Optimal scheduling of positive switched systems: application examples

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Summary

- F. Blanchini, P. Colaneri, M. E. Valcher, ”Switched Positive Linear Systems”, Now Publishing, 2015.
- P. Bolzern, P. Colaneri, ”Positive Markov Jump Linear Systems”, Now Publishing, 2015.

- Biological system: Therapy scheduling for HIV load mitigatiion
- Traffic system: Traffic light scheduling for congestion mitigation
- Energy system: Time optimal battery scheduling in public stations

Introduction

- Continuous-time

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$$

- Discrete-time

$$\mathbf{x}(k+1) = A_{\sigma(k)}\mathbf{x}(k)$$

A_i are Metzler matrices (in continuous time), A_i are nonnegative matrices (in discrete-time), $i = 1, 2, \cdots, M$. The signal $\sigma(.)$ is the switching signal to be used for control purposes.

Positivity constraint

$$\mathbf{x} \gg 0$$

OPTIMAL CONTROL

Optimal control - switched systems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \gg 0$$

$$\min_{\sigma} J(\mathbf{x}_0) = \mathbf{c}^\top \mathbf{x}(t_f), \quad \mathbf{c} \gg 0$$

Motivation: simplified model of the treatment of HIV infection dynamics, where x represents the concentrations of various viral mutants in a patient, and σ represents the selection of a suitable therapy. Alternatively, in the widespread SI (Susceptible Infective) models of epidemiology over a network, in the initial infection phase (epidemic outbreak) the concentration of susceptible individuals is approximately constant and the dynamics of infected individuals is linear.

Filippov trajectories, sliding modes?

Filippov-Wazewski relaxation: the set of possible trajectories of the switched system is dense in the set of trajectories generated by the bilinear system:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= A(\mathbf{u}(t))\mathbf{x}(t) = \left(\sum_{i=1}^{M} A_i u_i(t)\right)\mathbf{x}(t) \\
1 &= \mathbf{1}^\top u(t), \quad \text{symplex} \quad \mathscr{U} \\
u_i(t) &\geq 0 \quad \forall i \\
x(0) &= x_0
\end{aligned}$$

$$\min_{\mathbf{u}} J(\mathbf{x}_0) = \mathbf{c}^\top \mathbf{x}(t_f), \quad \mathbf{c} \gg 0$$

The optimal solution does exist: the set of velocities $F(\mathbf{x},\mathbf{u}) := \{A(\mathbf{u})\mathbf{x}, u \in \mathscr{U}\}$ is convex and the vector field is bounded by an affine function of the norm of the state variable, i.e.$\|A(\mathbf{u})\mathbf{x}\| \leq \alpha(1+\|\mathbf{x}\|)$.

Pontryagin triple

Optimal solutions satisfy:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= A(\mathbf{u}(t))\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \\
-\dot{\tilde{\mathbf{p}}}(\mathbf{t})^\top &= \tilde{\mathbf{p}}(t)^\top A(\mathbf{u}(t)), \quad \tilde{\mathbf{p}}(t_f) = \mathbf{c} \\
\mathbf{u}(t) &= \arg\min_{\mathbf{u}\in\mathscr{U}} \mathbf{x}(t)^\top A(\mathbf{u})^\top \tilde{\mathbf{p}}(t) \\
J(\mathbf{x}_0) &= \tilde{\mathbf{p}}(0)^\top \mathbf{x}_0
\end{aligned}$$

Hernandez-Varga, Colaneri, Middleton, Blanchini. Discrete-time control for switched positive sstems with application to mitigating viral escape. IJRN, 2011.

Lyapunov Metzler differential equation

Matrix Λ is a Metzler matrix with $\Lambda\mathbf{1} = 0$.

$$-\dot{\mathbf{p}}_i(t)^\top = \mathbf{p}_i^\top(t)A_i + \sum_{j\neq i}^{N} \lambda_{ij}(\mathbf{p}_j^\top(t) - \mathbf{p}_i^\top(t)) = 0, \quad \mathbf{p}_i(t_f) = \mathbf{c}$$

$$\sigma(t) = \arg\min_{j\in I(x(t))} D_+V, \quad V(x,t) = \min_i \mathbf{p}_i^\top(t)\mathbf{x}, \quad J(x_0) \leq \min_i \mathbf{p}_i(0)^\top \mathbf{x}_0$$

