On the Periodic Trajectories of Boolean Control Networks ⋆

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Abstract

In this note we first characterize the periodic trajectories (or, equivalently, the limit cycles) of a Boolean network, and their global attractiveness. We then investigate under which conditions all the trajectories of a Boolean control network may be forced to converge to the same periodic trajectory. If every trajectory can be driven to such a periodic trajectory, this is possible by means of a feedback control law.

Key words: Boolean logic, Boolean control networks, directed graphs, feedback stabilization, limit cycles, stabilizability.

1 Introduction

Boolean networks (BNs) are state-space models whose state variables attain two possible values (0 and 1, true or false) and whose update is governed by logic functions. The recent interest in BNs is motivated by the large number of natural and artificial systems whose describing variables display only two distinct configurations. Originally introduced to model simple neural networks, BNs recently proved to be suitable to describe and simulate the behavior of genetic regulatory networks [9,14]. In addition, BNs are fruitfully used to describe the interactions among agents and hence to investigate consensus problems [8,13]. Boolean control networks (BCNs) were subsequently introduced in the literature to keep into account that many biological systems have exogenous inputs. So, by adding Boolean inputs to a BN, it is possible to formally define a BCN. Indeed, a BCN can be seen as a switched system, switching among different BNs.

In addition to the increasingly large number of applications where BNs and BCNs proved their effectiveness, another reason for their recent success is the powerful algebraic framework, developed by D. Cheng and coauthors [2,3,5], where both BNs and BCNs can be recast. The main idea underlying this approach is that a Boolean network with n state variables exhibits 2^n possible configurations, and if any such configuration is represented by means of a canonical vector of size 2^n, all the logic maps that regulate the state-updating can be equivalently described by means of $2^n \times 2^n$ Boolean matrices. As a result, every Boolean network can be described as a discrete-time linear system. In a similar fashion, a Boolean control network can be converted into a discrete-time bilinear system or, more conveniently, it can be seen as a family of BNs, each of them associated with a specific value of the input variables.

In this paper, we investigate the periodic structure of the state trajectories of BNs and BCNs. In detail, we first characterize the periodic trajectories (or, equivalently, the limit cycles) of a Boolean network, and their global attractiveness. We then investigate under which conditions all the trajectories of a BCN may be forced to converge to the same periodic trajectory. If this is the case, this goal can be achieved by means of a feedback control. The stabilization problem to an equilibrium point, a topic first investigated in [4,6] (see also [11,12] for recent contributions about the stability and stabilizability problems for BCNs and BNs with impulsive effects), follows as a special case.

Notation. Given two nonnegative integers k, n, with $k \le n$, by the symbol $[k, n]$ we denote the set of integers $\{k, k+1, \dots, n\}$. We consider Boolean vectors and matrices, taking values in $\mathcal{B} = \{0, 1\}$, with the usual logical operations (And $\wedge$, Or $\vee$, Negation $^-$). δ_k^i denotes the ith canonical vector of size k, $\mathcal{L}_k$ the set of all k-dimensional canonical vectors, and $\mathcal{L}_{k\times n} \subset \mathcal{B}^{k\times n}$ the set

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of all $k \times n$ matrices whose columns are canonical vectors of size k. Any matrix $L \in \mathcal{L}_{k\times n}$ can be represented as a row vector whose entries are canonical vectors in $\mathcal{L}_k$, namely $L = \left[\, \delta_k^{i_1} \quad \delta_k^{i_2} \quad \dots \quad \delta_k^{i_n} \,\right]$, for suitable indices $i_1, i_2, \dots, i_n \in [1,k]$. $[A]_{\ell j}$ is the (ℓ, j)th entry of the matrix A. A *permutation matrix* P is a nonsingular square matrix in $\mathcal{L}_{k\times k}$. In particular, a matrix

$$P = C = [\, \delta_k^2 \quad \delta_k^3 \quad \dots \quad \delta_k^k \quad \delta_k^1 \,] \tag{1}$$

