Positive Systems and Positive Switched Systems:

Basic Theoretical Results and Main Applications. An Overview

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Outline of the talk

- Introduction to the Tutorial Session

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- Motivating examples

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- Research in the eighties and nineties mainly focused on two problems: reachability/controllability and positive realization. At the same time, there was extensive research on monotone systems that generalize positive systems and find quite meaningful applications in biology, pharmacokinetics, thermodynamics, etc.

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- More recently, research on positive systems revealed that many results obtained for positive systems, in particular those regarding stability and stabilization with performances that can be expressed in terms of norms, can be fruitfully used to deal with large scale and interconnected systems. Indeed, such properties can be checked by resorting to algorithms/ conditions that scale linearly with the system dimension.

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- **Fourth Talk**: Patrizio Colaneri (30 min)
  Optimal scheduling of positive switched systems: application examples.

Motivating examples (1)

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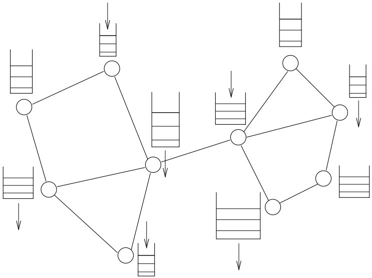

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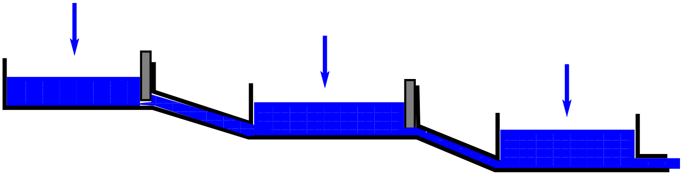
In a fluid network of the open-channel type, the fluid is stored in various reservoirs and transported from one reservoir to another by means of pipelines. This situation is typical, for instance, of water supply networks. The flow from one reservoir to another depends solely on the level of the upper reservoir (see Y. Ebihara's talk).



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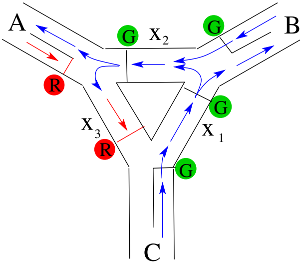

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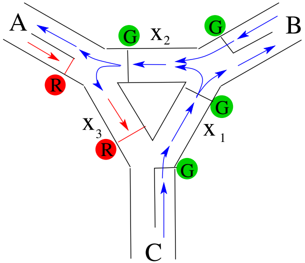

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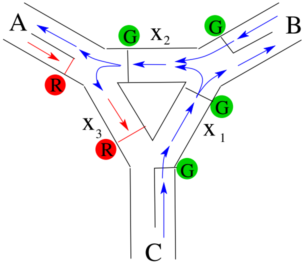

Three main roads (A, B and C) converge into a “triangular connection" governed by traffic lights. The buffer variable x_i represents the vehicles waiting at the ith traffic light inside the triangular loop.
If only one pair of adjacent traffic lights is red at any time, we have three different positive systems and hence a positive switched system (see P. Colaneri’s talk).

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- A matrix A is Metzler if all its off-diagonal entries are nonnegative

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- A square matrix A is Schur if all its eigenvalues have modulus strictly smaller than 1
- A symmetric matrix P is positive (semi)definite $P \succ 0$ $(P \succeq 0)$ if for every $\mathbf{x} \neq 0$, $\mathbf{x}^\top P \mathbf{x} > 0$ $(\mathbf{x}^\top P \mathbf{x} \geq 0)$

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in the continuous-time case. In these equations $\mathbf{x}$ represents the n-dimensional state variable, $\mathbf{u}$ the m-dimensional input variable and $\mathbf{y}$ the r-dimensional output variable.

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A linear state-space model is (internally) positive if

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For continuous-time systems: A is Metzler, B, C and D are nonnegative matrices

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2) By the monotonicity property of positive systems, asymptotic stability is equivalent to the convergence to zero of the state trajectory corresponding to $\mathbf{x}(0) = \mathbf{1}_n$

3) Conditions (1.4) and (2.4) amounts to saying that a positive (continuous-time or discrete-time) system is asymptotically stable if and only if it admits a diagonal quadratic Lyapunov function.

Copositive Lyapunov functions (1)

Definition 1 A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be copositive if $V(\mathbf{x}) > 0$ for every $\mathbf{x} > 0$. A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a linear copositive function if $V(\mathbf{x}) = \mathbf{z}^\top \mathbf{x}$, for some $\mathbf{z} \gg 0$.