Important notice: For $\Lambda = \alpha\bar{\Lambda}$ and $\alpha \to \infty$

- $\mathbf{p}_i(t) - \mathbf{p}_j(t) \to 0$
- $\mathbf{p}_i(t) \to \tilde{\mathbf{p}}(t)$
- $\mathbf{u}^\top(t)\bar{\Lambda} = 0$
- $-\dot{\tilde{\mathbf{p}}}(\mathbf{t})^\top = \tilde{\mathbf{p}}(\mathbf{t})^\top \mathbf{A}(\mathbf{u}(\mathbf{t}))$

Are the Pontryagin solutions optimal?

Class of switched (bilinear) systems for which the necessary Pontryagin conditions are also sufficient?

- First fact: The necessary condition for optimality given by Pontryagin is also sufficient if the cost functional is convex wrt the control u.
- Second fact:

$$A = D + \tilde{A}$$

with D diagonal and M Metzler. Using the Trotter formula:

$$exp(At) = lim_{k\to\infty}\left(exp(Dt/k)exp(\tilde{A}t/k)\right)^k$$

it is possible to prove that all entries of $exp(At)$ are convex functions of the diagonal entries of D.

Main Result

Assumption 1

$\sigma(t)$ affects only the diagonal entries of the Metzler matrices $A_{\sigma(t)}$, i.e. $A_i = D_i + \tilde{A}$, $i = 1, 2, \dots, M$.

$$\dot{\mathbf{x}}(t) = \tilde{A}\mathbf{x}(t) + \left(\sum_{i=1}^{M} D_i u_i(t)\right)\mathbf{x}(t), \quad J(\mathbf{x}_0, \mathbf{u}) = \mathbf{c}^\top \mathbf{x}(t_f)$$

Colaneri, Middleton, Chen, Caporale, Blanchini: Automatica 2014

Theorem 2

The global optimal control exists

The cost $J(x_0, u)$ is convex with respect to $\mathbf{u}(.)$.

The optimal cost J^0 is concave with respect to $\mathbf{x}_0$.

Rantzer, Bo. Control of convex-monotone systems. CDC 2014

Remarks

- Thanks to convexity the optimal control $\mathbf{u}^o(t)$ that minimizes $J_0(\mathbf{x}_0, \mathbf{u})$ can be found (numericaly) by discretization and gradient methods.
- In the same vein, the minimax problem $\min_{\mathbf{u}} \max_{\mathbf{x}_0} J(\mathbf{x}_0, \mathbf{u})$ can be solved.

Conjecture 3

$\exists\ \Lambda$ Metzler with $\Lambda \mathbf{1} = 0$ such that with the state-feedback switching rule given by the differential Lyapunov-Metzler equations

$$-\dot{\mathbf{p}}_i(t)^\top = \mathbf{p}_i^\top(t) A_i + \sum_{j \neq i}^{N} \lambda_{ij} (\mathbf{p}_j^\top(t) - \mathbf{p}_i^\top(t)) = 0, \quad \mathbf{p}_i(t_f) = \mathbf{c}$$

the optimal trajectory is found.

- More complex cost function can be considered, for instance

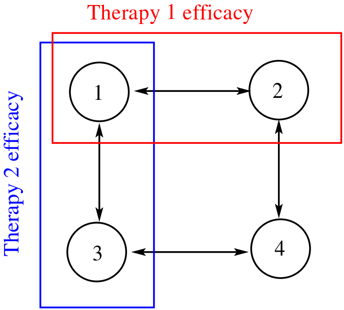
$$J(\mathbf{x}_0) = \mathbf{c}^\top \mathbf{x}(t_f) + \int_0^{t_f} \mathbf{q}_{\sigma(t)}^\top \mathbf{x}(t) dt, \quad q_i \gg 0$$