is a $k \times k$ *cyclic (permutation) matrix.* Given a matrix $L \in \mathcal{B}^{k\times k}$ (in particular, $L \in \mathcal{L}_{k\times k}$), we associate with it [1] a *digraph* $\mathcal{D}(L)$, with vertices $1, \dots, k$. There is an arc (j, ℓ) from j to ℓ if and only if the $[L]_{\ell j} = 1$. A sequence $j_1 \to j_2 \to \dots \to j_r \to j_{r+1}$ in $\mathcal{D}(L)$ is a *path* of length r from j_1 to j_{r+1} provided that $(j_1, j_2), \dots, (j_r, j_{r+1})$ are arcs of $\mathcal{D}(L)$. A closed path is called a *cycle.* In particular, a cycle γ with no repeated vertices is called *elementary,* and its length $|\gamma|$ coincides with the number of (distinct) vertices appearing in it. Note that a $k \times k$ cyclic matrix has a digraph that consists of one elementary cycle with length k.

There is a bijective correspondence between Boolean variables $X \in \mathcal{B}$ and vectors $\mathbf{x} \in \mathcal{L}_2$, defined by the relationship

$$\mathbf{x} = \begin{bmatrix} X \\ \overline{X} \end{bmatrix}.$$

We introduce the *(left) semi-tensor product* $\ltimes$ between matrices (and hence, in particular, vectors) as follows [5,10]: given $L_1 \in \mathbb{R}^{r_1\times c_1}$ and $L_2 \in \mathbb{R}^{r_2\times c_2}$ (in particular, $L_1 \in \mathcal{L}_{r_1\times c_1}$ and $L_2 \in \mathcal{L}_{r_2\times c_2}$), we set

$$L_1 \ltimes L_2 := (L_1 \otimes I_{T/c_1})(L_2 \otimes I_{T/r_2}), \quad T := \text{l.c.m.}\{c_1, r_2\}.$$

The semi-tensor product represents an extension of the standard matrix product, by this meaning that if $c_1 = r_2$, then $L_1 \ltimes L_2 = L_1 L_2$. Note that if $\mathbf{x}_1 \in \mathcal{L}_{r_1}$ and $\mathbf{x}_2 \in \mathcal{L}_{r_2}$, then $\mathbf{x}_1 \ltimes \mathbf{x}_2 \in \mathcal{L}_{r_1 r_2}$. By resorting to the semi-tensor product, we can extend the previous correspondence to a bijective correspondence [5] between $\mathcal{B}^n$ and $\mathcal{L}_{2^n}$. This is possible in the following way: given $X = [\, X_1 \quad X_2 \quad \dots \quad X_n \,]^\top \in \mathcal{B}^n$ set

$$\mathbf{x} := \begin{bmatrix} X_1 \\ \overline{X}_1 \end{bmatrix} \ltimes \begin{bmatrix} X_2 \\ \overline{X}_2 \end{bmatrix} \ltimes \dots \ltimes \begin{bmatrix} X_n \\ \overline{X}_n \end{bmatrix}.$$

This amounts to saying that

$$\mathbf{x} = \begin{bmatrix} X_1 X_2 \dots X_{n-1} X_n \\ X_1 X_2 \dots X_{n-1} \overline{X}_n \\ X_1 X_2 \dots \overline{X}_{n-1} X_n \\ \vdots \\ \overline{X}_1 \overline{X}_2 \dots \overline{X}_{n-1} \overline{X}_n \end{bmatrix}.$$

2 Limit cycles of a Boolean Network

A *Boolean Network* (BN) is described by the following equation

$$X(t+1) = f(X(t)), \qquad t \in \mathbb{Z}_+, \tag{2}$$

where $X(t)$ denotes the n-dimensional state variable at time t, taking values in $\mathcal{B}^n$. f is a (logic) function, namely a map $f : \mathcal{B}^n \to \mathcal{B}^n$. Upon representing the state vector $X(t)$ by means of its equivalent $\mathbf{x}(t)$ in $\mathcal{L}_{2^n}$, the BN (2) can be described [5] as

$$\mathbf{x}(t+1) = L \ltimes \mathbf{x}(t) \; = \; L\mathbf{x}(t), \tag{3}$$

where $L \in \mathcal{L}_{2^n \times 2^n}$ is a matrix whose columns are canonical vectors of size 2^n.

