Stability and stabilizability of continuous-time linear compartmental switched systems

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***Abstract*—In this paper we introduce continuous-time linear compartmental switched systems and investigate their stability and stabilizability properties. By their nature, these systems are always stable. Necessary and sufficient conditions for asymptotic stability for arbitrary switching functions, and sufficient conditions for asymptotic stability under certain dwell-time conditions on the switching functions are proposed. Finally, stabilizability is thoroughly investigated and proved to be equivalent to the existence of a Hurwitz convex combination of the subsystem matrices, a condition that for positive switched systems is only sufficient for stabilizability.**

***Index Terms*—Compartmental systems, Positive systems, Switched systems, Stability and Stabilizability.**

I. Introduction

LINEAR switched systems, with the positivity constraint on the state and input variables, have been the object of a significant number of contributions in the last ten years, e.g. [2], [4], [10], [11], [12], [15], [22], [29], [28], [42]. The interest in this class of systems is primarily motivated by the number of application areas where they have been fruitfully employed. To mention the most significant ones: wireless power control, congestion control, system biology, HIV mitigation therapy and pharmacokinetic [4], [17], [36], [46]. The investigation of positive switched systems offers a good number of challenging problems, since the positivity constraint on the state and input variables makes it impossible to resort to techniques and results based on finite-dimensional vector spaces, and demands for the involvement of less settled approaches, based on cones and polytopes. This is the case, for instance, when dealing with reachability and controllability of positive switched systems, two topics that are still mostly unexplored [33], [42].

When addressing stability and stabilizability problems, on the other hand, the results obtained for standard linear switched systems can be fruitfully employed but in general they represent conservative results, and ad-hoc tools, based on copositive (control) Lyapunov functions and on characterizations involving the convex combinations of the subsystem matrices, turn out to be more appropriate [2], [11], [12], [15], [22], [28], [48].

In many cases, in every single operating mode, namely in each of the configurations among which the system switches, the unforced dynamics of a positive switched system evolves in accordance with some conservation law (e.g. mass, energy, fluid), governing the exchange of material between different compartments. This is the case, for instance, when modeling a fluid network: state variables represent fluid levels in the various tanks and each subsystem corresponds to a different open/closed configuration of the pipes connecting the tanks [4]. Analogously, a compartmental switched system is oftentimes a well-suited choice for describing a thermal system that may undergo different working conditions, related to the fact that heat transmission coefficients may change (window/doors may be open or closed) [4]. Other practical examples arise, for instance, when describing economical systems [25] or the lung dynamics [19], [23]. Two motivating examples, one dealing with a thermal system and one dealing with the respiratory function, have been presented in detail in [43]. We refer to positive switched systems whose subsystems are described by autonomous compartmental state-space models as *compartmental switched systems*.

In this paper we first introduce continuous-time compartmental switched systems and then investigate their properties. Specifically, in Section II we investigate stability under arbitrary switching, and prove that for this class of systems the Hurwitz property of all the subsystem matrices is a necessary and sufficient condition for asymptotic stability. In Section III we determine necessary and sufficient conditions for the existence of quadratic positive definite Lyapunov functions of special types. In Section IV we drop the assumption that all the subsystem matrices are Hurwitz, and provide classes of switching functions with special persistence and/or dwell-time properties that ensure the asymptotic convergence of the associated system trajectories, independently of the initial condition. In Section V we introduce stabilizability and show that it is equivalent to the existence of a Hurwitz convex combination of the subsystems matrices. Also, by making use of some recent results appeared in [2], we show that stabilizability of compartmental switched systems may be related to the existence of special classes of copositive control Lyapunov functions. Finally, in Section VI conclusions are drawn. A preliminary version of Section V will appear in the conference paper [43], that will be presented at the next CDC 2015 conference.

Before proceeding we introduce some notation.

Notation. Given $k, n \in \mathbb{Z}$, with $k \le n$, the symbol $[k, n]$ denotes the integer set $\{k, k+1, \dots, n\}$. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. In the sequel, the (i, j)th entry of a matrix A is denoted by $[A]_{ij}$. If A is block partitioned, we denote its (i, j)th block by $block_{ij}[A]$. In the special case of a vector $\mathbf{v}$, its ith entry is $[\mathbf{v}]_i$ and its ith block is $block_i[\mathbf{v}]$. A matrix A_+ with entries in $\mathbb{R}_+$ is a *nonnegative matrix* ($A_+ \ge 0$); if $A_+ \ge 0$ and at least one entry is positive, A_+ is a *positive matrix* ($A_+ > 0$), while if all its entries are positive it is a *strictly positive matrix* ($A_+ \gg 0$). The same notation is adopted for nonnegative, positive and strictly positive vectors.

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We let $\mathbf{e}_i$ denote the ith vector of the canonical basis in $\mathbb{R}^n$ (where n is always clear from the context), whose entries are all zero except for the ith one that is unitary. $\mathbf{1}_n$ is the n-dimensional vector with all entries equal to 1, and $\mathbf{0}_n$ is the n-dimensional vector with all entries equal to 0 (the dimension n will be omitted if it is clear from the context). Given r vectors $\mathbf{v}_1, \dots, \mathbf{v}_r \in \mathbb{R}^n$, by $\mathrm{Cone}(\mathbf{v}_1, \dots, \mathbf{v}_r)$ we mean the set of nonnegative combinations of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_r$. A real square matrix A is *Hurwitz* if all its eigenvalues lie in the open left complex halfplane. We denote by $\sigma(A)$ the spectrum of A.

A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. If A is an $n \times n$ Metzler matrix, then [39] it exhibits a real dominant eigenvalue, known as *Frobenius eigenvalue* and denoted by $\lambda_F(A)$. This means that $\lambda_F(A) > \mathrm{Re}(\lambda), \forall\, \lambda \in \sigma(A), \lambda \neq \lambda_F(A)$, and there exists a positive eigenvector (*Frobenius eigenvector*) $\mathbf{v}_F$ corresponding to $\lambda_F(A)$. When no confusion may arise, we will use λ_F instead of $\lambda_F(A)$.

An $n \times n$ nonzero matrix A is *reducible* [13] "if we may partition $\{1, \dots, n\}$ into two non-empty subsets E, F such that $a_{ij} = 0$ if $i \in E, j \in F$." (see also [34]). This is equivalent to saying that there exists a permutation matrix Π such that (s.t.)

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices, otherwise it is *irreducible*. It follows that 1×1 nonzero matrices are always irreducible. In general, given a Metzler matrix A, a permutation matrix Π can be found s.t.

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1\ell} \\ 0 & A_{22} & \dots & A_{2\ell} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & A_{\ell\ell} \end{bmatrix}, \qquad (1)$$

where each diagonal block A_{ii}, of size $n_i \times n_i$, is either scalar ($n_i = 1$) or irreducible. (1) is usually known as *Frobenius normal form* of A [14], [30].

A (linear) *compartmental system* is an autonomous linear state-space model:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \qquad (2)$$

whose state matrix $A \in \mathbb{R}^{n \times n}$ is Metzler and the entries of each of its columns sum up to a nonpositive number, i.e. $\mathbf{1}_n^\top A \leq \mathbf{0}^\top$. A square matrix endowed with these two properties is called *compartmental matrix*. (Autonomous) compartmental systems are used to describe material flows between compartments. As $x_i(t)$, the ith entry of the state variable, represents the content of the ith compartment at time t, the system is intrinsically nonnegative and hence the matrix A must be Metzler. On the other hand, as the total amount of the material in the system $\sum_{i=1}^n x_i(t) = \mathbf{1}_n^\top \mathbf{x}(t)$ cannot increase with time (since there is no inflow, but there may be an outflow or losses) it follows that $\mathbf{1}_n^\top \dot{\mathbf{x}}(t) = \mathbf{1}_n^\top A\mathbf{x}(0) \leq 0$ for every $\mathbf{x}(0) > 0$, and hence $\mathbf{1}_n^\top A \leq \mathbf{0}^\top$ (see [16], [38]).

For any such matrix the Frobenius eigenvalue λ_F is nonpositive, and if $\lambda_F = 0$ then A is simply stable, by this meaning that it has the constant mode associated with $\lambda_F = 0$ but no unstable modes.

Given a Metzler matrix $A \in \mathbb{R}^{n \times n}$, we associate with it [6], [7], [35], [41] a *digraph* $\mathcal{D}(A) = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \dots, n\}$ is the set of vertices and $\mathcal{E}$ is the set of arcs (or edges). There is an arc $(j, \ell) \in \mathcal{E}$ from j to ℓ if and only if $[A]_{\ell j} > 0$. If so, $[A]_{\ell j}$ represents the *weight* of the arc. A sequence $j_1 \to j_2 \to \dots \to j_k \to j_{k+1}$ is a *path* of length k from j_1 to j_{k+1} provided that $(j_1, j_2), \dots, (j_k, j_{k+1})$ are elements of $\mathcal{E}$.

We say that vertex ℓ is *accessible* from j if there exists a path in $\mathcal{D}(A)$ from j to ℓ (equivalently, $\exists\; k \in \mathbb{N}$ s.t. $[A^k]_{\ell j} \neq 0$). Two distinct vertices ℓ and j are said to *communicate* if each of them is accessible from the other. Each vertex is assumed to communicate with itself. The concept of communicating vertices allows to partition the set of vertices $\mathcal{V}$ into *communicating classes*, say $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_\ell$. Class $\mathcal{C}_j$ accesses class $\mathcal{C}_i$ if there is a path from some vertex $k \in \mathcal{C}_j$ to some vertex $h \in \mathcal{C}_i$. Each class $\mathcal{C}_i$ has clearly access to itself. A class $\mathcal{C}_i$ that has access to no other class except for itself is called *recurrent*, otherwise it is called *transient* [21], [31]. If $\mathcal{C}_1, \dots, \mathcal{C}_s$ are the recurrent classes of $\mathcal{D}(A)$ and $\mathcal{C}_{s+1}, \dots, \mathcal{C}_\ell$ its transient classes, then there exists a permutation matrix Π s.t.

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & \dots & 0 & A_{1s+1} & & A_{1\ell} \\ & \ddots & & & \ddots & \\ 0 & \dots & A_{ss} & A_{ss+1} & & A_{s\ell} \\ & & & A_{s+1s+1} & \dots & A_{s+1\ell} \\ & & & & \ddots & \\ & & & A_{\ell s+1} & \dots & A_{\ell\ell} \end{bmatrix},$$

where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, \ell]$, are either scalar ($n_i = 1$) or irreducible matrices. $\mathcal{D}(A)$ is said to be *strongly connected* if every pair of vertices ℓ and j communicate, and hence it consists of a single communicating class. $\mathcal{D}(A)$ is strongly connected if and only if A is irreducible.

An $n \times n$ symmetric matrix $P = P^\top$ is said to be *quadratic positive definite* (and when so, we use the notation $P = P^\top \succ 0$) if $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$, and *quadratic copositive* if $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{R}^n_+, \mathbf{x} \neq \mathbf{0}$. Clearly, a quadratic positive definite matrix is quadratic copositive, but the converse is not true. Given a vector $\mathbf{v} \in \mathbb{R}^n$ and a set $\mathcal{N} \subset \mathbb{R}^n$, the *distance of the vector* $\mathbf{v}$ *from the set* $\mathcal{N}$ is $\mathrm{dist}(\mathbf{v}, \mathcal{N}) := \inf_{\mathbf{z} \in \mathcal{N}} \|\mathbf{v} - \mathbf{z}\|$.

II. Stability under arbitrary switching

In this paper, by a *Continuous-time Compartmental Switched System* (CCSS) we mean a system described by the following equation:

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \quad t \in \mathbb{R}_+, \qquad (3)$$

where $\mathbf{x}(t) \in \mathbb{R}^n_+$ denotes the value of the n-dimensional state variable at time t, $\sigma : \mathbb{R}_+ \to [1, M]$ is an arbitrary switching function, and, for each $i \in [1, M]$, A_i is an $n \times n$ compartmental matrix. We also assume that $\sigma(\cdot)$ is right continuous and in every finite interval it has a finite number of discontinuities.