Application example 1: HIV viral mitigation

- Highly Active Antiretroviral Therapy (HAART) for treating HIV: prevent immune deterioration, reduce morbidity and mortality, and prolong the life expectancy of people infected with HIV. Virological failure (HIV RNA levels less than 50 copies/ml), viral rebound, emergence of resistance-conferring mutations within the viral genome, virus with reduced susceptibility to one or more of the drugs.
- Alternating HAART regimens would further reduce the likelihood of the emergence of resistance. This proactive switching was evaluated in a clinical trial called SWATCH (SWitching Antiretroviral Therapy Combinations against HIV).
- Once under treatment, and until virological failure, macrophage and CD4+T cell counts are approximately constant. Under this assumption, most non-linear HIV models are rendered linear.

$$\dot{\mathbf{x}}(t) = \left(\mathrm{diag}(\rho_{i,\sigma(t)}) - \delta_v I\right)\mathbf{x}(t) + \mu M_u \mathbf{x}(t)$$

δ_V is the viral clearance, $\rho_{i,j}$ the replication rate of genotype i for theraphy j, μ the mutation rate, $[M_u]_{ij} = \{0,1\}$ the genetic connection between genotypes.



Simple model: 4 genotypes

Consider a system with 4 states, and two treatment options, $m = 2$, of the following structure

$$A_\sigma = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_{2\sigma} & 0 & 0 \\ 0 & 0 & \lambda_{3\sigma} & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} + \mu \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with (symmetric case)

$$\lambda_{21} = \lambda_{32} > 0, \quad \lambda_{22} = \lambda_{31} < 0, \quad \lambda_{21} - \lambda_{22} + \lambda_{31} - \lambda_{32} = 0$$

We want to minimize

$$J(\mathbf{x}_0) = \mathbf{c}^\top \mathbf{x}(t_f)$$

with

$$\mathbf{c} \gg 0$$

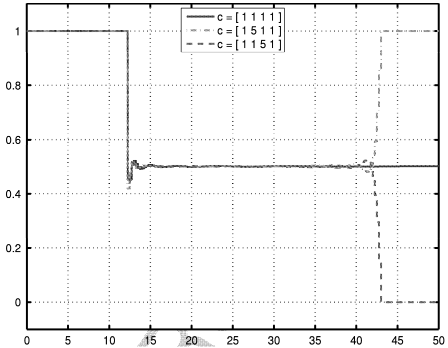
Simple model: 4 genotypes

$$\dot{\mathbf{x}}(t) = \left(\mathrm{diag}(\rho_{i,\sigma(t)}) - \delta_v I\right)\mathbf{x}(t) + \mu M_u \mathbf{x}(t)$$

$$\lambda_{i\sigma} = \rho_{i,\sigma} - \delta_v I$$

$$\begin{aligned}
\mu &= 10^4 \\
\lambda_1 &= 0.19 \\
\lambda_{21} &= \lambda_{32} = 0.03 \\
\lambda_{31} &= \lambda_{22} = 0.19 \\
\lambda_4 &= 0.03 \\
t_f &= 50 \\
x_0 &= [10^3\ 5\ 0\ 10^5]
\end{aligned}$$

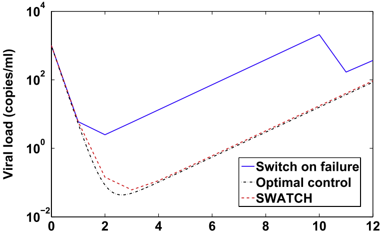
"Turnpike" solution. In the "symmetric case", the optimal control goes as fast as possible to a sliding mode and exit the sliding mode as closer as possible to the final condition.



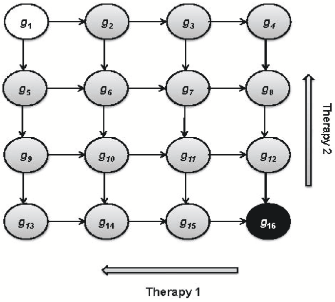
Simulations

Table 1
Symmetric replication rates for viral variants.

Variant	Therapy 1	Therapy 2
Wild type (x_1)	$\rho_{1,1} = 0.05$	$\rho_{1,2} = 0.05$
Genotype 1 (x_2)	$\rho_{2,1} = 0.27$	$\rho_{2,2} = 0.05$
Genotype 2 (x_3)	$\rho_{3,1} = 0.05$	$\rho_{3,2} = 0.27$
HR Genotype (x_4)	$\rho_{4,1} = 0.27$	$\rho_{4,2} = 0.27$



Extension - 16 genotypes



Network for 16 genotypes and two drug combinations.