Definition 1 *An ordered sequence of distinct vectors* $(\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \dots, \delta_{2^n}^{i_k})$ *is a* limit cycle $\mathcal{C}$ *of the BN (3) if* $\mathbf{x}(0) = \delta_{2^n}^{i_\ell}$ *for some* $\ell \in [1,k]$ *ensures that the corresponding state trajectory* $\mathbf{x}(t)$ *is periodic of period* k *and, for every* $t \in \mathbb{Z}_+$, $\mathbf{x}(t) = \delta_{2^n}^{i_j}$, *where* $j \in [1,k]$ *and* $j \equiv (t+\ell) \bmod k$. *A limit cycle of unitary length is an* equilibrium point *of the BN.*

Definition 2 *A* limit cycle $\mathcal{C}$ *of the BN (3) is* globally attractive *if for every* $\mathbf{x}(0) \in \mathcal{L}_{2^n}$ *there exists* $\tau \in \mathbb{Z}_+$ *such that* $\mathbf{x}(t)$ *is a state of* $\mathcal{C}$ *for every* $t \in \mathbb{Z}_+, t \geq \tau$.

Clearly, a BN has a globally attractive limit cycle if and only if all its state trajectories converge in a finite number of steps to the same periodic trajectory. In order to provide a characterization of globally attractive limit cycles, we introduce the following result.

Proposition 1 [7] *Given a BN (3), there exist* $r \in \mathbb{N}$ *and a permutation matrix* $P \in \mathcal{L}_{2^n\times 2^n}$ *such that*

$$P^\top L P = \text{blockdiag}\{D_1, D_2, \dots, D_r\}, \tag{4}$$

$$\text{with} \qquad D_i = \begin{bmatrix} N_i & 0 \\ T_i & C_i \end{bmatrix} \in \mathcal{L}_{n_i \times n_i}, \tag{5}$$

where N_i *is a* $(n_i - k_i) \times (n_i - k_i)$ *square nilpotent matrix, and* C_i *is a* $k_i \times k_i$ *cyclic permutation matrix.*

The permutation matrix P corresponds to a so called *change of basis* in the vector space of the logic functions of $x_1, x_2, \dots, x_n$ [5]. The previous proposition relates a number of properties of the BN to the algebraic structure of L: in the general case, a BN has r limit cycles. Every limit cycle (in particular, every equilibrium point) has a *domain of attraction*, namely a set of initial conditions $\mathbf{x}(0)$ that originate trajectories entering the cycle in a finite number of steps. The block structure of $P^\top L P$ clarifies the domain of attraction of each limit cycle. Finally, the number $T_r := \max_{i\in[1,r]}(n_i - k_i)$ represents an upper bound on the *transient time*, namely on the maximum number of steps after which $\mathbf{x}(t)$ steadily belongs to a limit cycle. Note that after T_r steps, every trajectory is periodic with (not necessarily minimum) period $T_p := l.c.m.\{k_i, i \in [1, r]\}$.

The previous comments immediately lead to the following characterization.