The definition of asymptotic stability for CCSSs is analogous to the one introduced for arbitrary switched systems. In addition, as for all linear switched systems, asymptotic stability is equivalent to exponential stability [37].

Definition 1. *The CCSS (3) is* asymptotically stable *if for every positive initial state* $\mathbf{x}(0)$ *and every switching function* $\sigma : \mathbb{R}_+ \to [1, M]$ *the state trajectory* $\mathbf{x}(t), t \in \mathbb{Z}_+$, *converges to zero.*

As it is well-known, a necessary condition for the stability under arbitrary switching of a linear switched system, and hence of a CCSS described as in (3), is that all individual subsystems are asymptotically stable. Hence, in this section we will steadily assume that all matrices A_i, $i \in [1, M]$, are Hurwitz. This implies, in particular, that $\mathbf{1}_n^\top A_i < \mathbf{0}^\top$, for every $i \in [1, M]$ (see Appendix).

A. Hurwitz property of the subsystem matrices

In the general (i.e. noncompartmental) case, the Hurwitz assumption on all the subsystem matrices is not sufficient to guarantee stability under arbitrary switching, not even when we deal with continuous-time positive switched systems and hence all matrices $A_i, i \in [1, M]$, are Metzler [27]. In the following we will show that for CCSSs stability under arbitrary switching is equivalent to the fact that all the subsystem matrices are Hurwitz.

Proposition 1. *Let* $A_i \in \mathbb{R}^{n\times n}$, $i \in [1, M]$, *be compartmental matrices. Then, the following facts are equivalent:*

i) *the CCSS (3) is asymptotically stable under arbitrary switching;*
ii) *for every choice of* $\alpha_i \geq 0$, $i \in [1, M]$, *with* $\sum_{i=1}^M \alpha_i = 1$, *the convex combination* $\sum_{i=1}^M \alpha_i A_i$ *is Hurwitz.*
iii) A_i *is Hurwitz for every* $i \in [1, M]$.

Proof. i)⇒ ii) It is a well known result for continuous-time switched systems [4], [24].

ii)⇒ iii) It is obvious.

iii) ⇒ i) The common Lyapunov function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is of class C^1, copositive and such that:

$$\nabla V(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) = \mathbf{1}_n^\top A_i \mathbf{x}(t) \leq 0, \quad \forall \mathbf{x}(t) > 0, \forall i \in [1, M],$$

and hence $V(\mathbf{x}(t))$ is a Common Weak Lyapunov Function in the sense of Definition 3 in [1] (of course, adjusting such definition to CCSSs, i.e. restricting it to the positive orthant $\mathbb{R}^n_+$, entails no loss of validity). Then, Proposition 1 in [1] ensures that the CCSS is stable under arbitrary switching (see also [5], [9]). However, we want to prove that the system is also asymptotically stable under arbitrary switching. To this aim, define the set $\mathcal{N}$ as:

$$\mathcal{N} := \{\mathbf{x} \in \mathbb{R}^n_+ : \exists i \in [1, M] \text{ s.t. } \nabla V(\mathbf{x}) A_i \mathbf{x} = \mathbf{1}_n^\top A_i \mathbf{x} = 0\} = \bigcup_{i=1}^M \mathcal{N}_i,$$

where $\mathcal{N}_i := \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{1}_n^\top A_i \mathbf{x} = 0\}$. For every $\mathbf{x} \in \mathcal{N}$, $\mathbf{x} \neq 0$, define as well

$$\begin{aligned} \mathcal{I}_\mathbf{x} &:= \{i \in [1, M] : \mathbf{x} \in \mathcal{N}_i\}, \\ d_\mathbf{x} &:= \min_{i \notin \mathcal{I}_\mathbf{x}} \text{dist}(\mathbf{x}, \mathcal{N}_i) > 0. \end{aligned}$$

By the compartmental property of each subsystem (see the proof of Proposition 9 in the Appendix), if $\mathbf{x}(0) \in \mathcal{N}_i$ for some $i \in [1, M]$, $\mathbf{x}(0) \neq \mathbf{0}$, then for every $\tau > 0$ sufficiently small $\mathbf{x}(\tau) \notin \mathcal{N}_i$. So, by choosing $\tau > 0$ sufficiently small we can ensure that $\mathbf{x}(\tau) \notin \mathcal{N}_i, \forall i \in \mathcal{I}_{\mathbf{x}(0)}$. On the other hand, since the distance $d_{\mathbf{x}(0)}$ is finite, it is also true that if τ is sufficiently small $\mathbf{x}(\tau) \notin \mathcal{N}_i$ for every $i \notin \mathcal{I}_{\mathbf{x}(0)}$. Therefore $\mathbf{x}(\tau) \notin \mathcal{N}$. This ensures that the only compact, weakly invariant set [1] contained in $\mathcal{N}$ is $\mathcal{M} = \{\mathbf{0}\}$, and by Theorem 1 in [1] every state trajectory is attracted by $\mathcal{M}$, i.e. converges to the origin. □

B. Characterizations in terms of Lyapunov functions

First of all, we introduce the definition of common Lyapunov function for the CCSS (3), as opposed to the notion of control Lyapunov function that we will explore in the following. Dealing with positive systems, we may loosen the constraints on the Lyapunov functions we are considering, by allowing them to take positive values only in the positive orthant and hence to be copositive.

Definition 2. *A function* $V : \mathbb{R}^n \to \mathbb{R}$ *is said to be* copositive *if* $V(0) = 0$ *and* $V(\mathbf{x}) > 0$ *for every* $\mathbf{x} > \mathbf{0}$. *A continuous and continuously differentiable copositive function* $V(\mathbf{x})$ *is a* common Lyapunov function *for the CCSS (3) if for every* $\mathbf{x} > \mathbf{0}$ *and every* $i \in [1, M]$ *the derivative of* V *in* $\mathbf{x}$ *along the direction of the* i*th subsystem is negative, namely*

$$\nabla V(\mathbf{x}) A_i \mathbf{x} < 0. \quad (4)$$

In this paper we will focus on two classes of common copositive Lyapunov functions: linear and quadratic positive definite Lyapunov functions. For these functions, the definition of Lyapunov function adjusts as follows.

Definition 3. *Given a CCSS (3), a continuous and continuously differentiable copositive function* $V(\mathbf{x})$ *is*

• *a* Common Linear Copositive Lyapunov Function *(CLCLF) for the CCSS (equivalently, for the matrices* $A_i, i \in [1, M]$*) if* $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, *for some strictly positive vector* $\mathbf{v} \in \mathbb{R}^n_+$, *and*

$$\mathbf{v}^\top A_i \ll \mathbf{0}^\top, \quad \forall i \in [1, M];$$

• *a* Common Quadratic Positive Definite Lyapunov Function *(CQPDLF) for the CCSS (equivalently, for the matrices* $A_i, i \in [1, M]$*) if* $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, *for some* $n \times n$ *positive definite matrix* $P = P^\top \succ 0$, *and*

$$\begin{gathered} \dot{V}_i(\mathbf{x}) = \mathbf{x}^\top [A_i^\top P + P A_i] \mathbf{x} < 0, \\ \forall \mathbf{x} > \mathbf{0}, \quad \forall i \in [1, M], \end{gathered}$$

[1]Consistently with [1], [26], we say that a set $\mathcal{M}$ is *weakly invariant* with respect to the ith mode of (3) if for every $\mathbf{x} \in \mathcal{M}$ there exists a real number $b > 0$ such that the solution of $\dot{\mathbf{x}}(t) = A_i \mathbf{x}(t)$ corresponding to the initial condition $\mathbf{x}(0) = \mathbf{x}$ is such that $\mathbf{x}(t) \in \mathcal{M}$ either for every $t \in [0, b]$ or for every $t \in [-b, 0]$. A compact set $\mathcal{M}$ is *weakly invariant* with respect to (3) if for every $\mathbf{x} \in \mathcal{M}$ there exists an index $i \in [1, M]$ such that $\mathcal{M}$ is weakly invariant with respect to the ith mode of (3).

namely the (symmetric) matrices $Q_i := -[A_i^\top P + PA_i], i \in [1, M]$, are quadratic copositive.

It is worth noticing that the function $V(\mathbf{x}) = \mathbf{1}_n^\top \mathbf{x}$ that we have used in the proof of Proposition 1 is a CLCLF for the CCSS but in a weak sense, since condition (4) holds with $<$ replaced by $\leq$.

First of all, we investigate the relationship between Linear Copositive and Quadratic Positive Definite Common Lyapunov functions. As a starting point, we have the following preliminary result, inherited from the general class of positive switched systems (see [4]).

Proposition 2. *Given a CCSS described as in (3), the following facts are equivalent:*

(i) *For every choice of M nonnegative diagonal matrices D_i, $i \in [1, M]$, with $\sum_{i=1}^M D_i = I_n$, the matrix $\sum_{i=1}^M A_i D_i$ is Metzler Hurwitz;*

(ii) *the CCSS admits a CLCLF.*

If any of the previous equivalent conditions holds, then

(iii) *the CCSS admits a CQPDLF.*

If (iii) holds, then

(iv) *the CCSS is asymptotically stable under arbitrary switching.*

For general positive switched systems, none of the latest implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ can be reversed. Under the compartmental assumption, it is still true that (iii) does not imply (ii), as the following example shows.

Example 1. *Consider the CCSS (3) with $M = 2$ and*

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}.$$

As $Q_i := -[A_i^\top + A_i] = -2A_i \succ 0, i = 1, 2$, condition (iii) holds for $P = I_2$. However, the CCSS does not admit a CLCLF, since for every $\mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^\top \gg \mathbf{0}$, we have

$$\begin{array}{rcl} \mathbf{v}^\top A_1 & = & \begin{bmatrix} -v_1 + v_2 & \star \end{bmatrix} \ll \mathbf{0}^\top \implies v_2 < v_1; \\ \mathbf{v}^\top A_2 & = & \begin{bmatrix} \star & v_1 - v_2 \end{bmatrix} \ll \mathbf{0}^\top \implies v_1 < v_2. \end{array}$$

At the current stage of our research, it is not clear whether condition (iv) is equivalent to condition (iii) or not, since we have not been able to find either a counterexample or a proof. In the next section we will show that under certain conditions on the matrices of the CCSS, condition (iv) implies (iii), namely the Hurwitz property of all the compartmental matrices $A_i, i \in [1, M]$, guarantees the existence of a CQPDLF for the associated CCSS.

III. Converse results about CQPDLFs

If the compartmental matrices $A_i, i \in [1, M]$, are Hurwitz, they all satisfy condition $\mathbf{1}_n^\top A_i < \mathbf{0}^\top$, and hence $V(\mathbf{x}) = \mathbf{x}^\top \mathbf{1}_n \mathbf{1}_n^\top \mathbf{x}$ represents a *Weak Common Quadratic Copositive Lyapunov Function* for the CCSS, since for every $\mathbf{x} > 0$ and every $i \in [1, M]$:

$$\begin{array}{rcl} V(\mathbf{x}) & = & \mathbf{x}^\top \mathbf{1}_n \mathbf{1}_n^\top \mathbf{x} > 0, \\ \dot{V}_i(\mathbf{x}) & = & -\mathbf{x}^\top [A_i^\top \mathbf{1}_n \mathbf{1}_n^\top + \mathbf{1}_n \mathbf{1}_n^\top A_i] \mathbf{x} \leq 0. \end{array}$$

However, in general this is not a CQPDLF, since there exist indices $i \in [1, M]$ and vectors $\mathbf{x} > 0$ such that $\dot{V}_i(\mathbf{x}) = 0$. In order to explore under what conditions the Hurwitz stability of the matrices $A_i, i \in [1, M]$, allows to construct a CQPDLF, we focus our attention on the class of positive definite Lyapunov functions described as $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ with

$$P = P^\top = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon D \succ 0, \tag{5}$$

where $\varepsilon > 0$ and $D \in \mathbb{R}_+^{n \times n}$ is a diagonal matrix with positive diagonal entries.

Before proceeding let us state a preliminary lemma that will be used in the proofs of the following propositions.