- E. Hernandez Vargas, R. Middleton, and P. Colaneri. Switching strategies to mitigate HIV mutation. IEEE Transactions on Control Systems Technology, 2014.
- E. Hernandez Vargas, R. Middleton. Modelling the three stages in HIV infection. Theoretical Biology 2013

Extension

Infected ($T^\star$) and uninfected (T) CD4+T cells, infected ($P^\star$) and uninfected (P) macrophages, viral load V_i of the i-th genotype, total viral load V_T.

$$
\begin{aligned}
\dot{T} &= s_T + \frac{\rho_T V_T}{C_T + V_T} T - \sum_{i=1}^{n} k_{Ti} T V_i - \delta_T T \\
\dot{T}_i^\star &= k_{Ti} T V_i - \delta_{T^\star} T_i^\star + \mu \sum_{i=1}^{n} m_{ij} V_j T \\
\dot{P} &= s_P + \frac{\rho_P V_T}{C_P + V_T} P - \sum_{i=1}^{n} k_{Pi} P V_i - \delta_P P \\
\dot{P}_i^\star &= k_{Pi} P V_i - \delta_{P^\star} P_i^\star + \mu \sum_{i=1}^{n} m_{ij} V_j P \\
\dot{V}_i &= p_{Ti} T_i^\star + p_{Pi} P_i^\star - \delta_V V_i \\
V_T &= \sum_{i=1}^{N} V_i
\end{aligned}
$$

where s_T and s_P are the generation rates of new T-cells and macrophages, respectively, C_T and C_P are proliferation parameters, ρ_T and ρ_P are the uninfected cell replication rates, k_{Ti} and k_{Pi} represent the infection rates, whereas p_{Ti} and p_{Pi} are the viral proliferation rates. The mutation rate is expressed by μ, and the coefficients $m_{ij} \in [0,1]$ represent the genetics connection between genotypes. Finally, the parameters δ are the death rates for the relevant species.

Linear approximation

$$x = \begin{bmatrix} T_1^\star P_1^\star V_1^\star & T_2^\star P_2^\star V_2^\star & \cdots & T_n^\star P_n^\star V_n^\star \end{bmatrix}^\top$$

$$\dot{x} = \begin{bmatrix} A_{1\sigma} & 0 & \cdots & 0 \\ 0 & A_{2\sigma} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n\sigma} \end{bmatrix} x + \mu M x$$

$$A_{i\sigma} = \begin{bmatrix} -\delta_{T^\star} & 0 & k_{Ti} T \\ 0 & -\delta_{P^\star} & k_{Pi} P \\ p_{Ti} & p_{Pi} & -\delta_V \end{bmatrix}, \quad M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & T \\ 0 & 0 & P \\ 0 & 0 & 0 \end{bmatrix}$$

Extension - 16 genotypes

- Virological failure: Introduce a new regimen if there is detectable viremia (HIV RNA $>$ 1000 copies/ml), SWATCH: Alternate every 3 months.

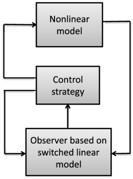


TABLE I

SIMULATION RESULTS AFTER SIX YEARS OF HAART

Strategy	Viral load	CD4+T cells
Switch on virologic failure	342	549
SWATCH	222	652
Costate control	179	701
Guaranteed Cost	234	696
MPC	7.5	857

6 year treatment period and 3 month decision time: Proactive switching overperforms the treatment based on virological failure. Linear switched strategies present values close to the SWATCH treatment, where MPC exhibits the best performance compared to the other strategies. See Hernandez-Varga, Middleton, Colaneri, IFAC 2011, for 64 genotypes, 3 drugs.

STABILIZATION

Stabilization: stable linear combination

- There exists a Hurwitz linear combination of matrices A_i, i.e.

$$A(\alpha) = \sum_{i=1}^{M} A_i \alpha_i$$

is Hurwitz. Then take $\mathbf{p} \gg 0$ satisfying

$$\mathbf{p}^\top A(\alpha) \ll 0$$

and switch according to

$$\sigma(\mathbf{x}(t)) = \min_i \mathbf{p}^\top A_i \mathbf{x}(t)$$

There are stabilizable systems for which there are no Hurwitz matrices in the convex hull of A_i.