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Conditions (1.3) and (2.3) in Proposition 1 correspond to saying that a positive matrix (a Metzler matrix) is Schur (Hurwitz) if and only if it admits a linear copositive Lyapunov function, i.e., there exists $V(\mathbf{x}) = \mathbf{z}^\top \mathbf{x}$, with $\mathbf{z} \gg 0$, such that $\Delta V(\mathbf{x}) < 0$ ($\dot{V}(\mathbf{x}) < 0$) for every $\mathbf{x} > 0$.

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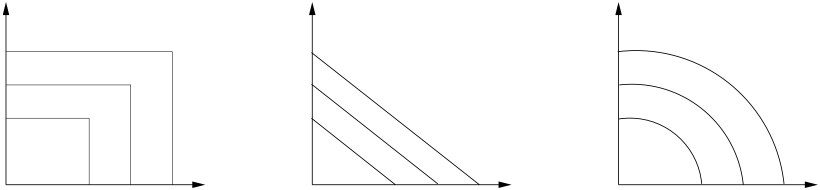

Figure: Level curves of Lyapunov functions - conditions (1.2), (1.3) and (1.4) in Proposition 1.

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$$\|\mathbf{x}(t; \mathbf{x}(0), \sigma)\| \leq Ce^{-\beta t}\|\mathbf{x}(0)\|,$$

for every $\mathbf{x}(0) \in \mathbb{R}^n_+, t \geq 0,$ and every switching signal σ.

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Lyapunov functions (1)

Definition 3 A differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the continuous-time (positive) switched system if it is positive definite and

$$\nabla V(\mathbf{x}) A_i \mathbf{x} < 0, \qquad \forall\, \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0, \forall\, i \in \{1, 2, \dots, M\}.$$

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Theorem 3 The following facts are equivalent:

i) the continuous-time positive switched system is exponentially stable;

ii) there exists a (differentiable) Lyapunov function V for the switched system, homogeneous of order 2 (i.e., $V(\alpha\mathbf{x}) = \alpha^2 V(\mathbf{x})$ for every $\alpha > 0$ and every $\mathbf{x} \in \mathbb{R}^n$).

Lyapunov functions (2)

Theorem 4 Given a continuous-time positive switched system, the following facts are equivalent:

1) $\exists\ \mathbf{v} \gg 0$ such that $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is an linear copositive Lyapunov function for the system;
2) $\exists\ P = P^\top$ of rank 1 such that $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is a quadratic copositive Lyapunov function for the system;
3) for each map $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, M\}$, the matrix

$$A_\pi := \begin{bmatrix} \mathrm{col}_1(A_{\pi(1)}) & \mathrm{col}_2(A_{\pi(2)}) & \ldots & \mathrm{col}_n(A_{\pi(n)}) \end{bmatrix}$$

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If any of the previous conditions holds, the continuous-time positive switched system is exponentially stable.

Stabilization definitions

Theorem 5 Given a continuous-time positive switched system, the following facts are equivalent:

1) $\exists\ \bar{\mathbf{x}}_0 \gg 0$ and a switching signal $\bar{\sigma}(t), t \in \mathbb{R}_+,$ such that the trajectory $\bar{\mathbf{x}}(t), t \in \mathbb{R}_+,$ generated corresponding to $\bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0$ and $\bar{\sigma}(t), t \in \mathbb{R}_+,$ exponentially converges to zero.

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3) The switched system is consistently stabilizable, i.e., there exists a switching signal $\sigma(t), t \in \mathbb{R}_+,$ that drives $\mathbf{x}(t)$ to zero exponentially, independently of the positive initial condition $\mathbf{x}(0) > 0$.

Control Lyapunov functions (1)

Definition 4 A copositive function $V(\mathbf{x})$ is a control Lyapunov function for the continuous-time positive switched system, if it is decreasing along the system trajectories, provided that a certain feedback switching strategy is applied.

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Theorem 6 A continuous-time positive switched system is stabilizable if and only it admits a concave copositive control Lyapunov function $V(\mathbf{x})$, positively homogeneous of order one (i.e. $V(\alpha\mathbf{x}) = \alpha V(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ and every $\alpha > 0$).

Control Lyapunov functions (2)

Theorem 7 Given a continuous-time positive switched system, the following facts are equivalent:

1) There exists a convex combination $\sum_{i=1}^{M} \alpha_i A_i$ that is Hurwitz.
2) The system admits a linear copositive control Lyapunov function.

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Theorem 7 Given a continuous-time positive switched system, the following facts are equivalent:

1) There exists a convex combination $\sum_{i=1}^{M} \alpha_i A_i$ that is Hurwitz.
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If any of the previous equivalent conditions holds, the continuous-time positive switched system is stabilizable.

Conclusions

Thanks for your attention!