Lemma 1. *Let $M = M^\top \in \mathbb{R}^{n \times n}$ have the following block structure:*

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$$

with $A = A^\top \in \mathbb{R}^{k \times k}, B \in \mathbb{R}^{k \times (n-k)}, C = C^\top \in \mathbb{R}^{(n-k) \times (n-k)}$. If $B \geq 0$ and both A and C are copositive, then M is copositive.

Proof. For every nonzero vector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top \end{bmatrix}^\top$, $\mathbf{x}_1 \in \mathbb{R}_+^k$, $\mathbf{x}_2 \in \mathbb{R}_+^{n-k}$, one has

$$\begin{aligned} &\begin{bmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top \end{bmatrix} \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \\ &= \mathbf{x}_1^\top A \mathbf{x}_1 + \mathbf{x}_2^\top B^\top \mathbf{x}_1 + \mathbf{x}_2^\top C \mathbf{x}_2 + \mathbf{x}_1^\top B \mathbf{x}_2 \\ &\geq \mathbf{x}_1^\top A \mathbf{x}_1 + \mathbf{x}_2^\top C \mathbf{x}_2 > 0, \end{aligned}$$

where the last inequality follows from the properties of A and C, and fact that either $\mathbf{x}_1$ or $\mathbf{x}_2$ (or both) are nonzero. □

We can now provide a characterization of CCSSs admitting a Common Quadratic Positive Definite Lyapunov function of type (5). We first consider a single compartmental Hurwitz matrix in Lemma 2 and then we generalize this result to the case of a CCSS described as in (3).

Lemma 2. *Let $A \in \mathbb{R}^{n \times n}$ be a compartmental Hurwitz matrix s.t.*

$$\mathbf{1}_n^\top A = \mathbf{1}_n^\top \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_k^\top & -\mathbf{v}^\top \end{bmatrix}, \quad \mathbf{v} \gg \mathbf{0},$$

with $A_{11} \in \mathbb{R}^{k \times k}$ and $\mathbf{v} \in \mathbb{R}_+^{n-k}$. The following facts are equivalent:

(i) *$\exists\ \bar{\varepsilon} > 0$ s.t., for every $0 < \varepsilon < \bar{\varepsilon}$, the matrix $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov Function for A;*

(ii) *the matrix $-(A_{11} + A_{11}^\top)$ is copositive.*

Proof. Set $P := \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$, with $\varepsilon > 0$, and notice that the matrix $Q := -(A^\top P + PA)$ takes the following form:

$$\begin{aligned} Q &= -(A^\top P + PA) \\ &= \begin{bmatrix} \mathbf{0}_{k \times k} & \mathbf{0}_{k \times (n-k)} \\ \mathbf{v} \mathbf{1}_k^\top & \mathbf{v} \mathbf{1}_{n-k}^\top \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{k \times k} & \mathbf{1}_k \mathbf{v}^\top \\ \mathbf{0}_{(n-k) \times k} & \mathbf{1}_{n-k} \mathbf{v}^\top \end{bmatrix} \\ &\quad -\varepsilon \begin{bmatrix} A_{11} + A_{11}^\top & A_{12} + A_{21}^\top \\ A_{12}^\top + A_{21} & A_{22} + A_{22}^\top \end{bmatrix} \\ &= \begin{bmatrix} -\varepsilon(A_{11} + A_{11}^\top) & \mathbf{1}_k \mathbf{v}^\top - \varepsilon(A_{12} + A_{21}^\top) \\ \mathbf{v} \mathbf{1}_k^\top - \varepsilon(A_{12}^\top + A_{21}) & (\mathbf{v} \mathbf{1}_{n-k}^\top + \mathbf{1}_{n-k} \mathbf{v}^\top) - \varepsilon(A_{22} + A_{22}^\top) \end{bmatrix}. \end{aligned}$$

(i) $\Rightarrow$ (ii) If there exists $\varepsilon > 0$ s.t. Q is copositive, then for every positive vector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^\top & \mathbf{0}_{n-k}^\top \end{bmatrix}^\top$ it holds:

$$0 < -\mathbf{x}^\top(A^\top P + PA)\mathbf{x} = -\varepsilon \mathbf{x}_1^\top(A_{11} + A_{11}^\top)\mathbf{x}_1, \quad \forall \mathbf{x}_1 > \mathbf{0},$$

and hence the matrix $-(A_{11} + A_{11}^\top)$ is copositive.
(ii) $\Rightarrow$ (i) Assume that $-(A_{11} + A_{11}^\top)$ is copositive and notice that there always exists $\bar\varepsilon > 0$ such that for every $\varepsilon \in (0, \bar\varepsilon)$

$$\begin{aligned} \mathbf{1}_k \mathbf{v}^\top - \varepsilon(A_{12} + A_{21}^\top) &\geq 0 \\ (\mathbf{v}\mathbf{1}_{n-k}^\top + \mathbf{1}_{n-k}\mathbf{v}^\top) - \varepsilon(A_{22} + A_{22}^\top) &\gg 0. \end{aligned}$$

Then, recalling that a symmetric strictly positive matrix is also a copositive matrix, by Lemma 1, for every such ε, matrix Q is copositive and hence $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov Function for A. $\square$

Remark 1. *If the compartmental matrix $A \in \mathbb{R}^{n\times n}$ is such that $\mathbf{1}_n^\top A \ll \mathbf{0}^\top$, then for every $\varepsilon > 0$ sufficiently small the matrix $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov function for A since $Q = -(A^\top P + PA)$ is a strictly positive matrix. This immediately follows from the proof of the previous proposition in the case $k = 0$.*

Proposition 3. *Consider the CCSS (3), and assume that $A_i \in \mathbb{R}^{n\times n}$, $i \in [1, M]$, are compartmental Hurwitz matrices. Define the following sets:*

$$\mathcal{J}_i := \{j \in [1, n] \colon \mathbf{1}_n^\top col_j(A_i) = 0\}, \quad i \in [1, M], \tag{6}$$

and, if $\mathcal{J}_i \neq \emptyset$, denote by $A_{\mathcal{J}_i}$ the submatrix of A_i obtained by selecting rows and columns of A_i indexed by $\mathcal{J}_i$. The following facts are equivalent:

- (i) *$\exists\, \bar\varepsilon > 0$ s.t., for all $0 < \varepsilon < \bar\varepsilon$, the matrix $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a CQPDLF for the CCSS;*
- (ii) *for every $i \in [1, M]$ with $\mathcal{J}_i \neq \emptyset$ the matrix $-\left(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i}\right)$ is copositive.*

Proof. If $i \in [1, M]$ is such that $\mathcal{J}_i = \emptyset$, then $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov Function for A_i for every $\varepsilon > 0$ sufficiently small (see Remark 1). Otherwise, let Π_i be a permutation matrix such that:

$$\mathbf{1}_n^\top \underbrace{\Pi_i^\top A_i \Pi_i}_{\tilde A_i} = \mathbf{1}_n^\top \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{k_i}^\top & -\mathbf{v}_i^\top \end{bmatrix},$$

with $\mathbf{v}_i \in \mathbb{R}_+^{n-k_i}$ and $\mathbf{v}_i \gg \mathbf{0}$, and notice that $A_{11}^{(i)} = A_{\mathcal{J}_i}$. Moreover notice that $P = P^\top \succ 0$ defines a QPDLF for A_i, i.e. $-(A_i^\top P + PA_i)$ is copositive, if and only if the matrix

$$-\Pi_i^\top(A_i^\top P + PA_i)\Pi_i$$

is copositive. This, in turn amounts to saying that the matrix $-[(\Pi_i^\top A_i^\top \Pi_i)(\Pi_i^\top P \Pi_i) + (\Pi_i^\top P \Pi_i) \cdot (\Pi_i^\top A_i \Pi_i)] = -(\tilde A_i^\top P + P\tilde A_i)$ is copositive, and by Lemma 2 such condition holds for every $\varepsilon > 0$ sufficiently small if and only if the matrix $-(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i})$ is copositive. Hence, for every $\varepsilon > 0$ sufficiently small the matrix $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a CQPDLF for the CCSS if and only if for every $i \in [1, M]$, with $\mathcal{J}_i \neq \emptyset$, the submatrix $-(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i})$ is copositive. $\square$

Corollary 1. *Consider the CCSS (3), and assume that $A_i \in \mathbb{R}^{n\times n}$, $i \in [1, M]$, are compartmental Hurwitz matrices. If for every $i \in [1, M]$ the matrix A_i is s.t. the vector $\mathbf{1}_n^\top A_i < \mathbf{0}^\top$ has at most one entry equal to 0, then $\exists\ \varepsilon > 0$ such that $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a CQPDLF for the CCSS.*

Proof. If A_i is such that $\mathbf{1}_n^\top A_i \ll \mathbf{0}^\top$, then $\forall \varepsilon > 0$ sufficiently small the matrix $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov Function for A_i (see Remark 1). If the vector $\mathbf{1}_n^\top A_i < \mathbf{0}^\top$ has exactly one entry, say the jth one, equal to 0, then $A_{\mathcal{J}_i} = [A_i]_{jj}$. Since A_i is Metzler Hurwitz, it must be $[A_i]_{jj} < 0$, but then $-\left(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i}\right)$ is copositive and the thesis follows directly from Proposition 3. $\square$

Remark 2. *In the particular case where $A_i \in \mathbb{R}^{2\times 2}$, $i \in [1, M]$, the previous corollary implies that there always exists $\varepsilon > 0$ such that $P = \mathbf{1}_2 \mathbf{1}_2^\top + \varepsilon I_2$ defines a CQPDLF for the CCSS with subsystem matrices $A_1, \dots, A_M$. Indeed, since each A_i is compartmental and Hurwitz, condition $\mathbf{1}_2^\top A_i \neq \mathbf{0}^\top$ ensures that $\mathbf{1}_2^\top A_i$ has either zero or 1 entries equal to 0. This proves that when dealing with two-dimensional CCSSs, conditions (iii) and (iv) in Proposition 2 are equivalent.*

We now explore a slightly bigger class of CQPDLFs with respect to those addressed in the previous results, since we replace the identity matrix in P with a diagonal matrix.

Proposition 4. *Consider the CCSS (3), and assume that $A_i \in \mathbb{R}^{n\times n}$, $i \in [1, M]$, are compartmental Hurwitz matrices. Define the sets $\mathcal{J}_i$, $i \in [1, M]$, as in (6). If $\mathcal{J}_i \cap \mathcal{J}_j = \emptyset$ for every $i \neq j$, then there exists $\varepsilon > 0$ and a diagonal matrix D, with positive diagonal entries, such that the matrix $P := \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon D$ defines a CQPDLF for the CCSS.*

Proof. Let $i \in [1, M]$ be such that $\mathcal{J}_i \neq \emptyset$. Set $k_i := |\mathcal{J}_i|$. It entails no loss of generality assuming that $\mathcal{J}_i$ is an ordered k_i-tuple, with entries sorted in ascending order. Since by Lemma 6 in the Appendix the submatrix $A_{\mathcal{J}_i}$ is compartmental and Hurwitz, there always exists [18] a diagonal matrix $D_{\mathcal{J}_i} \in \mathbb{R}^{k_i \times k_i}$, with positive diagonal entries, such that

$$-\left[A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} + D_{\mathcal{J}_i} A_{\mathcal{J}_i}\right] \succ 0, \tag{7}$$

and hence, in particular, $-(A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} + D_{\mathcal{J}_i} A_{\mathcal{J}_i})$ is copositive. Define the following positive diagonal matrix $D \in \mathbb{R}^{n\times n}$:

$$[D]_{jj} = \begin{cases} 1, & \text{if } \nexists i \in [1, M] \text{ s.t. } j \in \mathcal{J}_i; \\ \left[D_{\mathcal{J}_i}\right]_{kk}, & \text{if } j \in \mathcal{J}_i \text{ and } j \text{ is the } k\text{th entry of } \mathcal{J}_i. \end{cases}$$