- There exist Λ Metzler, with $\Lambda \mathbf{1} = 0$ such that

$$A_{BIG}(\Lambda) = \mathrm{diag}(A_i) + \Lambda^\top \otimes I_n$$

is Hurwitz. Then take $\mathbf{p} = [\mathbf{p}_1^\top \, \mathbf{p}_2^\top \, \ldots \mathbf{p}_M^\top]^\top \gg 0$ satisfying

$$\mathbf{p}^\top A_{BIG}(\Lambda) \ll 0 \quad \Longleftrightarrow \quad \mathbf{p}_i^\top A_i + \sum_{i=1}^{M} \lambda_{ij}(\mathbf{p}_j^\top - \mathbf{p}_i^\top) \ll 0$$

and switch according to

$$\sigma(\mathbf{x}(t^+)) = \arg \min_{j \in I(\mathbf{x}(t))} \mathbf{p}_{\sigma(t)}^\top(t) A_j \mathbf{x}(t),$$

There are stabilizable systems for which no feasible solution of the LM inequalities exists.

$$\mathbf{p}_i^\top A_i + \sum_{j\neq i} \lambda_{ij}(\mathbf{p}_j^\top e^{A_j T} - \mathbf{p}_i^\top) < 0, \quad i = 1, 2, \cdots, N$$

State-feedback control law with dwell time

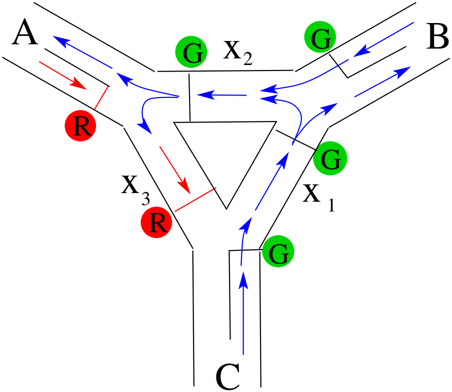
$$\begin{aligned}
\sigma(t) &= i, \quad t \in [t_k, t_k + T) \\
\sigma(t) &= i, \quad \text{if } \mathbf{x}(t)^\top \mathbf{p}_i \leq \mathbf{x}(t)^\top e^{A_j^\top T} \mathbf{p}_j \mathbf{x}(t)\ \forall j \neq i, \text{ and } t > t_k + T \\
\sigma(t_{k+1}) &= \arg\min_j \mathbf{x}(t_{k+1})^\top e^{A_j^\top T} \mathbf{p}_j, \quad \text{otherwise}
\end{aligned}$$

L. Allerhand, U. Shaked. Robust State-Dependent Switching of Linear Systems With Dwell Time. IEEE TAC 2013.

Conjecture 4

A positive switched linear system is stabilizable iff there exists T and Λ Metzler with $\Lambda \mathbf{1} = 0$ such that the DTLM inequalities are feasible.

Application example 2: Traffic junction



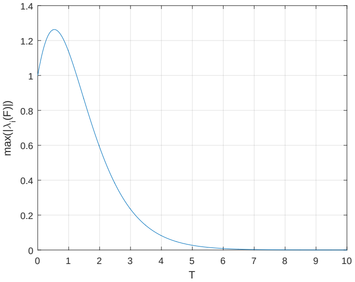
Simple model of a traffic junction

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$$

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- There does not exist a Hurwitz stable linear combination of A_1, A_2, A_3.
- There exist no solution of the Lyapunov-Metzler inequalities for any Λ.

Periodic switching law



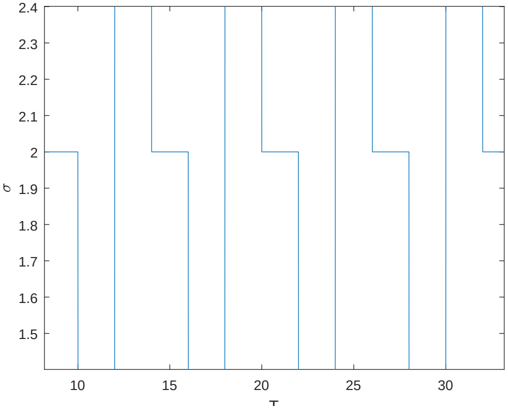
The matrix product $F = e^{A_1 T} e^{A_2 T} e^{A_3 T}$ is Schur stable for $T > 1.257$.

