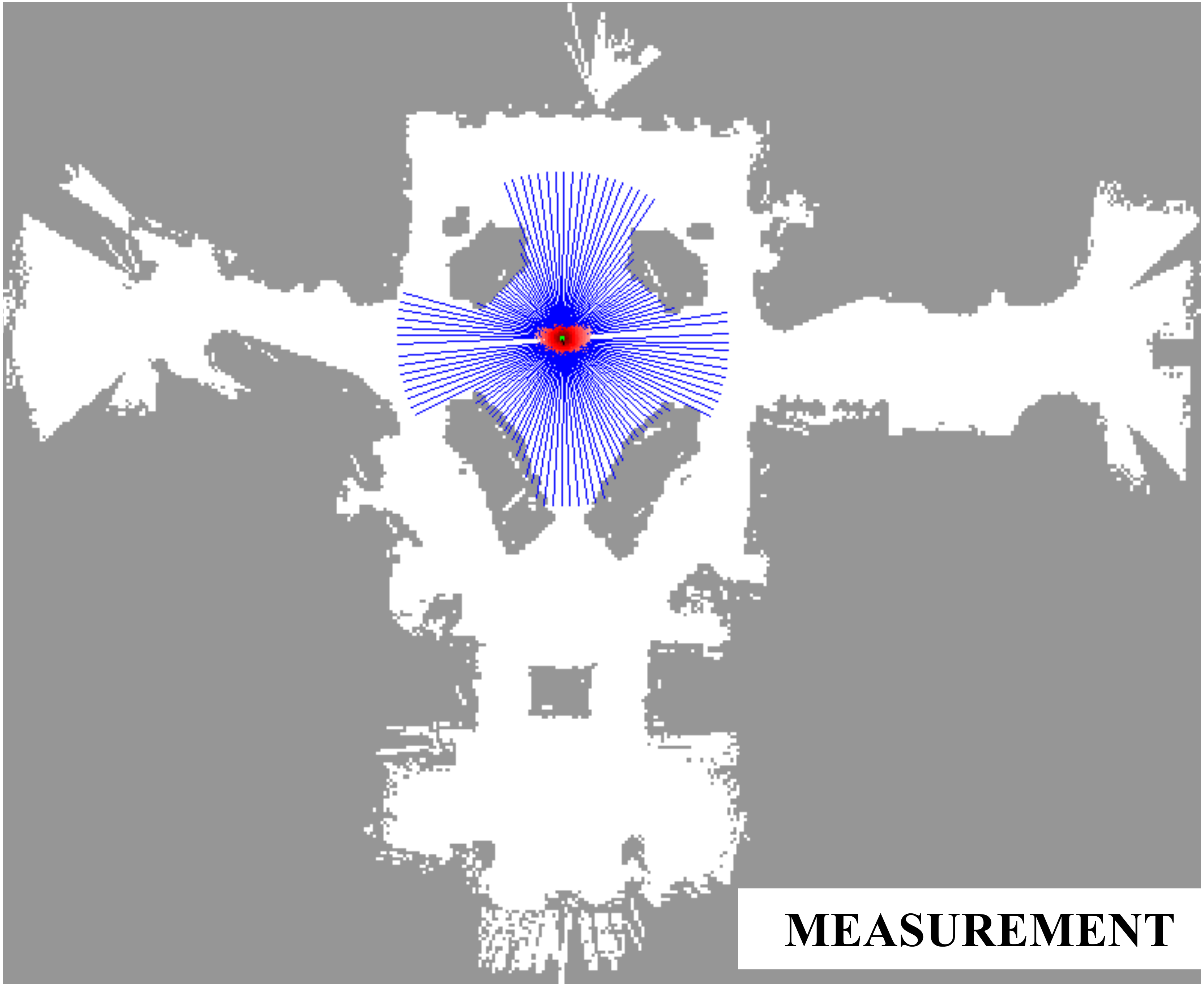


EXAMPLE: LOCALIZATION



UPDATE FOR MOTION

EXAMPLE: LOCALIZATION



OTHER APPLICATIONS: DETECTION/TRACKING