Proposition 2 *Given a BN (3), an ordered set of distinct canonical vectors $\mathcal{C} = (\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_k})$ is a globally attractive limit cycle of the BN if and only if there exists a permutation matrix $P \in \mathcal{L}_{2^n \times 2^n}$ such that $P^\top LP$ can be described as in (4)-(5) for $r = 1$, with C_1 a cyclic permutation matrix of size k and, possibly upon a circular permutation of the indices i_ℓ, $P^\top \delta_{2^n}^{i_\ell} = \delta_{2^n}^{2^n-k+\ell}$, for every $\ell \in [1, k]$.*

By Proposition 1, the *characteristic polynomial* of the matrix L takes the form

$$\Delta_L(z) := \det(zI_{2^n} - L) = \left(z^{2^n - \sum_{i=1}^r k_i}\right) \prod_{i=1}^r (z^{k_i} - 1).$$

Consequently, we have the following corollary.

Corollary 1 *A BN (3) has a globally attractive limit cycle (of length k) if and only if $\Delta_L(z) = z^{2^n-k}(z^k - 1)$.*

3 Boolean Control Networks: basic definitions and stabilizability to a limit cycle

A *Boolean Control Network* (BCN) is described by the following equation

$$X(t+1) = f(X(t), U(t)), \qquad t \in \mathbb{Z}_+, \tag{6}$$

where $X(t)$ and $U(t)$ are the n-dimensional state variable and the m-dimensional input variable at time t, taking values in $\mathcal{B}^n$ and in $\mathcal{B}^m$, respectively, and f is a logic function, i.e. $f : \mathcal{B}^n \times \mathcal{B}^m \to \mathcal{B}^n$. By resorting to the semi-tensor product $\ltimes$, the BCN (6) can be described as [5]

$$\mathbf{x}(t+1) = L \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{7}$$

where $L \in \mathcal{L}_{2^n \times 2^{n+m}}$. For every choice of the input variable at time t, namely for every $\mathbf{u}(t) = \delta_{2^m}^j$, $L \ltimes \mathbf{u}(t) =: L_j$ is a matrix in $\mathcal{L}_{2^n \times 2^n}$. So, we can think of the BCN (7) as a Boolean switched system,

$$\mathbf{x}(t+1) = L_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{8}$$

where $\sigma(t), t \in \mathbb{Z}_+$, is a switching sequence taking values in $[1, 2^m]$. For every $j \in [1, 2^m]$, we refer to the BN

$$\mathbf{x}(t+1) = L_j\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{9}$$

as the jth subsystem of the Boolean switched system (8).

Definition 3 [5] *Given a BCN (7), we say that $\mathbf{x}_f = \delta_{2^n}^i$ is* reachable *from $\mathbf{x}_0 = \delta_{2^n}^j$ if there exists $\tau \in \mathbb{Z}_+$ and an input $\mathbf{u}(t), t \in [0, \tau - 1]$, that leads the state trajectory from $\mathbf{x}(0) = \mathbf{x}_0$ to $\mathbf{x}(\tau) = \mathbf{x}_f$. The BCN is* controllable *if $\mathbf{x}_f$ is reachable from $\mathbf{x}_0$, for every choice of $\mathbf{x}_0, \mathbf{x}_f \in \mathcal{L}_{2^n}$.*

A state $\mathbf{x}_f = \delta_{2^n}^i$ is reachable from $\mathbf{x}_0 = \delta_{2^n}^j$ if and only if [5] there exists $\tau \in \mathbb{Z}_+$ such that the Boolean sum of the matrices $L_\nu, \nu \in [1, 2^m]$, namely

$$L_{tot} := \bigvee_{\nu=1}^{2^m} L_\nu,$$

satisfies $[L_{tot}^\tau]_{ij} = 1$. Consequently, by the theory of positive matrices [1], the BCN is controllable if and only if L_{tot} is an irreducible matrix, or, equivalently, the Boolean matrix

$$\mathbb{L} := \bigvee_{i=0}^{2^n-1} (L_{tot})^i \tag{10}$$

has all unitary entries. In the sequel, we will denote the set of states reachable from $\mathbf{x}_0$ as $\mathcal{R}(\mathbf{x}_0)$.