The switched system is stabilizable with a periodic switching law.

$$\sigma(t) = \begin{cases} 3, & t \in [kT, kT+T) \\ 2, & t \in [kT+T, kT+2T) \\ 1, & t \in [kT+2T, kT+3T) \end{cases}$$

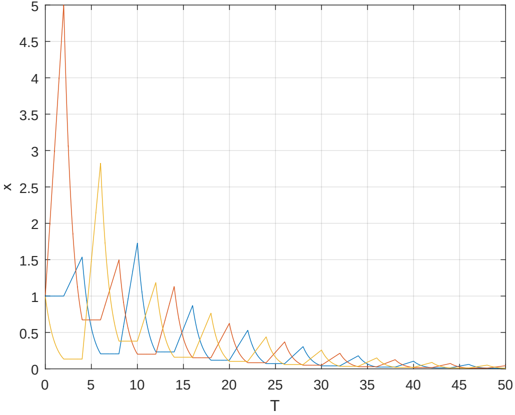
Lyapunov Metzler inequalities with dwell time are feasible!

$T = 2,\ x_0 = [1\ 1\ 1]^\top$



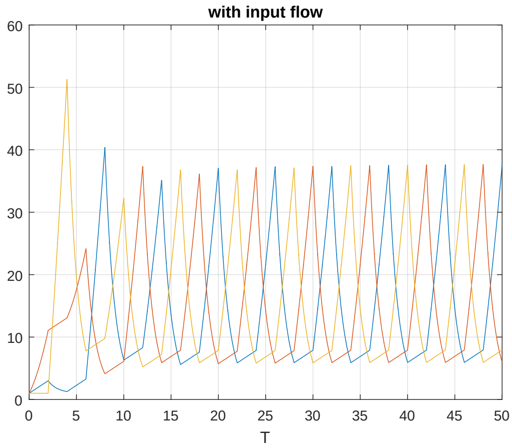
Simulation

$T = 2,\ x_0 = [1\ 1\ 1]^\top$



Simulation

$$T = 2,\ x_0 = [1\ 1\ 1]^\top,\ \dot{x} = A_\sigma x + \mathbf{1}$$



AIMD

Additive Increasing Multiplicative Decreasing

Classical AIMD

Distributed resource allocation problem: applications in the context of smart grids and smart transportation systems.

Multiagent system with limited communication between agents and limited feedback to the agents concerning aggregate utilization.
At time t the limited resource cannot be overused and each user $i = 1, 2, ..., N$ receives a share $p_i(t)$ such that

$$\sum_{i=1}^{N} p_i(t) \le P$$

Chiu, Jain. Analysis of the increase and decrease algorithm for congestion avoidance in computer networks. Computer networks and ISDN systems, 1989.

Classical AIMD

In classical AIMD there is no explicit exchange between agents and agents are informed via binary feedback when

$$\sum_{i=1}^{N} p_i(t) = P$$

This is called "capacity event". Each agents operates an algorithm that consists of a probing phase where an agents takes more and more resource (additive increase phase, AI) and a response phase where agents respond to the capacity events (multiplicative decrease phase, MD).

In the AI phase each agents linearly increase, with slope $\alpha_i > 0$, their share until the next capacity event. Upon the receipt of the CE signal, each agents execute the multiplicative decrease phase by instantaneously reducing their resorce share by a factor $\beta_i < 1$.

$$p_i(t) = \beta_i p_i(\tau_k) + \alpha_i(t - \tau_k), \quad \tau_k < t \le \tau_{k+1}$$

where τ_k denotes the time instant when the k-th CE occurs.

