Scalable synthesis for positive systems

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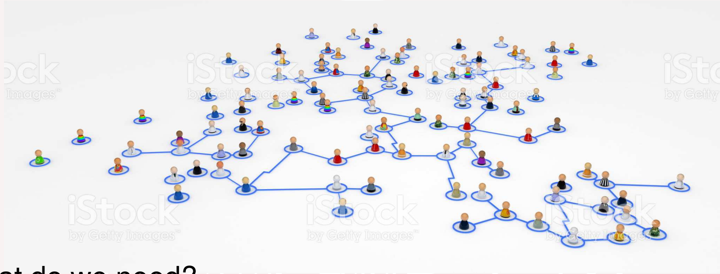
Towards a Scalable Theory of Control



What do we need?

- Scalable Synthesis
- Scalable Verification
- Scalable Modeling
- Scalable Objectives

A Scalable Stability Test

Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

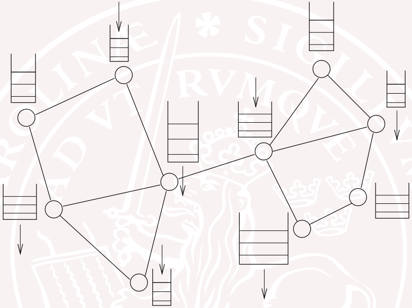
The first node verifies the inequality of the first row.

The second node verifies the inequality of the second row.

...

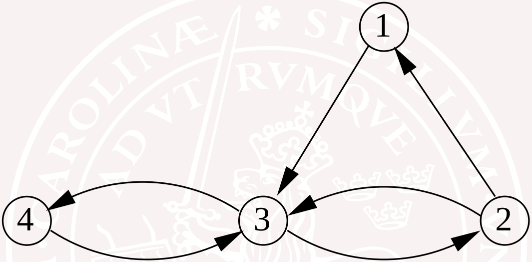
Verification is scalable!

Dynamic Buffer Networks



- Producers, consumers and storages
- Examples: water, power, traffic, data
- Discrete/continuous, stochastic/deterministic
- Multiple commodities, human interaction

Control Synthesis for a Buffer Network



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}-1-\ell_{31} & \ell_{12} & 0 & 0\\ 0 & -\ell_{12}-\ell_{32} & \ell_{23} & 0\\ \ell_{31} & \ell_{32} & -\ell_{23}-\ell_{43} & \ell_{34}\\ 0 & 0 & \ell_{43} & -4-\ell_{34}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix}$$

How do we select ℓ_{ij} to minimize some gain from w to x?

Outline

- **Norms and Gains**
- Control Synthesis for Positive Systems
- Nonlinear Monotone Systems
- Bilinear Positive Systems

Norms and Gains

Given $\mathbf{x} \in \mathbb{R}_+^n$ and $p \in (0, +\infty)$ define $|\mathbf{x}|_p := \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$ and $|\mathbf{x}|_\infty := \max_{i \in [1,n]} |x_i|$.

Consider $M \in \mathbb{R}^{l \times m}$, $f : \mathbb{R}_+ \to \mathbb{R}^n$ and $p \in (0, +\infty]$. Let $\mathcal{G}$ be an operator from L_p^m to L_p^r, while $G(s) \in \mathbb{R}(s)^{r \times m}$ is a proper stable rational matrix. Then

$$
\begin{aligned}
\|M\|_{p-\text{ind}} &:= \sup_{\mathbf{x}:|\mathbf{x}|_p=1} |M\mathbf{x}|_p \\
\|f\|_{L_p} &:= \left(\int_0^{+\infty} |f(t)|_p^p dt\right)^{\frac{1}{p}} \\
\|f\|_{L_\infty} &:= \text{ess sup}_{t \geq 0} |f(t)|_\infty \\
\|\mathcal{G}\|_{L_p - L_p} &:= \sup_{\|\mathbf{w}\|_{L_p}=1} \|\mathcal{G}\mathbf{w}\|_{L_p} \\
\|G\|_{H_\infty} &:= \sup_{\omega} \|G(i\omega)\|_{2-\text{ind}}.
\end{aligned}
$$