Definition 4 *A BCN (7) is* stabilizable *to the elementary cycle $\mathcal{C} = (\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_k})$ if for every $\mathbf{x}(0) \in \mathcal{L}_{2^n}$ there exist $\mathbf{u}(t), t \in \mathbb{Z}_+$, and $\tau \in \mathbb{Z}_+$ such that $\mathbf{x}(t) = \delta_{2^n}^{i_j}$ for every $t \geq \tau$, where $j \in [1, k]$ and $j \equiv (t-\tau+1) \bmod k$.*

The proof of the following result is quite straightforward.

Proposition 3 *A BCN (7) is stabilizable to the elementary cycle $\mathcal{C} = (\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_k})$ if and only if the following two conditions hold*

1) *for every $(i_\ell, i_{\ell+1}), \ell \in [1, k]$, (with $i_{k+1} = i_1$) there exists $\delta_{2^m}^{j_\ell}$ such that $\delta_{2^n}^{i_{\ell+1}} = L \ltimes \delta_{2^m}^{j_\ell} \ltimes \delta_{2^n}^{i_\ell} = L_{j_\ell}\delta_{2^n}^{i_\ell}$;*
2) *$\delta_{2^n}^{i_1}$ is reachable from every initial state $\mathbf{x}(0)$, which amounts to saying that*

$$\delta_{2^n}^{i_1} \in \bigcap_{\mathbf{x}(0) \in \mathcal{L}_{2^n}} \mathcal{R}(\mathbf{x}(0)).$$

Corollary 2 *A BCN (7) is stabilizable to the state $\mathbf{x}_e := \delta_{2^n}^i \in \mathcal{L}_{2^n}$ if and only if the following two conditions hold*

1) *$\mathbf{x}_e$ is an equilibrium point of the jth subsystem (9), for some $j \in [1, 2^m]$;*
2) *$\mathbf{x}_e$ is reachable from every initial state $\mathbf{x}(0)$, i.e., $\mathbf{x}_e \in \cap_{\mathbf{x}(0) \in \mathcal{L}_{2^n}} \mathcal{R}(\mathbf{x}(0))$.*

To understand what are the states to which we may stabilize a BCN (7), we have first to look into the set of all equilibrium points of the various subsystems. This requires to determine all the indices $i \in [1, 2^n]$ such that there exists $j \in [1, 2^m]$ for which $\mathrm{col}_i(L_j) = \delta_{2^n}^i$. Once we have identified the set $\mathcal{X}_e$ of all such canonical vectors, we have to verify which of them are reachable from any initial state. This amounts to checking which rows of $\mathbb{L}$ have all unitary entries. So, to conclude, the BCN will be stabilizable to all vectors $\delta_{2^n}^i \in \mathcal{X}_e$ such that the ith row of $\mathbb{L}$ is all unitary.

4 Feedback stabilization to a limit cycle

In the previous section we have investigated the stabilization problem to some cyclic trajectory. This problem has been addressed by assuming that at every time instant $t \in \mathbb{Z}_+$ the input variable $\mathbf{u}(t)$ can be freely chosen in $\mathcal{L}_{2^m}$. We want to show now that this stabilization problem can be solved by means of a feedback law, by this meaning that at every time instant t the input $\mathbf{u}(t)$ can be expressed as $\mathbf{u}(t) = K\mathbf{x}(t)$, for some matrix $K \in \mathcal{L}_{2^m \times 2^n}$. We provide the following result, whose proof is based on some ideas appearing in [5] (see page 358, Theorem 15.2).