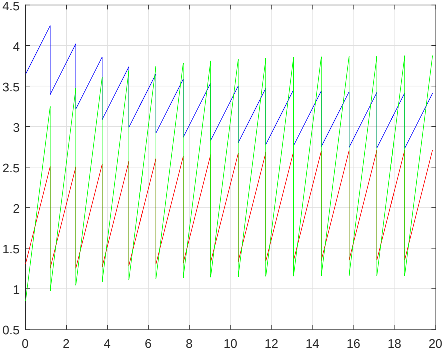
Dynamics at the capacity events. Column stochastic positive discrete-time system:

$$\begin{aligned}\mathbf{p}(\tau_{k+1}) &= A\mathbf{p}(\tau_k)\\ A &= \mathrm{diag}(\boldsymbol{\beta}) + (\mathbf{1}^\top \boldsymbol{\alpha})^{-1}\boldsymbol{\alpha}(\mathbf{1} - \boldsymbol{\beta})^\top\end{aligned}$$

Simulations

Example 5

$$P = 10, \quad \boldsymbol{\beta} = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.3 \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix}, \quad \mathbf{p}(\tau_k) \longrightarrow \begin{bmatrix} 2.72 \\ 3.40 \\ 3.88 \end{bmatrix}$$



Equilibrium (Frobenius-Perron)

$$A\bar{\mathbf{p}} = \bar{\mathbf{p}}$$

$$\bar{p}_i = \frac{\gamma_i}{\sum_{j=1}^N \gamma_j} P, \quad \gamma_i = \frac{\alpha_i}{1-\beta_i}$$

How to achieve a target $\mathbf{p}^o$:

$$\boldsymbol{\beta} = \mathbf{1} - \mathrm{diag}(\mathbf{p}^o)^{-1}\boldsymbol{\alpha}\eta, \; \eta \in (0, \, \eta*], \; \eta* = \min_i \frac{p_i^o}{\alpha_i}$$

Place dependent stochastic AIMD systems

Random shares

$$
\begin{aligned}
\mathbf{\Pi}(\tau_{k+1}) &= A_{\sigma(\tau_k)}\mathbf{\Pi}(\tau_k) \\
A_\sigma &= \mathrm{diag}(\boldsymbol{\beta}_\sigma) + (\mathbf{1}^\top \boldsymbol{\alpha})^{-1}\boldsymbol{\alpha}(\mathbf{1}-\boldsymbol{\beta}_\sigma)' \\
\mathrm{Prob}(A_\sigma = A_i \,|\, \mathbf{\Pi} = \mathbf{p}) &= \rho_i(\mathbf{p}), \quad i = 1, 2, \cdots, M
\end{aligned}
$$

Convergence of $\mathbf{\Pi}(\tau_k)$? Under technical assumptions ($\rho_i(\mathbf{p})$ Lipshitz continuous, A_σ average contractive, A_σ drop matrices), it can be shown that $\mathbf{\Pi}(\tau_k)$ has a unique invariant (attractive) distribution Π^*.

M. Corless, C. King, R. Shorten, F. Wirth, ''AIMD dynamics and distributed resourses allocation'', SIAM book, 2015.

Application example 3: EV public charging station

N Electrical vehicles. Energy required E_i^* for each vehicle.

- Optimal total charging time

$$\min_p \sum_i t_i : \quad \sum_{k=0}^{t_i-1} p_i(k) = E_i^*$$

- Optimal operation time

$$\min_{p_i} \max_i \frac{E_i^*}{p_i}$$

- Optimal shares

$$\min_{\gamma_i} \sum_i \frac{E_i^*}{\gamma_i P}, \quad \sum_i \gamma_i = 1$$

Optimal shares

Optimal shares

$$\min_{\gamma_i} \sum_i \frac{E_i^*}{\gamma_i P}, \quad \sum_i \gamma_i = 1$$

Centralized solution

$$\gamma_i^o = \frac{\sqrt{E_i^*}}{\sum_i \sqrt{E_i^*}} \quad \to \quad p_i^o = \frac{\sqrt{E_i^*}}{\sum_i \sqrt{E_i^*}} \bar{P}$$

$$\boldsymbol{\beta}^o = \mathbf{1} - \mathrm{diag}(\mathbf{p}^o)^{-1} \boldsymbol{\alpha} \boldsymbol{\varepsilon}$$

How to implement a distributed adaptive algorithm? No knowledge of the total available power. The algorithm can be implemented by each EV without any communication at all, except that of a broadcast of the capacity event notification (P reached).