Now we show that there always exists $\varepsilon > 0$ such that $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon D$ defines a CQPDLF for the CCSS. Again, as in the proof of Lemma 6, we can assume w.l.o.g. that $\mathcal{J}_i = [1, r]$, $r \in [1, n]$, and hence:

$$\mathbf{1}_n^\top A_i = \mathbf{1}_n^\top \begin{bmatrix} A_{\mathcal{J}_i} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{k_i}^\top & -\mathbf{v}_i^\top \end{bmatrix},$$

for some $\mathbf{v}_i \in \mathbb{R}_+^{n-k_i}$, $\mathbf{v}_i \gg \mathbf{0}$, and

$$D = \begin{bmatrix} D_{\mathcal{J}_i} & \\ & \bar D_i \end{bmatrix}, \quad \bar D_i \in \mathbb{R}_+^{(n-k_i)\times(n-k_i)}.$$

The matrix $Q_i := -(A_i^\top P + PA_i)$ takes the form in (8).

$$
\begin{aligned}
Q_i &= -(A_i^\top P + P A_i)\\
&= -\left(\begin{bmatrix} \mathbf{0}_{k_i\times k_i} & \mathbf{0}_{k_i\times(n-k_i)}\\ -\mathbf{v}_i\mathbf{1}_{k_i}^\top & -\mathbf{v}_i\mathbf{1}_{n-k_i}^\top\end{bmatrix} + \begin{bmatrix} \mathbf{0}_{k_i\times k_i} & -\mathbf{1}_{k_i}\mathbf{v}_i^\top\\ \mathbf{0}_{(n-k_i)\times k_i} & -\mathbf{1}_{n-k_i}\mathbf{v}_i^\top\end{bmatrix} + \varepsilon\begin{bmatrix} A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} & (A_{21}^{(i)})^\top \bar{D}_i\\ (A_{12}^{(i)})^\top D_{\mathcal{J}_i} & (A_{22}^{(i)})^\top \bar{D}_i\end{bmatrix} + \begin{bmatrix} D_{\mathcal{J}_i}A_{\mathcal{J}_i} & D_{\mathcal{J}_i}A_{12}^{(i)}\\ \bar{D}_iA_{21}^{(i)} & \bar{D}_iA_{22}^{(i)}\end{bmatrix}\right)\\
&= \begin{bmatrix} -\varepsilon\left(A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} + D_{\mathcal{J}_i}A_{\mathcal{J}_i}\right) & -\varepsilon\left((A_{21}^{(i)})^\top\bar{D}_i + D_{\mathcal{J}_i}A_{12}^{(i)}\right) + \mathbf{1}_{k_i}\mathbf{v}_i^\top\\ \mathbf{v}_i\mathbf{1}_{k_i}^\top - \varepsilon\left((A_{12}^{(i)})^\top D_{\mathcal{J}_i} + \bar{D}_iA_{21}^{(i)}\right) & \mathbf{v}_i\mathbf{1}_{n-k_i}^\top + \mathbf{1}_{n-k_i}\mathbf{v}_i^\top - \varepsilon\left((A_{22}^{(i)})^\top\bar{D}_i + \bar{D}_iA_{22}^{(i)}\right)\end{bmatrix}.
\end{aligned}
\tag{8}
$$

Notice that there always exists $\bar{\varepsilon} > 0$ s.t. the following conditions are satisfied for every $\varepsilon \in (0, \bar{\varepsilon})$:

$$
\begin{aligned}
-\varepsilon\left((A_{21}^{(i)})^\top\bar{D}_i + D_{\mathcal{J}_i}A_{12}^{(i)}\right) + \mathbf{1}_{k_i}\mathbf{v}_i^\top &\geq 0;\\
\mathbf{v}_i\mathbf{1}_{n-k_i}^\top + \mathbf{1}_{n-k_i}\mathbf{v}_i^\top - \varepsilon\left((A_{22}^{(i)})^\top\bar{D}_i + \bar{D}_iA_{22}^{(i)}\right) &\gg 0.
\end{aligned}
$$

Since, by assumption, condition (7) holds, by making use of Lemma 1 we can claim that for every $\varepsilon > 0$ sufficiently small the matrix Q_i is copositive, and hence $P = \mathbf{1}_n\mathbf{1}_n^\top + \varepsilon D$ defines a Common Quadratic Positive Definite Lyapunov Function for A_i for every $i \in [1, M]$, such that $\mathcal{J}_i \neq \emptyset$. On the other hand, if $i \in [1, M]$ is such that $\mathcal{J}_i = \emptyset$, then $k_i = 0$ and

$$
Q_i = \mathbf{v}_i\mathbf{1}_n^\top + \mathbf{1}_n\mathbf{v}_i^\top - \varepsilon(A_i^\top D + DA_i) \gg 0
$$

for sufficiently small $\varepsilon > 0$. So, the result is proved. □

IV. Stability under dwell-time switching

We have seen in Section II that if all matrices A_i, $i \in [1, M]$, are compartmental and Hurwitz, then for every initial condition $\mathbf{x}(0) > \mathbf{0}$ and for every switching function $\sigma\colon \mathbb{R}_+ \to [1, M]$ the state trajectory of the CCSS asymptotically converges to $\mathbf{0}$. In this section we relax the assumption that all the subsystem matrices are Hurwitz, and investigate which switching functions drive to zero the state trajectory independently of the initial condition.

To this purpose, we introduce the following ergodicity condition [8], [26], [44], [49]) (a sort of dwell-time condition) on the switching functions $\sigma(\cdot)$ we will consider:

Property 1. *There exists a real number $\tau_\sigma > 0$ and a subset $\Omega_\sigma \subseteq [1, M]$ such that for every $\bar{t} \geq 0$ and for every $p \in \Omega_\sigma$, consecutive switching instants $t_{k+1} > t_k \geq \bar{t}$ can be found, satisfying the following conditions:*

a) $t_{k+1} - t_k \geq \tau_\sigma$;
b) $\forall t \in [t_k, t_{k+1}), \sigma(t) = p$.

Notice that Property 1 amounts to requiring that the switching function $\sigma(\cdot)$ admits a set $\Omega_\sigma \subseteq [1, M]$ of *persistent modes* (see Definition 2.1 in [26]).

Proposition 5. *Consider a CCSS described as in (3). Define the sets:*

$$
\mathcal{J}_i := \{j \in [1, n] \colon \mathbf{1}_n^\top col_j(A_i) = 0\}, \quad i \in [1, M].
$$

Then, $\forall\sigma(\cdot)$ satisfying Property 1 for a set Ω_σ such that $\cap_{i\in\Omega_\sigma}\mathcal{J}_i = \emptyset$, the state trajectory $\mathbf{x}(t)$, $t \in \mathbb{Z}_+$, asymptotically converges to $\mathbf{0}$, $\forall\mathbf{x}(0) > \mathbf{0}$.

Proof. As remarked in Proposition 1, $V(\mathbf{x}(t)) = \mathbf{1}_n^\top\mathbf{x}(t)$ is a Common Weak Lyapunov function. Let $\sigma(\cdot)$ be an arbitrary switching function satisfying Property 1 for a set Ω_σ such that $\cap_{i\in\Omega_\sigma}\mathcal{J}_i = \emptyset$ and let $p \in \Omega_\sigma$ be any (persistent) mode of $\sigma(\cdot)$. Denote by $\mathcal{M}_p$ the largest weakly invariant set with respect to the pth mode in $\mathcal{N}_p := \{\mathbf{x} \in \mathbb{R}_+^n \colon \mathbf{1}_n^\top A_p\mathbf{x} = 0\}$ and notice that $\mathcal{M}_p \subseteq \text{Cone}(\mathbf{e}_j, j \in \mathcal{J}_p)$. Then, by Theorem 4.1 in [26], the state trajectory $\mathbf{x}(t)$ weakly approaches $\mathcal{M}_p^* := \mathcal{M}_p \cap V^{-1}(c)$, for some c, in the pth mode as $t \to +\infty$, meaning that:

$$
\lim_{\substack{t\to+\infty\\ t\in W_p^\sigma}} \text{dist}(\mathbf{x}(t), \mathcal{M}_p^*) = 0, \tag{9}
$$

where W_p^σ is the union of all the intervals $[t_k, t_{k+1})$ of length at least τ_σ and such that $\sigma(t) = p$ for every $t \in [t_k, t_{k+1})$. Now, if $n_p := |\mathcal{J}_p|$ and $n_p^c := n - n_p = |\mathcal{J}_p^c|$, $\mathcal{J}_p^c$ being the complementary set of $\mathcal{J}_p$ in $[1, n]$, denote by $\mathbf{x}_{\mathcal{J}_p} \in \mathbb{R}_+^{n_p}$ and $\mathbf{x}_{\mathcal{J}_p^c} \in \mathbb{R}_+^{n_p^c}$ the vectors formed by selecting the components of $\mathbf{x}$ indexed by the sets $\mathcal{J}_p$ and $\mathcal{J}_p^c$, respectively. By definition of $\mathcal{M}_p$, for every $\mathbf{x} \in \mathcal{M}_p^*$ we have $\mathbf{x}_{\mathcal{J}_p^c} = \mathbf{0}$, and hence equation (9) implies that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $t \geq t_\varepsilon$ with $\sigma(t) = p$ we have $\mathbf{1}_{n_p^c}^\top\mathbf{x}_{\mathcal{J}_p^c}(t) < \varepsilon$. But then, since for every $t \geq 0$ such that $\sigma(t) \neq p$ the function $V_p(\mathbf{x}(t)) = \mathbf{1}_{n_p^c}^\top\mathbf{x}_{\mathcal{J}_p^c}(t)$ is not increasing, it must be:

$$
\lim_{t\to+\infty} \mathbf{1}_{n_p^c}^\top\mathbf{x}_{\mathcal{J}_p^c}(t) = 0, \tag{10}
$$

and hence $\mathbf{x}_{\mathcal{J}_p^c}(t) \to \mathbf{0}$ as $t \to +\infty$. Now, since this is true for every $p \in \Omega_\sigma$, the thesis follows from the fact that by hypothesis $\cap_{i\in\Omega_\sigma}\mathcal{J}_i = \emptyset$, i.e. $(\cap_{i\in\Omega_\sigma}\mathcal{J}_i)^c = [1, n]$, and hence by the De Morgan's law $\cup_{i\in\Omega_\sigma}\mathcal{J}_i^c = [1, n]$. □

Example 2. *Consider the matrices:*

$$
A_1 = \begin{bmatrix} 0 & 0\\ 0 & -1\end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0\\ 0 & 0\end{bmatrix}
$$

and note that $\mathcal{J}_1 = \{1\}$ and $\mathcal{J}_2 = \{2\}$. If $\bar{\sigma}(\cdot)$ is a switching function with a finite number of switchings, i.e. there exist $\bar{t} \geq 0$ and $p \in \{1, 2\}$ such that $\bar{\sigma}(t) = p$ for every $t \geq \bar{t}$, then Property 1 holds for $\Omega_\sigma = \{p\}$, but then $\cap_{i\in\Omega_\sigma}\mathcal{J}_i = \mathcal{J}_p = \{p\} \neq \emptyset$. Indeed, if we consider the state trajectory starting from any initial condition $\mathbf{x}(0) \gg \mathbf{0}$, clearly $[\mathbf{x}(t)]_p = [\mathbf{x}(\bar{t})]_p > 0$ for every $t \geq \bar{t}$, and hence the state trajectory cannot converge to zero, as shown in Figure 1, on the left.

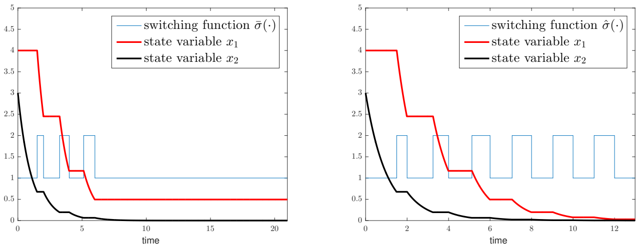

Figure 1. State trajectories corresponding to a switching function $\bar{\sigma}(\cdot)$ with a finite number of switchings (left) and to the switching function $\hat{\sigma}(\cdot)$ (right).