Proposition 4 *If a BCN (7) is stabilizable to some elementary cycle $\mathcal{C} = (\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_k})$, then it is stabilizable by means of a feedback law.*

Proof. If the BCN is stabilizable to $\mathcal{C}$, then conditions 1) and 2) of Proposition 3 hold. We want to make use of these two conditions to define the columns of K, one by one. We first consider the indices $i_1, i_2, \ldots, i_k \in \mathcal{C}$. By condition 1), we know that, for every $\mathbf{x} = \delta_{2^n}^{i_\ell}, \ell \in [1, k]$, there exists $\mathbf{u} = \delta_{2^m}^{j_\ell}, j_\ell \in [1, 2^m]$, such that $\delta_{2^n}^{i_{\ell+1}} = L \ltimes \delta_{2^m}^{j_\ell} \ltimes \delta_{2^n}^{i_\ell} = L \ltimes \mathbf{u} \ltimes \mathbf{x}$ (where $i_{k+1} = i_1$), and hence it is sufficient to impose

$$\mathbf{u} = \delta_{2^m}^{j_\ell} = K\delta_{2^n}^{i_\ell} = K\mathbf{x}, \qquad \forall\, \ell \in [1, k],$$

which amounts to imposing $\mathrm{col}_{i_\ell}(K) := \delta_{2^m}^{j_\ell}$.

Let $\mathcal{S}_t, t \in \mathbb{Z}_+$, denote the set of all states $\delta_{2^n}^i, i \in [1, 2^n]$, whose minimum distance from the cycle $\mathcal{C}$ is t, by this meaning that the length of the shortest part from the state $\delta_{2^n}^i$ to any state of $\mathcal{C}$ is just t. Clearly, $\mathcal{S}_0 = \mathcal{C}$, and $\mathcal{S}_{t+1} \neq \emptyset$ implies $\mathcal{S}_t \neq \emptyset$. On the other hand, for every $t > 2^n - k$, $\mathcal{S}_t = \emptyset$. Finally, by assumption 2),

$$\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_{2^n - k} = \mathcal{L}_{2^n},$$

and all sets $\mathcal{S}_t$ are disjoint. Since for every $\mathbf{x} = \delta_{2^n}^i \in \mathcal{S}_{t+1}$ there exists $\mathbf{u} = \delta_{2^m}^j$ such that $L \ltimes \delta_{2^m}^j \ltimes \delta_{2^n}^i \in \mathcal{S}_t$, it is easy to see that by assuming $\mathrm{col}_i(K) := \delta_{2^m}^j$, for every $\delta_{2^n}^i \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_{2^n-k}$, we assign all the remaining columns of K. Therefore, the feedback law $\mathbf{u}(t) = K\mathbf{x}(t)$ allows to converge to $\mathcal{C}$ and to remain therein. $\square$

Example 1 *Consider a BCN (7), with $n = 3$ and $m = 1$, and suppose that*

$$L_1 := L \ltimes \delta_2^1 = [\,\delta_8^2 \quad \delta_8^3 \quad \delta_8^4 \quad \delta_8^5 \quad \delta_8^2 \quad \delta_8^7 \quad \delta_8^8 \quad \delta_8^3\,],$$
$$L_2 := L \ltimes \delta_2^2 = [\,\delta_8^5 \quad \delta_8^1 \quad \delta_8^6 \quad \delta_8^2 \quad \delta_8^4 \quad \delta_8^7 \quad \delta_8^8 \quad \delta_8^8\,].$$

The BCN can be described by the following digraph, where the $2^3 = 8$ states of the BCN are represented by vertices, blue continuous arcs correspond to the unitary entries of L_1, while red dashed arcs to entries of L_2.

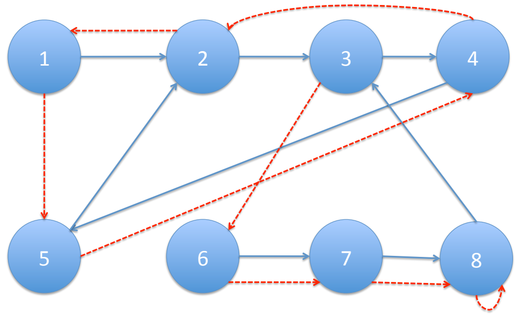

FIG. 1. Digraph corresponding to the BCN of Example 1.