Stochastic distributed algorithm

Two values of $\beta_i \in \{\beta^{1]}, \beta^{2]}\}$. Let ρ_i the probability that agent i selects β^1. Then

$$\rho_i(\tau_{k+1}) = \rho_i(\tau_k) - \eta f_i(p_i(\tau_k), E_i^*, \delta \hat{f}_i(\tau_k))$$

where $\delta \hat{f}_i(\tau_k))$ is an estimate of the gradient of the utility function. Then in the average $E(\mathbf{p}) = \mathbf{p}^o$.

Shah, Incremona, Bolzern, Colaneri. Optimization Based AIMD Saturated Algorithms for Public Charging of Electric Vehicles. EJC 2019.

Wirth, Stuedli, Yu, Yuan, Corless, Shorten. Nonhomogeneous Place-Dependent Markov Chains, Unsynchronized AIMD, and Network Utility Maximization. ACM, 2018.

Example

Three agents, $N = 3$. Capacity, $P = 7.5$. $p(0) = [2, 5, 0.5]$. $\beta^{[1]} = 0.8$ and $\beta^{[2]} = 0.95$. $E = [2.19, 5.22, 8.58]$. Furthermore, in order to examine the process, 50000 iterations of the system are performed and ergodicity is exploited. A histogram of the values p_1, p_2 and p_3 is then constructed. This would be an estimate of the steady-state distribution of the random variables Π_1, Π_2 and Π_3. The resulting histograms are illustrated in Figure 1 together with the sample averages of the shares p_i.

Example

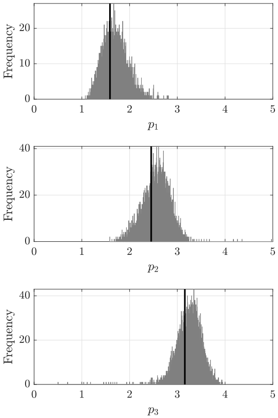


Figure 2: Estimation of the steady state distribution of $\bar{\Pi}_i$, $i = 1, 2, 3$.

Example

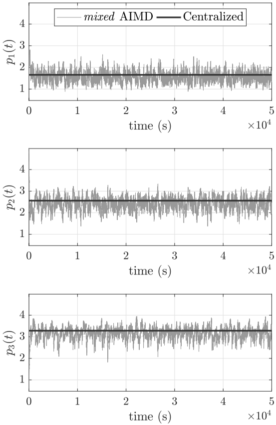


Figure 3: Time evolution of the $p_i(t)$, $i = 1, 2, 3$

Example

	$i=1$	$i=2$	$i=3$
p_i^o	1.658	2.559	3.282
$\widetilde{p}_i$	1.691	2.565	3.248

Figure 4: Comparizon between the centralized solution and the distributed solution.

Case study: public charging of electric vehicles

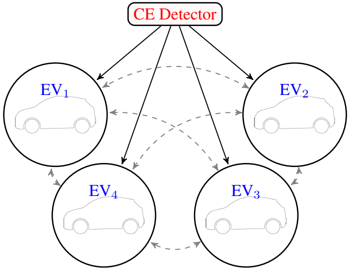


Figure 5: Schematic representation of the considered AIMD infrastructure

Random arrival of vehicles. Each vehicle has random energy demand (E_i^*) and the total available power is equal to $P = 2.5NskW$, with $Ns = 4$ being the total number of available charging spots. Average arrival rate of the vehicles $=$ 3 vehicles per hour.

Simulations

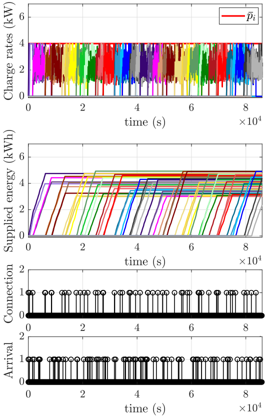


Figure 6: Time evolution of the charge rates and supplied energy in a random arrival scenario when the optimal share stochastic AIMD algorithm is used. Flag signals equal to 1 indicate the connection or the arrival time instants.

Conclusions

Application-driven theory of positive switched linear systems.

- Optimal control (Virus load mitigtion)
- Stabilization (Traffic light scheduling)
- AIMD (EV distributed battery charge)

Thanks

Think positive