Now consider the switching function:

$$\hat{\sigma}(t) = \begin{cases} 1, & \text{if } t \in \left[2k, 2k+1+\frac{1}{2^{k+1}}\right),\ k = 0,1,\dots \\ 2, & \text{elsewhere.} \end{cases}$$

and notice that in this case $\hat{\sigma}(\cdot)$ satisfies Property 1 for the set $\Omega_\sigma = \{1,2\}$ and $\cap_{i\in\Omega_\sigma}\mathcal{J}_i = \emptyset$. Therefore, by Proposition 5, the state trajectory corresponding to any positive initial condition converges to the origin (see Figure 1, on the right).

In the special case when the set

$$\mathcal{I}_{AS} := \{i \in [1,M] : A_i \text{ is Hurwitz}\}$$

is not empty, we can derive a similar result to the one given in Proposition 5, by considering all the switching functions $\sigma(\cdot)$ for which the set Ω_σ intersects $\mathcal{I}_{AS}$.

Proposition 6. *Consider a CCSS described as in (3), for which the index set $\mathcal{I}_{AS}$ is not empty. Then, for every switching $\sigma(\cdot)$ satisfying Property 1 for a set $\Omega_\sigma \subseteq [1,M]$, such that $\Omega_\sigma \cap \mathcal{I}_{AS} \neq \emptyset$, the state trajectory $\mathbf{x}(t)$, $t \in \mathbb{Z}_+$, asymptotically converges to $\mathbf{0}$, $\forall\, \mathbf{x}(0) > 0$.*

Proof. The proof proceeds as the proof of Proposition 5, but in this case we consider $p \in \Omega_\sigma \cap \mathcal{I}_{AS}$ as a persistent mode of $\sigma(\cdot)$ and hence we can claim that $\mathcal{M}_p = \{\mathbf{0}\}$. Then, by Theorem 4.1 in [26], the state trajectory $\mathbf{x}(t)$ weakly approaches $\mathcal{M}_p$ in the pth mode as $t \to +\infty$, meaning that $\lim_{\substack{t\to+\infty \\ t\in W_p^\sigma}} \mathrm{dist}(\mathbf{x}(t), \mathcal{M}_p) = 0$, where W_p^σ is the union of all the intervals $[t_k, t_{k+1})$ of length at least τ_σ and such that $\sigma(t) = p$ for every $t \in [t_k, t_{k+1})$. This implies that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $t \geq t_\varepsilon$ for which $\sigma(t) = p$ we have $\mathbf{1}_n^\top \mathbf{x}(t) < \varepsilon$. But then, since for every $t \geq 0$ such that $\sigma(t) \neq p$ the function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is not increasing, it must be $\lim_{t\to+\infty} \mathbf{1}_n^\top \mathbf{x}(t) = 0$, and hence $\mathbf{x}(t) \to \mathbf{0}$ as $t \to +\infty$. ☐

Now we consider switching functions having a common dwell-time $\tau^* > 0$, by this meaning that for every $\sigma(\cdot)$ and every pair of consecutive switching instants t_k and t_{k+1} we have $t_{k+1} - t_k \geq \tau^*$ (this definition of dwell-time is consistent with the one introduced in [4], [37], [47]).

Corollary 2 below follows directly from Proposition 6.

Corollary 2. *Consider the CCSS (3) and consider any set of switching functions $\mathcal{S}_{dwell,AS}$ satisfying the following two conditions:*

- *there exists $\tau^* > 0$ such that all the switching functions have dwell-time τ^*;*
- *for every $\sigma(\cdot) \in \mathcal{S}_{dwell,AS}$ and every $\bar{t} \geq 0$ it holds true that $\mu(\{t \geq \bar{t} \colon \sigma(t) \in \mathcal{I}_{AS}\}) \neq 0$.*

Then, $\forall \sigma \in \mathcal{S}_{dwell,AS}$ and $\forall \mathbf{x}(0) > \mathbf{0}$, the state trajectories of the CCSS converge to $\mathbf{0}$.

V. Stabilizability

Definition 4. *[12] The CCSS (3) is* stabilizable *if $\forall\, \mathbf{x}(0) > \mathbf{0}$ there exists a switching function $\sigma : \mathbb{R}_+ \to [1,M]$ s.t. the corresponding state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.*

Remark 3. *Clearly, the stabilization problem is a non-trivial one only if all matrices $A_i, i \in [1,M]$, are not Hurwitz, and hence in the following we steadily make this assumption. On the other hand, if all matrices $A_i, i \in [1,M]$, would fulfill condition $\mathbf{1}_n^\top A_i = \mathbf{0}^\top$, stabilization would not be possible, since at every time $t \geq 0$ one would have $\mathbf{1}_n^\top \mathbf{x}(t) = \mathbf{1}_n^\top \mathbf{x}(0)$, and the state would never converge to zero. So, it must be $\mathbf{1}_n^\top A_i < 0$ for at least one index $i \in [1,M]$.*

A. Existence of a Hurwitz convex combination

It is a well-known result that the existence of a Hurwitz convex combination of the subsystem matrices $A_i, i \in [1,M]$, is a sufficient condition for the stabilizability of the switched system (3) even without any positivity assumption, namely even when the matrices A_i are arbitrary $n \times n$ real matrices [45]. In recent times, it has been proved [2] that, when dealing with two-dimensional continuous-time positive switched systems, the existence of a Hurwitz convex combination of the system matrices is equivalent to stabilizability, but this is no longer the case when dealing with positive switched systems of arbitrary dimension n [3].

In the following we will show that the compartmental property of the system matrices makes the difference, and hence stabilizability of a CCSS is equivalent to the existence of a Hurwitz convex combination of the matrices $A_i, i \in [1,M]$. To this end, we first consider, in Lemma 3, the case in which the matrix sum $\sum_{i=1}^M A_i$ is irreducible and then we remove this hypothesis in Lemma 4.

Lemma 3. *Consider the CCSS (3) and assume that the matrix sum $A := \sum_{i=1}^M A_i$ of the compartmental matrices $A_i \in \mathbb{R}^{n\times n}$, $i \in [1,M]$, is irreducible. If the CCSS is stabilizable, then there exists a Hurwitz convex combination of $A_1, \dots, A_M$, i.e. $\exists\, \alpha_i \geq 0, i \in [1,M]$, with $\sum_{i=1}^M \alpha_i = 1$, such that $\sum_{i=1}^M \alpha_i A_i$ is Hurwitz.*

Proof. Suppose by contradiction that for every choice of $\alpha_i \geq 0$, $i \in [1,M]$, with $\sum_{i=1}^M \alpha_i = 1$, the matrix $\sum_{i=1}^M \alpha_i A_i$ is not Hurwitz. Then, in particular, this is the case if we impose that all coefficients α_i are positive. So, assume $\bar{\alpha}_i > 0$, $i \in [1,M]$, with $\sum_{i=1}^M \bar{\alpha}_i = 1$, and consider the compartmental matrix $\bar{A} := \sum_{i=1}^M \bar{\alpha}_i A_i$. By assumption, $\bar{A}$ is irreducible, too, and not Hurwitz. By Lemma 5 in the Appendix this implies:

$$\mathbf{0}^\top = \mathbf{1}_n^\top \left(\sum_{i=1}^M \bar{\alpha}_i A_i\right) = \sum_{i=1}^M \bar{\alpha}_i (\mathbf{1}_n^\top A_i),$$

and hence $\mathbf{1}_n^\top A_i = \mathbf{0}^\top$ for all $i \in [1, M]$. But this contradicts (see Remark 3) the stabilizability assumption. □

Lemma 4. *If the CCSS (3) is stabilizable, then there exists a Hurwitz convex combination of $A_1, \ldots, A_M$.*

Proof. If the matrix $\sum_{i=1}^M A_i$ is irreducible, then the statement follows from Lemma 3. If $\sum_{i=1}^M A_i$ is reducible, there exists a permutation matrix Π such that:

$$\begin{aligned} B &:= \Pi^\top \left(\sum_{i=1}^M A_i\right) \Pi = \sum_{i=1}^M \underbrace{\Pi^\top A_i \Pi}_{B^{(i)}} \\ &= \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1\ell} \\ 0 & B_{22} & \ldots & B_{2\ell} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & B_{\ell\ell} \end{bmatrix}, \end{aligned}$$

where $B_{jj} \in \mathbb{R}^{n_j \times n_j}$, $j \in [1, \ell]$, are irreducible matrices. Notice that, accordingly, each $B^{(i)}$, $i \in [1, M]$, takes the following form:

$$B^{(i)} := \Pi^\top A_i \Pi = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} & \ldots & A_{1\ell}^{(i)} \\ 0 & A_{22}^{(i)} & \ldots & A_{2\ell}^{(i)} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & A_{\ell\ell}^{(i)} \end{bmatrix}.$$

We want to prove that, under the stabilizability assumption, B is a Hurwitz matrix. To this aim, suppose by contradiction that B is not Hurwitz and let $k \in [1, \ell]$ be such that B_{kk} is (irreducible) compartmental and not Hurwitz. By Lemma 5 in the Appendix it must be $\mathbf{1}_{n_k}^\top B_{kk} = \mathbf{0}^\top$ and hence $\sum_{i=1}^M \mathbf{1}_{n_k}^\top A_{kk}^{(i)} = \mathbf{0}^\top$. Since all matrices $A_{jj}^{(i)}$, $j \in [1, \ell]$, $i \in [1, M]$, are compartmental, the previous identity implies:

$$\mathbf{1}_{n_k}^\top A_{kk}^{(i)} = \mathbf{0}^\top, \quad \forall i \in [1, M]. \tag{11}$$

Now consider the initial condition $\bar{\mathbf{x}}(0)$ (w.r.t. to the new coordinate system, namely $\bar{\mathbf{x}}(0) = \Pi^\top \mathbf{x}(0)$) whose kth block is $\mathbf{1}_{n_k}$, while all the other blocks are zero, i.e.

$$\bar{\mathbf{x}}(0) = \begin{bmatrix} \mathbf{0}^\top & \ldots & \mathbf{1}_{n_k}^\top & \ldots & \mathbf{0}^\top \end{bmatrix}^\top.$$

We want to show that, independently of the switching function $\sigma(\cdot)$, the corresponding state trajectory $\bar{\mathbf{x}}(t), t \geq 0$, cannot converge to $\mathbf{0}$. Set $\bar{\mathbf{x}}_k(t) := block_k[\bar{\mathbf{x}}(t)]$, and notice that its time evolution is described by the equation $\dot{\bar{\mathbf{x}}}_k(t) = A_{kk}^{\sigma(t)} \bar{\mathbf{x}}_k(t)$. By condition (11), for every $t > 0$ it holds:

$$0 = \int_0^t \mathbf{1}_{n_k}^\top \dot{\bar{\mathbf{x}}}_k(\tau) d\tau = \mathbf{1}_{n_k}^\top \bar{\mathbf{x}}_k(t) - \mathbf{1}_{n_k}^\top \bar{\mathbf{x}}_k(0) = \mathbf{1}_{n_k}^\top \bar{\mathbf{x}}_k(t) - n_k,$$

where n_k is the dimension of B_{kk}. Hence, for every switching function $\sigma(\cdot)$, we have: $\mathbf{1}_{n_k}^\top \bar{\mathbf{x}}_k(t) = n_k, \forall t \geq 0$, that contradicts stabilizability. Therefore, B must be Hurwitz and, by similarity, also $\sum_{i=1}^M A_i$ is Hurwitz. Hence, the positive convex combination $\sum_{i=1}^M \frac{1}{M} A_i$ is Hurwitz. □

The previous lemmas immediately lead to the following characterization of stabilizability.

Proposition 7. *The following facts are equivalent:*
(i) the CCSS (3) is stabilizable;
(ii) there is a Hurwitz convex combination of $A_1, \ldots, A_M$.

Proof. i) $\Rightarrow$ ii) follows from Lemma 4, while ii) $\Rightarrow$ i) is a well known result for switched systems [45]. □

B. Characterizations in terms of Lyapunov functions

In this section we aim to provide additional characterizations of stabilizability by making use of copositive control Lyapunov functions.