A possible limit cycle is $\mathcal{C} = (\delta_8^2, \delta_8^3, \delta_8^4)$. The transition from δ_8^2 to δ_8^3 and from δ_8^3 to δ_8^4 is due to blue continuous arcs (and hence to L_1),while the transition from δ_8^4 to δ_8^2 to a red dashed arc (to L_2). Accordingly we have that

$$K\delta_4^2 = \delta_2^1, \qquad K\delta_4^3 = \delta_2^1, \qquad K\delta_4^4 = \delta_2^2.$$

If we consider now the vertices that have distance t from $\mathcal{C}$, we find

i) $\mathcal{S}_1 = \{\delta_8^1, \delta_8^5, \delta_8^8\}$;
ii) $\mathcal{S}_2 = \{\delta_8^7\}$;
iii) $\mathcal{S}_3 = \{\delta_8^6\}$.

Keeping in mind what are the arcs that belong to the shortest paths from each of these vertices to $\mathcal{C}$, we obtain as possible feedback matrices all matrices

$$K = [\,\delta_2^1 \quad \delta_2^1 \quad \delta_2^1 \quad \delta_2^2 \quad * \quad * \quad * \quad \delta_2^1\,],$$

where $*$ denotes columns that can be either δ_2^1 or δ_2^2.

Corollary 3 *Given a controllable BCN (7), for every elementary cycle $\mathcal{C} = (\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_k})$ appearing in $\mathcal{D}(L_{tot})$, there exists a state-feedback matrix $K_{\mathcal{C}}$ that drives all state trajectories to $\mathcal{C}$.*

Proof. If $\mathcal{C}$ is an elementary cycle in $\mathcal{D}(L_{tot})$, for every pair of vertices $(i_\ell, i_{\ell+1}) \in \mathcal{D}(L_{tot})$, there is an arc connecting them. This means that for every pair of states

$(\delta_{2^n}^{i_\ell}, \delta_{2^n}^{i_{\ell+1}})$, $\ell \in [1, k]$, with $i_{k+1} = i_1$, there exists a value of $\mathbf{u}(t)$, say $\delta_{2^m}^{j_\ell}$, that allows the transition from $\mathbf{x}(t) = \delta_{2^n}^{i_\ell}$ to $\mathbf{x}(t+1) = \delta_{2^n}^{i_{\ell+1}}$. On the other hand, controllability ensures that $\delta_{2^n}^{i_1}$ is surely reachable from every $\mathbf{x}(0)$. Since both conditions 1) and 2) of Proposition 3 are satisfied, by proceeding as in the previous proof, we can explicitly construct one such matrix $K_{\mathcal{C}}$. $\square$

If the BCN is not controllable, the matrix L_{tot} is not irreducible, and hence $\mathcal{D}(L_{tot})$ is not strongly connected. Consequently, $\mathcal{D}(L_{tot})$ can be partitioned into communication classes, and the BCN can be stabilized to some limit cycle if and only if there exists a communication class that is accessible from every other class of $\mathcal{D}(L_{tot})$.

Remark 1 *If we stabilize a BCN (7) to some elementary cycle by means of a feedback law $\mathbf{u}(t) = K\mathbf{x}(t)$, the resulting BCN can be described as*

$$\mathbf{x}(t+1) = L \ltimes K \ltimes \mathbf{x}(t) \ltimes \mathbf{x}(t) = L \ltimes K \ltimes \Phi \ltimes \mathbf{x}(t),$$

where $\Phi \in \mathcal{L}_{(2^n)^2 \times 2^n}$ is the group power reducing matrix *[5] acting as $\mathbf{x}(t) \ltimes \mathbf{x}(t) = \Phi\mathbf{x}(t)$. The effect of the stabilization is that the matrix $L \ltimes K \ltimes \Phi \in \mathcal{L}_{2^n \times 2^n}$ can be reduced, by means of a permutation matrix to the form*

$$\begin{bmatrix} N & 0 \\ T & C \end{bmatrix},$$

where C is a cyclic permutation matrix, and N a nilpotent matrix.

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