Gains of Positive Systems

Suppose that $\mathcal{G}$ is the input-output operator of an asymptotically stable positive system. Then for $p = 1, 2$ and $+\infty$ we have

$$\|\mathcal{G}\|_{L_p - L_p} = \|G(0)\|_{p-\text{ind}}$$

In particular, if $\mathcal{G}_1$ and $\mathcal{G}_2$ have transfer functions $G(s)$ and $G(s)^\top$ respectively, then

$$\|\mathcal{G}_1\|_{L_1 - L_1} = \|\mathcal{G}_2\|_{L_\infty - L_\infty}.$$

Moreover, if the system is SISO, i.e. $r = m = 1$, then

$$\|\mathcal{G}\|_{L_p - L_p} = G(0), \qquad \forall\, p \in [1, +\infty].$$

L_∞ Gain Verification

Given the positive system (A, B, C, D) and any $\gamma > 0$, the following statements are equivalent:

(1) A is Hurwitz and $\|\mathcal{G}\|_{L_\infty - L_\infty} < \gamma$.

(2) A is Hurwitz and $G(0)\mathbf{1}_m \ll \gamma\mathbf{1}_r$.

(3) There exists $\xi \gg 0, \xi \in \mathbb{R}^n$, such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ \mathbf{1}_m \end{bmatrix} \ll \begin{bmatrix} 0 \\ \gamma\mathbf{1}_r \end{bmatrix}.$$

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Verfication by linear programming.

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In the scalar case: All L_p-gains are the same.

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More about L_1-gain in Ebihara's presentation.

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A ScalablSynthesis Method

Problem: Find $\ell_1, \ell_2 \in [0,1]$ to minimize L_2-gain of

$$\dot{x} = \begin{bmatrix} a_{11}-\ell_1 & a_{12} & 0 & a_{14} \\ a_{21}+\ell_1 & a_{22}-\ell_2 & a_{23} & 0 \\ 0 & a_{32}+\ell_2 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} x$$

Solution: Find ℓ_1, ℓ_2 with $0 \le \mu_k \le \xi_k$ such that

$$\begin{bmatrix} a_{11}\xi_1-\mu_1 & a_{12}\xi_2 & 0 & a_{14}\xi_4 & 1 \\ a_{21}\xi_1+\mu_1 & a_{22}\xi_2-\mu_2 & a_{23}\xi_3 & 0 & 0 \\ 0 & a_{32}\xi_2+\mu_2 & a_{33}\xi_3 & a_{34}\xi_4 & 0 \\ a_{41}\xi_1 & 0 & a_{43}\xi_3 & a_{44}\xi_4 & 0 \\ 0 & 0 & \xi_3 & \xi_4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \ll \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma \end{bmatrix}$$

and set $\ell_1 = \mu_1/\xi_1$ and $\ell_2 = \mu_2/\xi_2$. (Distributed lin. prog.)

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and set $\ell_1 = \mu_1/\xi_1$ and $\ell_2 = \mu_2/\xi_2$. (Distributed lin. prog.)

H_∞ Optimal Control of Buffer Networks

Problem:

Given a graph $(\mathcal{V}, \mathcal{E})$ and

$$\dot{x}_i = a_i x_i + \sum_{(i,j)\in\mathcal{E}} (u_{ij} - u_{ji}) + w_i \qquad\qquad i \in \mathcal{V}$$

find control law $u = Kx$ that minimizes the H_∞ norm of the map from w to (x, u).

Solution:

An optimal control law when $a_i < 0$ is given by

$$u_{ij} = x_i/a_i - x_j/a_j \qquad\qquad (i,j) \in \mathcal{E}.$$

The closed loop system is a positive system!