Definition 5. *A continuous and continuously differentiable copositive function $V(\mathbf{x})$ is a* control Lyapunov function *for the CCSS (3) if for every $\mathbf{x} > \mathbf{0}$ there exists $i = i(\mathbf{x}) \in [1, M]$ such that the derivative of V in $\mathbf{x}$ along the direction of the ith subsystem is negative, namely (4) holds.*

In [2] (see Theorem 3) the following result was proved.

Theorem 1. *Consider the continuous-time switched system (3), and assume that all the matrices $A_i, i \in [1, M]$, are Metzler. The following facts are equivalent:*
(i) There is a Hurwitz convex combination of of $A_1, \ldots, A_M$;
(ii) the positive switched system (3) admits a linear copositive control Lyapunov function $V_L(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, with $\mathbf{v} \gg \mathbf{0}$.
(iii) the positive switched system (3) admits a quadratic positive definite control Lyapunov function $V_Q(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, with $P = P^\top \succ 0$.

Thanks to Proposition 7, conditions ii) and iii) in Theorem 1 become equivalent characterizations of stabilizability for CCSS. By making use of them, we can provide an additional characterization that allows to draw a very complete picture of the nature of stabilizability for CCSSs. To this goal we need a preliminary result.

Proposition 8. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices. If there exist indices $i_1, \ldots, i_n \in [1, M]$ such that the matrix $\tilde{A} := \begin{bmatrix} col_1(A_{i_1}) & \ldots & col_n(A_{i_n}) \end{bmatrix}$ is Hurwitz, then there exist $\alpha_i \geq 0$, $i \in [1, M]$, with $\sum_{i=1}^M \alpha_i = 1$, such that the convex combination $\sum_{i=1}^M \alpha_i A_i$ is Hurwitz.*

Proof. Suppose, first, that $\tilde{A}$ is irreducible. Let $\mathcal{D}(\tilde{A}) := (\mathcal{V}, \mathcal{E}_{\tilde{A}})$ and $\mathcal{D}(A_\Sigma) := (\mathcal{V}, \mathcal{E}_\Sigma)$ denote the digraphs associated with $\tilde{A}$ and $A_\Sigma := \sum_{i=1}^M A_i$ respectively, and notice that by construction $\mathcal{E}_{\tilde{A}} \subseteq \mathcal{E}_\Sigma$. Recalling that a matrix is irreducible if and only if its associated directed graph is strongly connected, the irreducibility assumption on $\tilde{A}$ guarantees that also A_Σ is irreducible. By hypothesis $\tilde{A}$ is Hurwitz, and hence there exists $k \in [1, n]$ such that $\mathbf{1}_n^\top \tilde{A} \mathbf{e}_k = \mathbf{1}_n^\top col_k(A_{i_k}) < 0$, but then $\mathbf{1}_n^\top col_k(A_\Sigma) = \sum_{i=1}^M \mathbf{1}_n^\top col_k(A_i) < 0$. This in turn implies, by Lemma 5 of the Appendix, that A_Σ is Hurwitz, and hence also the convex combination $\sum_{i=1}^M \frac{1}{M} A_i$ is Hurwitz.

If $\tilde{A}$ is a reducible matrix, then the matrix $A_\Sigma := \sum_{i=1}^M A_i$ may either be irreducible or not and in the following the two cases will be considered separately. Assume first that A_Σ is irreducible and let Π be a permutation matrix that reduces $\tilde{A}$

to Frobenius normal form:

$$\Pi^\top \tilde{A}\Pi = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \dots & \tilde{A}_{1\ell} \\ 0 & \tilde{A}_{22} & \dots & \tilde{A}_{2\ell} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & \tilde{A}_{\ell\ell} \end{bmatrix},$$

where $\tilde{A}_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1,\ell]$, are irreducible matrices. Accordingly, all matrices A_i are replaced by $\Pi^\top A_i \Pi$. Since, by hypothesis, $\tilde{A}_{11}$ is Hurwitz, there exists $k \in [1, n_1]$ such that $\mathbf{1}_{n_1}^\top col_k(\tilde{A}_{11}) < 0$, but then also $\mathbf{1}_n^\top col_k\left(\Pi^\top A_\Sigma \Pi\right) = \sum_{i=1}^M \mathbf{1}_n^\top col_k(\Pi^\top A_i \Pi) < 0$. This in turn implies, by Lemma 5 of the Appendix, that A_Σ is Hurwitz, and hence also the convex combination $\sum_{i=1}^M \frac{1}{M} A_i$ is Hurwitz.

Consider now the case when A_Σ is reducible (this implies that every A_i, $i \in [1,M]$, is reducible). Let Π be a permutation matrix that reduces A_Σ to Frobenius normal form:

$$\hat{A}_\Sigma := \Pi^\top A_\Sigma \Pi = \sum_{i=1}^M \underbrace{\Pi^\top A_i \Pi}_{A_i^\Pi} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \dots & \hat{A}_{1\ell} \\ 0 & \hat{A}_{22} & \dots & \hat{A}_{2\ell} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & \hat{A}_{\ell\ell} \end{bmatrix},$$

where $\hat{A}_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1,\ell]$, are irreducible compartmental matrices. By hypothesis there exist indices $\hat{i}_1, \dots, \hat{i}_n$ (related to $i_1, \dots, i_n$ by the same permutation described by Π) such that the matrix $\tilde{B} := \begin{bmatrix} col_1(A^\Pi_{\hat{i}_1}) & \dots & col_n(A^\Pi_{\hat{i}_n}) \end{bmatrix}$ is Hurwitz and reducible. Moreover, $\tilde{B}$ takes the following form:

$$\tilde{B} = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \dots & \tilde{B}_{1\ell} \\ 0 & \tilde{B}_{22} & \dots & \tilde{B}_{2\ell} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & \tilde{B}_{\ell\ell} \end{bmatrix},$$

with $\tilde{B}_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1,\ell]$. However, $\tilde{B}$ might not be in Frobenius normal form and, if this is the case, there exists a permutation matrix $\tilde{\Pi}$ such that:

$$\bar{B} := \tilde{\Pi}^\top \tilde{B} \tilde{\Pi} = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} & \dots & \bar{B}_{1\ell} \\ 0 & \bar{B}_{22} & \dots & \bar{B}_{2\ell} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & \bar{B}_{\ell\ell} \end{bmatrix},$$

where each diagonal block $\bar{B}_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1,\ell]$, has the following form:

$$\bar{B}_{ii} = \begin{bmatrix} \bar{B}^{(i)}_{11} & \bar{B}^{(i)}_{12} & \dots & \bar{B}^{(i)}_{1s_i} \\ 0 & \bar{B}^{(i)}_{22} & \dots & \bar{B}^{(i)}_{2s_i} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & \bar{B}^{(i)}_{s_i s_i} \end{bmatrix},$$

with $\bar{B}^{(i)}_{jj} \in \mathbb{R}^{\bar{n}^{(i)}_j \times \bar{n}^{(i)}_j}$, $j \in [1, s_i]$, irreducible compartmental matrices and $\sum_{j=1}^{s_i} \bar{n}^{(i)}_j = n_i$. Now, notice that for every $j \in [1, s_i]$, $i \in [1,\ell]$, the matrix $\bar{B}^{(i)}_{jj}$ is Hurwitz. In particular, $\bar{B}^{(i)}_{11}$ is irreducible, compartmental and Hurwitz and therefore $\mathbf{1}^\top_{\bar{n}^{(i)}_1} \bar{B}^{(i)}_{11} < \mathbf{0}^\top$, i.e. there exists $\bar{k} \in [1, \bar{n}^{(i)}_1]$ such that $\mathbf{1}^\top_{\bar{n}^{(i)}_1} col_{\bar{k}}\left(\bar{B}^{(i)}_{11}\right) < 0$. But then, there also exists $\tilde{k} \in [1, n_i]$, with $\tilde{k}$ possibly different from $\bar{k}$, such that $\mathbf{1}^\top_{n_i} col_{\tilde{k}}\left(\tilde{B}_{ii}\right) < 0$. Hence, it is also true that $\mathbf{1}^\top_{n_i} col_{\tilde{k}}\left(\hat{A}_{ii}\right) < 0$ and this in turn implies, by Lemma 5 of the Appendix, that every diagonal block $\hat{A}_{ii}$, $i \in [1,\ell]$, is Hurwitz. So, finally, the matrix $\hat{A}_\Sigma$ and also the convex combination $\sum_{i=1}^M \frac{1}{M} A_i$ are Hurwitz. □

By putting together, Propositions 7 and 8, and Theorem 1, we finally derive the following set of necessary and sufficient conditions for stabilizability.

Theorem 2. *The following facts are equivalent:*

i) *the CCSS (3) is stabilizable;*
ii) *there is a Hurwitz convex combination of $A_1, \dots, A_M$;*
iii) *$\exists\, \mathbf{v} \gg \mathbf{0}$ s.t. for every $\mathbf{x} > \mathbf{0}$ there exists $i \in [1,M]$ such that $\mathbf{v}^\top A_i \mathbf{x} < 0$;*
iv) *$\exists\, P = P^\top \succ 0$ s.t. for every $\mathbf{x} > \mathbf{0}$ there exists $i \in [1,M]$ such that $\mathbf{x}^\top [A_i^\top P + P A_i]\mathbf{x} < 0$;*
v) *$\exists\, i_1, \dots, i_n \in [1,M]$ such that the matrix: $\tilde{A} := \begin{bmatrix} col_1(A_{i_1}) & \dots & col_n(A_{i_n}) \end{bmatrix}$ is Hurwitz;*
vi) *there exist M nonnegative diagonal matrices D_i, $i \in [1,M]$, with $\sum_{i=1}^M D_i = I_n$ such that the matrix $\sum_{i=1}^M A_i D_i$ is Hurwitz.*

Proof. i) ⇔ ii) It follows from Proposition 7.

ii) ⇔ iii) ⇔ iv) It follows from Theorem 1.

iii) ⇒ v) Assume that a vector $\mathbf{v} \gg \mathbf{0}$ can be found, such that for every $\mathbf{x} > \mathbf{0}$ there exists $i \in [1,M]$ such that $\mathbf{v}^\top A_i \mathbf{x} < 0$. Then, in particular, for every $j \in [1,n]$, there exists $i_j \in [1,M]$ such that $\mathbf{v}^\top A_{i_j} \mathbf{e}_j < 0$. So, the matrix $\tilde{A} := \begin{bmatrix} col_1(A_{i_1}) & \dots & col_n(A_{i_n}) \end{bmatrix}$ satisfies $\mathbf{v}^\top \tilde{A} \ll \mathbf{0}^\top$, and this ensures that $\tilde{A}$ is Hurwitz.

v) ⇒ ii) It follows from Proposition 8.

v) ⇒ vi) For every $i \in [1,M]$ define the nonnegative diagonal matrix D_i as follows:

$$[D_i]_{kk} = \begin{cases} 1, & \text{if } i = i_k, \\ 0, & \text{otherwise,} \end{cases}$$

and notice that $\sum_{i=1}^M D_i = I_n$. Moreover, $\tilde{A} = \sum_{i=1}^M A_i D_i$ and hence by hypothesis it is Hurwitz.

vi) ⇒ v) By hypothesis the matrix $\sum_{i=1}^M A_i D_i$ is Hurwitz, and hence there exists $\mathbf{v} \gg \mathbf{0}$ such that

$$\mathbf{z}^\top := \mathbf{v}^\top \left(\sum_{i=1}^M A_i D_i\right) \ll \mathbf{0}^\top,$$

i.e. for every $k \in [1,n]$ it holds $[\mathbf{z}]_k = \sum_{i=1}^M [\mathbf{v}^\top A_i D_i]_k < 0$. This implies that for every $k \in [1,n]$ there is $i_k \in [1,M]$ such that $[\mathbf{v}^\top A_{i_k} D_{i_k}]_k < 0$. As D_{i_k} is a nonnegative diagonal matrix, the previous inequality implies that $[\mathbf{v}^\top A_{i_k}]_k = \mathbf{v}^\top col_k(A_{i_k}) < 0$. Hence, $\tilde{A} := \begin{bmatrix} col_1(A_{i_1}) & \dots & col_n(A_{i_n}) \end{bmatrix}$ is such that $\mathbf{v}^\top \begin{bmatrix} col_1(A_{i_1}) & \dots & col_n(A_{i_n}) \end{bmatrix} \ll \mathbf{0}^\top$, i.e. $\tilde{A}$ is Hurwitz. □

Remark 4. *Notice that in order to prove v) ⇒ vi) and vi) ⇒ v) the compartmental assumption on the subsystem matrices is not required, and hence the equivalence between statements v) and vi) of Theorem 2 holds in the general (i.e. non-compartmental) case.*

Remark 5. *It is worth noticing an interesting consequence of the characterization provided in Theorem 2. Condition v) involves up to n matrices, since the indices $i_1, i_2, \dots, i_n \in [1, M]$ are not necessarily distinct. If we look into the proof of Proposition 8, we easily realize that the existence of indices $i_1, i_2, \dots, i_n \in [1, M]$ such that $\begin{bmatrix} col_1(A_{i_1}) & \dots & col_n(A_{i_n}) \end{bmatrix}$ is Hurwitz allows to say that the compartmental matrix $\sum_{k=1}^{n} \frac{1}{n} A_{i_k}$ is Hurwitz. Correspondingly, we can find $\mathbf{v} \gg \mathbf{0}$ such that for every $\mathbf{x} > \mathbf{0}$*

$$\min_{k\in[1,n]} \mathbf{v}^\top A_{i_k}\mathbf{x} < 0,$$

and therefore the switching law

$$\sigma(t) = \text{argmin}_{k\in[1,n]}\mathbf{v}^\top A_{i_k}\mathbf{x}$$

is stabilizing. This shows that even when $M > n$, a stabilizable CCSS can always be stabilized by switching among a number of subsystems not bigger than the system dimension n.