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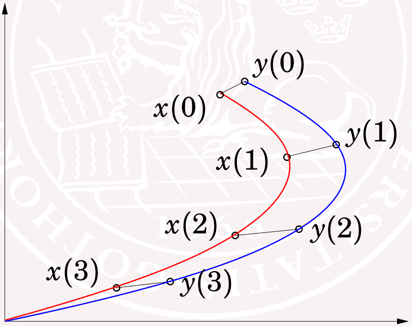
Nonlinear Monotone Systems

The system

$$\dot{x}(t) = f(x(t), u(t)), \qquad\qquad x(0) = a$$

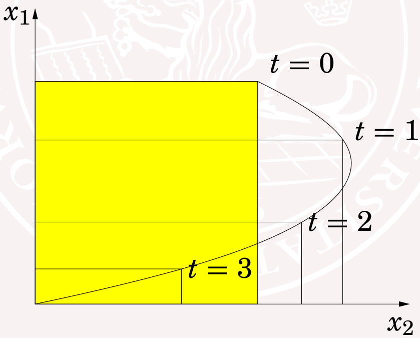
is a *monotone system* if its linearization is a positive system.



Max-separable Lyapunov Functions

Let $\dot{x} = f(x)$ be a globally asymptotically stable monotone system with invariant compact set $X \subset \mathbb{R}^n_+$. Then there exist strictly increasing functions $V_k : \mathbb{R}_+ \to \mathbb{R}_+$ for $k = 1, \dots, n$ with $\frac{d}{dt}V(x(t)) = -V(x(t))$ in X where $V(x)$ is equal to

$$V(x) = \max\{V_1(x_1), \dots, V_n(x_n)\}.$$



Convex-Monotone Systems

The system

$$\dot{x}(t) = f(x(t), u(t)), \qquad\qquad x(0) = a$$

is a *monotone system* if its linearization is a positive system. It is a *convex monotone system* if every row of f is also convex.

Theorem.

For a convex monotone system $\dot{x} = f(x, u)$, each component of the trajectory $\phi_t(a, u)$ is a convex function of (a, u).

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Example: Combination Therapy

Evolutionary dynamics:

$$\dot{x} = \left(A - \sum_i u_i D^i\right) x$$

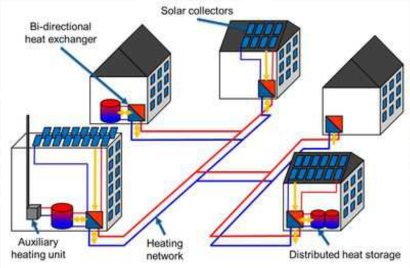
Each state x_k is the concentration of a mutant. (There can be hundreds!) Each input u_i is a drug dosage.

A describes the mutation dynamics without drugs, while $D^1, \ldots, D^m \geq 0$ are diagonal matrices modeling drug effects.

Determine $u_1, \ldots, u_m \geq 0$ with $u_1 + \cdots + u_m \leq 1$ such that x decays as fast as possible!

[Hernandez-Vargas, Colaneri and Blanchini, JRNC 2011]
[Jonsson, Rantzer,Murray, ACC 2014]

Example: District Heating



Energy balance for heat exchanger:

$$V\frac{dT}{dt} = (\widehat{T} - T)q - P$$

Positive system when flows are constant.

Bilinear positive system when flows are controlled.

Optimizing Decay Rate

Stability of the matrix $A - \sum_i u_i D^i + \gamma I$ is equivalent to existence of $\xi > 0$ with

$$(A - \sum_i u_i D^i + \gamma I)\xi < 0$$

For row k, this means

$$A_k \xi - \sum_i u_i D^i_k \xi_k + \gamma \xi_k < 0$$

or equivalently

$$\frac{A_k \xi}{\xi_k} - \sum_i u_i D^i_k + \gamma < 0$$

Maximizing γ is convex optimization in $(\log \xi_i, u_i, \gamma)$!

Using Measurements of Virus Concentrations

The evolutionary dynamics can be written as a convex monotone system:

$$\frac{d}{dt}\log x_k(t) = \frac{A_k x(t)}{x_k(t)} - \sum_i u_i(t) D_k^i$$

Hence the decay of $\log x_k$ is a convex function of the input and optimal trajectories can be found even for large systems.

Summary

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