The following corollary provides a sufficient condition for the stabilizability of the CCSS (3).

Corollary 3. *Let $A_i \in \mathbb{R}^{n\times n}$, $i \in [1, M]$, be compartmental matrices. If for every $j \in [1, n]$ there exists $i_j \in [1, M]$ such that $\mathbf{1}_n^\top col_j(A_{i_j}) < 0$, then the CCSS (3) is stabilizable.*

Proof. By hypothesis $\tilde{A} := \begin{bmatrix} col_1(A_{i_1}) & \dots & col_n(A_{i_n}) \end{bmatrix}$ satisfies $\mathbf{1}_n^\top \tilde{A} \ll \mathbf{0}^\top$, and hence is Hurwitz. So, by Theorem 2, the CCSS (3) is stabilizable. □

The sufficient condition stated in the previous Corollary 3 is not necessary, as shown by the following example.

Example 3. *Consider the matrices:*

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}$$

and notice that every convex combination of A_1 and A_2 is Hurwitz:

$$\alpha A_1 + (1-\alpha)A_2 = \begin{bmatrix} -1 & 1 \\ 1 & \alpha - 3 \end{bmatrix}, \quad \alpha \in [0, 1].$$

Hence, by Proposition 7, the CCSS (3) is stabilizable. However, the previous sufficient condition does not hold, since $\mathbf{1}_2^\top col_1(A_1) = \mathbf{1}_2^\top col_1(A_2) = 0$.

VI. Conclusions

In this paper we have investigated asymptotic stability and stabilizability of continuous-time linear compartmental switched systems. We have shown that asymptotic stability is equivalent to the Hurwitz property of all the system matrices $A_i, i \in [1, M]$. Also, sufficient conditions for stability based on the existence of various kinds of copositive Lyapunov functions have been mutually related. On the other hand, we have shown that even when the matrices A_i are not (all) Hurwitz, convergence to zero can always be guaranteed, for every choice of the initial conditions, by resorting to switching functions that satisfy certain ergodicity/dwell time properties. Finally, we have investigated stabilizability and shown that it is equivalent to the existence of a convex Hurwitz combination of the system matrices. In addition, stabilizability can always be tested by resorting to linear copositive control Lyapunov functions or quadratic positive definite control Lyapunov functions. The question of whether asymptotic stability implies the existence of a common quadratic positive definite Lyapunov function is still an open problem that deserves further investigation.

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Appendix
Some technical results about compartmental matrices and compartmental systems

By definition, an $n\times n$ Metzler matrix A is compartmental if $\mathbf{1}_n^\top A \le \mathbf{0}^\top$. Condition $\mathbf{1}_n^\top A \ll \mathbf{0}^\top$ ensures that A is Hurwitz [18]. On the other hand, condition $\mathbf{1}_n^\top A = \mathbf{0}^\top$ means that $\mathbf{1}_n$ is a left eigenvector of A corresponding to $\lambda_F = 0$ and hence A is not Hurwitz. The intermediate case when $\mathbf{1}_n^\top A < \mathbf{0}^\top$, but at least one of the entries of $\mathbf{1}_n^\top A$ is zero, does not allow to draw any conclusion on the Hurwitz property of A, unless A is irreducible.

Lemma 5. *[20], [40] An irreducible compartmental matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz if and only if $\mathbf{1}_n^\top A < \mathbf{0}^\top$.*

The following result is used in Section II.

Lemma 6. *If $A \in \mathbb{R}^{n\times n}$ is a compartmental Hurwitz matrix, every principal submatrix of A is compartmental and Hurwitz.*

Proof. Let $A_{\mathcal{J}}$ denote the principal submatrix of A obtained by selecting rows and columns of A indexed by the set $\mathcal{J} = \{j_1, \dots, j_r\} \subseteq [1, n]$, $\mathcal{J} \neq \emptyset$. Since for every permutation matrix Π the matrix $\Pi^\top A \Pi$ is still compartmental and Hurwitz, it entails no loss of generality assuming that $\mathcal{J} = [1, r]$, $r \in [1, n]$, and hence the matrix A takes the following form:

$$A = \begin{bmatrix} A_{\mathcal{J}} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Clearly, $A_{\mathcal{J}}$ is Metzler. Now notice that for every $j \in \mathcal{J}$ it holds: $\mathbf{1}_r^\top col_j(A_{\mathcal{J}}) \le \mathbf{1}_r^\top col_j(A_{\mathcal{J}}) + \mathbf{1}_{n-r}^\top col_j(A_{21}) = \mathbf{1}_n^\top col_j(A) \le 0$, and hence $A_{\mathcal{J}}$ is compartmental. Moreover, let $[A]_{ii} =: d_i$, $i \in [r+1, n]$, and notice that the following relation holds:

$$A \ge \bar{A} := \left[\begin{array}{c|ccc} A_{\mathcal{J}} & & \mathbf{0}_{1\times(n-r)} & \\ \hline & d_{r+1} & \dots & 0 \\ \mathbf{0}_{n-r} & \vdots & \ddots & \vdots \\ & 0 & \dots & d_n \end{array}\right]$$

Then, by the Metzler matrix properties and recalling that A is Hurwitz, one has $0 > \lambda_F(A) \ge \lambda_F(\bar{A}) = \max\{\lambda_F(A_{\mathcal{J}}), d_{r+1}, \dots, d_n\}$, and hence $A_{\mathcal{J}}$ is Hurwitz. □

We now consider linear compartmental systems described as in (2), for some compartmental matrix $A \in \mathbb{R}^{n\times n}$, and prove that when A is a Hurwitz matrix, the linear copositive Lyapunov function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is strictly decreasing along the system trajectories independently of the positive initial condition. We consider the *irreducible* case in Lemma 7 and then we remove the irreducibility hypothesis in Proposition 9.

Lemma 7. *Consider the compartmental system (2) and assume that $A \in \mathbb{R}^{n\times n}$ is an irreducible, compartmental, Hurwitz matrix. Then, if $\mathbf{x}(0) \neq \mathbf{0}$, the Lyapunov function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is strictly decreasing along the system trajectories, independently of the positive initial condition, i.e.*

$$\mathbf{1}_n^\top \mathbf{x}(t) < \mathbf{1}_n^\top \mathbf{x}(0), \qquad \forall t > 0, \forall \mathbf{x}(0) > \mathbf{0}. \tag{12}$$

Proof. Since A is an irreducible compartmental and Hurwitz matrix, by Lemma 5, $\mathbf{1}_n^\top A < \mathbf{0}^\top$. Now, if $\mathbf{1}_n^\top A \ll \mathbf{0}^\top$, then $\dot V(\mathbf{x}(t)) = \mathbf{1}_n^\top A\mathbf{x}(t) < 0$ for every $t > 0$, independently of $\mathbf{x}(0) > 0$, and hence $V(\mathbf{x}(t))$ is strictly decreasing along the system trajectories, independently of the positive initial condition, namely (12) holds.
If $\mathbf{1}_n^\top A < \mathbf{0}^\top$, then, possibly by resorting to row and column permutations on A, we can assume w.l.o.g. that it takes the form $\mathbf{1}_n^\top A = \begin{bmatrix}\mathbf{0}_k^\top & -\mathbf{v}^\top\end{bmatrix}, \mathbf{v} \in \mathbb{R}^{n-k}, \mathbf{v} \gg \mathbf{0}$. Set

$$\begin{aligned}\mathcal{N} &:= \left\{\mathbf{x} \geq 0 \colon \dot V(\mathbf{x}) = \mathbf{1}_n^\top A\mathbf{x} = 0\right\} \\ &= \operatorname{Cone}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k).\end{aligned} \tag{13}$$

We want to show that $\mathcal{N}$ contains no system trajectory except for the zero trajectory, i.e. $\mathbf{x}(t) = \mathbf{0}$ for all $t \geq 0$. If $\mathbf{x}(0) \in \mathcal{N}$, $\mathbf{x}(0) \neq \mathbf{0}$, then $\mathbf{x}(0)^\top = \begin{bmatrix}\mathbf{x}_{10}^\top & \mathbf{0}_{1\times k}\end{bmatrix}, \mathbf{x}_{10} \in \mathbb{R}_+^{n-k}, \mathbf{x}_{10} > \mathbf{0}$. By the irreducibility assumption on A, $e^{At} \gg 0$ for all $t > 0$ [32], and hence for every $t > 0$ $\mathbf{x}(t) = e^{At}\mathbf{x}(0) = \begin{bmatrix}\mathbf{x}_1(t)^\top & \mathbf{x}_2(t)^\top\end{bmatrix}^\top$ with $\mathbf{x}_2(t) \gg \mathbf{0}$. So, for every $t > 0$, $\mathbf{x}(t) \notin \mathcal{N}$, and therefore $V(\mathbf{x}(t))$ is strictly decreasing over any arbitrarily small time interval $[0, t]$. □

Proposition 9. *Consider the compartmental system (2) and assume that $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix. Then, if $\mathbf{x}(0) \neq \mathbf{0}$, the Lyapunov function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is strictly decreasing along the system trajectories independently of the positive initial condition, i.e. (12) holds.*

Proof. The case when A is irreducible has been addressed in Lemma 7. So, we assume now that A is reducible. Consider the directed graph associated with A, and let $\mathcal{C}_1, \dots, \mathcal{C}_s$ be its recurrent classes and $\mathcal{C}_{s+1}, \dots, \mathcal{C}_\ell$ be its transient classes. It entails no loss of generality assuming that

$$A = \begin{bmatrix} A_{11} & \dots & 0 & A_{1s+1} & & A_{1\ell} \\ & \ddots & & & \ddots & \\ 0 & \dots & A_{ss} & A_{ss+1} & & A_{s\ell} \\ & & & A_{s+1s+1} & \dots & A_{s+1\ell} \\ & & & & \ddots & \\ & & & A_{\ell s+1} & \dots & A_{\ell\ell} \end{bmatrix},$$

where $A_{ii} \in \mathbb{R}^{n_i\times n_i}$, $i \in [1, \ell]$, are irreducible matrices, since we can always reduce ourselves to this situation by resorting to a suitable permutation of the rows and columns of A that does not affect the compartmental property of A. Accordingly (see [32], Proposition 1) e^{At} takes the following form:

$$e^{At} =: \mathcal{A}(t) = \begin{bmatrix} \mathcal{A}_{11}(t) & \dots & 0 & \mathcal{A}_{1s+1}(t) & & \mathcal{A}_{1\ell}(t) \\ & \ddots & & & \ddots & \\ 0 & \dots & \mathcal{A}_{ss}(t) & \mathcal{A}_{ss+1}(t) & & \mathcal{A}_{s\ell}(t) \\ & & & \mathcal{A}_{s+1s+1}(t) & \dots & \mathcal{A}_{s+1\ell}(t) \\ & & & & \ddots & \\ & & & \mathcal{A}_{\ell s+1}(t) & \dots & \mathcal{A}_{\ell\ell}(t) \end{bmatrix},$$

where, for every $t > 0$, the matrix $\mathcal{A}_{ij}(t) = \text{block}_{ij}[e^{At}] \in \mathbb{R}^{n_i\times n_j}$, $i \in [1, s]$, $j \in [s+1, \ell]$, is strictly positive if the class $\mathcal{C}_j$ has access to the class $\mathcal{C}_i$ and the null matrix otherwise. By definition of transient class, for every $j \in [s+1, \ell]$, there exists $i \in [1, s]$ s.t. $\mathcal{A}_{ij}(t) \in \mathbb{R}^{n_i\times n_j}$ is strictly positive. Moreover, since A_{ii}, $i \in [1, s]$, are irreducible matrices, the matrices $\mathcal{A}_{ii}(t) = e^{A_{ii}t}$, $i \in [1, s]$, are strictly positive for every $t > 0$. The state vector $\mathbf{x}(t)$ can be partitioned into ℓ blocks, according to the partition of A. Now, define the set $\mathcal{N}$ as in (13), and let $\mathbf{x}(0) > \mathbf{0}$ be in $\mathcal{N}$. We first note that, by the irreducibility assumption on $A_{ii}, i \in [1, s]$, in every set of indices $\{(\sum_{k=0}^{i-1} n_k) + 1, \dots, (\sum_{k=0}^{i-1} n_k) + n_i\}, i \in [1, s]$, (with $n_0 := 0$), there is at least one index j such that $\mathbf{1}_n^\top A\mathbf{e}_j < 0$. This implies that if $\mathbf{x}(t) > \mathbf{0}$ and $[\mathbf{x}(t)]_j > 0$ then $\mathbf{1}_n^\top A\mathbf{x}(t) < 0$, and hence $\mathbf{x}(t) \notin \mathcal{N}$. So, by the same reasoning adopted in Lemma 7, we can claim that every $\mathbf{x}(0) \in \mathcal{N}, \mathbf{x}(0) > \mathbf{0}$, whose nonzero entries belong only to the first s blocks, necessarily generates a state trajectory that exits $\mathcal{N}$.
Assume, now, that there exists $k \in [s+1, l]$ such that $\text{block}_k[\mathbf{x}(0)] > \mathbf{0}$. Then, by the previous reasoning, there exists $i \in [1, s]$ s.t. $\text{block}_i[e^{At}\mathbf{x}(0)] \gg \mathbf{0}, \forall\, t > 0$. But this implies that $\mathbf{x}(t)$ cannot belong to $\mathcal{N}$. Thus, as $\dot V(\mathbf{x}(t)) = \mathbf{1}_n^\top A\mathbf{x}(t) \leq 0$ and $\mathcal{N}$ does not include system trajectories apart from the identically zero one, it follows that $V(\mathbf{x}(t))$ is strictly decreasing with t. □

Remark 6. *It is worth noticing that even if $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is strictly decreasing along the system trajectories, nonetheless it is not true that $\mathbf{1}_n^\top A \ll \mathbf{0}^\top$, which is the property a linear copositive Lyapunov function has to satisfy. However, it is clear that since A is a Metzler Hurwitz matrix, there is always a vector $\mathbf{v} \gg \mathbf{0}$ such that $\mathbf{v}^\top A \ll \mathbf{0}^\top$. So, a linear copositive Lyapunov function always exists but it is not necessarily the one associated with the vector $\mathbf{1}_n$.*

REFERENCES

[1] A. Bacciotti and L. Mazzi. An invariance principle for nonlinear switched systems. *Systems Control Letters*, 54:1109 – 1119, 2005.
[2] F. Blanchini, P. Colaneri, and M.E. Valcher. Co-positive Lyapunov functions for the stabilization of positive switched systems. *IEEE Trans.Aut. Contr.*, 57 (12):3038–3050, 2012.
[3] F. Blanchini, P. Colaneri, and M.E. Valcher. A stabilizable switched linear system does not necessarily admit a smooth homogeneous Lyapunov function. In *Proc. of the 52nd IEEE CDC*, pages 5969–5974, Firenze, Italy, 2013.
[4] F. Blanchini, P. Colaneri, and M.E. Valcher. Switched positive linear systems. *submitted to Foundations and Trends in Systems and Control*, 2015.
[5] M. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Aut. Contr.*, 43:475–482, 1998.
[6] R. Bru, S. Romero, and E. Sánchez. Canonical forms for positive discrete-time linear control systems. *Linear Algebra & its Appl.*, 310:49–71, 2000.

[7] R. A. Brualdi and H. J. Ryser. *Combinatorial matrix theory*. Cambridge Univ. Press, 1991.

[8] D. Cheng, J. Wang, and X. Hu. An extension of LaSalle's invariance principle and its application to multi-agent consensus. *IEEE Trans. Autom. Control*, 53:1765–1770, 2008.

[9] W.P. Dayawansa and C.F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Trans. Aut. Contr.*, 44:751 –760, 1999.

[10] L. Fainshil, M. Margaliot, and P. Chigansky. On the stability of positive linear switched systems under arbitrary switching laws. *IEEE Trans. Automat. Control*, 54:807–899, 2009.

[11] E. Fornasini and M.E. Valcher. Linear copositive Lyapunov functions for continuous-time positive switched systems. *IEEE Trans. Aut. Contr.*, 55 (8):1933–1937, 2010.

[12] E. Fornasini and M.E. Valcher. Stability and stabilizability criteria for discrete-time positive switched systems. *IEEE Trans. Aut. Contr.*, 57 (5):1208–1221, 2012.

[13] G.F. Frobenius. Über Matrizen aus nicht Negativen Elementen. *Ges. Abh.*. Springer, 3 (92):546-567, 1968.

[14] F.R. Gantmacher. *The theory of matrices*. Chelsea Pub. Co., 1960.

[15] L. Gurvits, R. Shorten, and O. Mason. On the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52 (6):1099–1103, 2007.

[16] W.M. Haddad, V. Chellaboina, and Q. Hui. *Nonnegative and compartmental dynamical systems*. Princeton University Press, 2010.

[17] E. Hernandez-Vargas, R. Middleton, P. Colaneri, and F. Blanchini. Discrete-time control for switched positive systems with application to mitigating viral escape. *Int. J. Robust Nonlinear Control*, 21, (10):1093–1111, 2011.

[18] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge (UK), 1991.

[19] S.P. Hou, N. Meskin, and W.M. Haddad. A general multicompartment lung mechanics model with nonlinear resistance and compliance respiratory parameters. In *Proc. of the 2014 American Control Conference*, pages 566–571, Portland, Oregon, 2014.

[20] J. Jacquez and C.P. Simon. Qualitative theory of compartmental systems. *SIAM Review*, 35:43–79, 1993.

[21] J.G. Kemeny and J.L. Snell. *Finite Markov Chains*. Princeton N.J. van Nostrand Company, 1960.

[22] F. Knorn, O. Mason, and R.N. Shorten. On linear co-positive Lyapunov functions for sets of linear positive systems. *Automatica*, 45, Issue 8:1943–1947, August 2009.

[23] H. Li and W.M. Haddad. Optimal determination of respiratory airflow patterns using a nonlinear multicompartment model for a lung mechanics system. *Computational and Mathematical Methods in Medicine*, 2012.

[24] D. Liberzon. *Switching in Systems and Control*. Volume in series Systems and Control: Foundations and Applications, Birkhauser, Boston (MA), 2003.

[25] L. Lin. Stabilization analysis for economic compartmental switched systems based on quadratic Lyapunov function. *Nonlinear Analysis: Hybrid Systems*, 2:1187–1197, 2008.

[26] J. Liu, X. Liu, and W.C. Xie. Extending LaSalle's invariance principle to impulsive switched systems with an application to hybrid epidemic dynamics. In *Proceedings of Chinese Control and Decision Conference*, pages 136–141, Xuzhou, China, 2010.

[27] O. Mason and R.N. Shorten. On the simultaneous diagonal stability of a pair of positive linear systems. *Linear Algebra and its Appl.*, 413:13–23, 2006.

[28] O. Mason and R.N. Shorten. On linear copositive Lyapunov functions and the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52, no. 7:1346 – 1349, 2007.

[29] O. Mason and R.N. Shorten. Quadratic and copositive Lyapunov functions and the stability of positive switched linear systems. In *Proc. of the American Control Conference (ACC 2007)*, pages 657–662, New York, 2007.

[30] H. Minc. *Nonnegative Matrices*. J.Wiley & Sons, New York, 1988.

[31] J.R. Norris. *Markov Chains*. Cambridge University Press, 1997.

[32] P. Santesso and M.E. Valcher. On the zero pattern properties and asymptotic behavior of continuous-time positive system trajectories. *Linear Algebra and its Applications*, 425:283–302, 2007.

[33] P. Santesso and M.E. Valcher. Monomial reachability and zero-controllability of discrete-time positive switched systems. *Systems and Control Letters*, 57:340–347, 2008.

[34] H. Schneider. The concept of irreducibility and full indecomposability of a matrix in the works of Frobenius, König and Markov. *Lin. Alg. Appl.*, 18:139–162, 1977.

[35] H. Schneider. The influence of the marked reduced graph of a nonnegative matrix on the jordan form and on related properties. *Lin. Alg. Appl.*, 84:161–189, 1986.

[36] R. Shorten, F. Wirth, and D. Leith. A positive systems model of TCP-like congestion control: asymptotic results. *IEEE/ACM Transactions on Networking*, 14 (3):616–629, 2006.

[37] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49:545–592, 2007.

[38] H.L. Smith. Linear compartmental systems - The basics, 2006. available online at `https://math.la.asu.edu/ halsmith/compartment.pdf`.

[39] N.K. Son and D. Hinrichsen. Robust stability of positive continuous time systems. *Numer. Funct. Anal. and Optimiz.*, 17 (5 & 6):649–659, 1996.

[40] O. Taussky. A recurring theorem on determinants. *American Math. Monthly*, 56 (10):672–676, 1949.

[41] M.E. Valcher. Controllability and reachability criteria for discrete time positive systems. *Int. J. of Control*, 65:511–536, 1996.

[42] M.E. Valcher and P. Santesso. Reachability properties of single-input continuous-time positive switched systems. *IEEE Transactions on Automatic Control*, 55, no.5:1117–1130, 2010.

[43] M.E. Valcher and I. Zorzan. On the stabilizability of continuous-time compartmental switched systems. In *Proc. of the 54th IEEE Conf. on Decision and Control*, pages 4246-4251, Osaka, Japan, 2015.

[44] J. Wang, D. Cheng, and X. Hu. An extension of LaSalle's invariance principle for a class of switched linear systems. *Systems Control Letters*, 58:754–758, 2009.

[45] M.A. Wicks, P. Peleties, and R.A. De Carlo. Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems. *European J. of Control*, 4, no. 2:140–147, 1998.

[46] A. Zappavigna, T. Charalambous, and F. Knorn. Unconditional stability of the Foschini-Miljanic algorithm. *Automatica*, 48 (1):219–224, 2012.

[47] A. Zappavigna, P. Colaneri, J.C. Geromel, and R. Shorten. Dwell time analysis for continuous-time switched linear positive systems. In *Proceedings of American Control Conference*, pages 6256–6261, Baltimore, USA, 2010.

[48] X. Zhao, X. Liu, S. Yin, and H. Li. Improved results on stability of continuous-time switched positive linear systems. *Automatica*, 50(2):614–621, 2014.

[49] X. Zhao, S. Yin, H. Li, and B. Niu. Switching stabilization for a class of slowly switched systems. *IEEE Trans. Automat. Control*, 60 (1):221–226, 2015.

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