State-feedback stabilization of multi-input compartmental systems

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Abstract. In this paper we address the positive (state-feedback) stabilization of multi-input compartmental systems, i.e., the design of a state-feedback matrix that preserves the compartmental property of the resulting feedback system, while achieving stability. We first provide necessary and sufficient conditions for the positive stabilizability of compartmental systems whose state matrix is irreducible. Then we address the case when the state matrix is reducible, identify two sufficient conditions for the problem solution, and then extend them to a general algorithm that allows to verify when the problem is solvable and to produce a solution.

1 Introduction

The stabilization of positive systems and the dual problem of positive observer design have been the subject of several papers (see, e.g., [1, 5, 7, 8, 9, 13, 18, 19, 26]). Most of the literature focused on the general class of positive systems and translated the positive stabilization problem either into a Linear Matrix Inequality (LMI) [13], or into a Linear Programming (LP) problem [18], by making use of the fact that the positive/Metzler matrix of the system obtained by means of a state-feedback is Schur/Hurwitz if and only if it admits a positive diagonal Lyapunov function (condition that leads to the LMI formulation) or a linear copositive Lyapunov function (condition that leads to an LP condition). The solution in terms of LP, even if equivalent from a theoretical viewpoint, is preferable due to its lower computational complexity. Moreover, it lends itself to be easily extended to cope with robust stabilization in the presence of polytopic uncertainties, stabilization with restricted sign controls and stabilization with bounded controls [18].

Alternative approaches to the positive stabilization problem have been proposed in [19] and [5]. The characterization derived in [19] is based on the construction of certain polytopes and on verifying whether a selection of their vertices can be used to construct a stabilizing state-feedback matrix. On the other hand, in [5] the problem of achieving by means of a state-feedback not only positivity and stability, but also certain L_1 and L_∞ performance, has been investigated. Also in this case, necessary and sufficient conditions for the existence of a solution have been expressed as LPs.

(Linear) compartmental systems are a special class of positive state-space models that represent physical systems in which units, called compartments, exchange material and are subject to the law of mass conservation. Such systems were first introduced in physiology [15] and they

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are characterized by the fact that their state variables are nonnegative and their sum, $\sum_{i=1}^{n} x_i(t)$, cannot increase with time. For a general introduction to compartmental systems we refer the interested reader to [14, 16].

The (positive) stabilization of single-input compartmental systems has been thoroughly investigated in [8] (see also [24]): very strong characterisations, that rely only on the nonzero patterns of the matrices involved in the system description, have been derived. These characterizations do not find a straighforward extension to the class of multi-input compartmental systems, for which positive stabilizability also depends on the specific entries of the involved matrices and not only on their nonzero patterns (see Example 10 below). On the other hand, the only results available in the multi-input case are simply the aforementioned ones, derived for the general class of multi-input positive systems. It turns out that the compartmental property allows to obtain much stronger characterizations of the positive stabilizability property. Even more, it allows to considerably simplify the LPs that provide conditions that are equivalent to the existence of a solution.

In this paper we investigate the positive stabilization of multi-input compartmental systems, by first showing that when the original system matrix A is irreducible, the positive stabilization problem is solvable if and only if it can be solved by resorting to a state-feedback that depends on a single compartment. Necessary and sufficient conditions for this to be the case are given, in the form of Linear Programming: since these conditions involve only a single column, they are quite simpler than the general ones obtained for multi-input positive systems.

When the system matrix is reducible on the other hand, we first provide two sets of sufficient conditions for positive stabilizability that involve a very low number of system compartments, and are based on the property that a compartmental matrix is Hurwitz if and only if all its compartments are outflow connected. The intuition behind these two sufficient conditions is then formalized in graph terms and this allows to provide a necessary and sufficient condition for positive stabilization in the form of an algorithm. The algorithm provides a solution having a number of nonzero columns that does not exceed the number of communication classes of the original compartmental state matrix. This means that the solution modifies the outflow of a minimal number of compartments, specifically, at most one per communication class. From a practical point of view, this means that the state-feedback law is expressed in terms of the values of a small subset of the state variables, a property that may be extremely convenient when sensors are expensive or quite difficult to locate. For instance, it may be the case that very few state variables are actually available for measurements and in this sense the proposed solution is extremely convenient. On the other hand, in order to be able to exploit the proposed algorithm, the knowledge of the communication classes of the digraph associated with the matrix A is required. If this information is available, the algorithm can impose a significantly lower computational burden with respect to the one required to solve the LPs proposed in [18] for the general case of an unstructured pair of matrices (A, B).

The paper is organised as follows. In section 2, notation, mathematical preliminaries and problem statement are given. In section 3, three technical lemmas are given. As a starting point, in section 4, the class of compartmental systems whose system matrix is irreducible is thoroughly investigated. In section 5, the analysis is extended to address the case when the system matrix is reducible, and conditions that ensure positive stabilizability are given. Finally, section 6 presents a necessary and sufficient condition for positive stabilization of multi-input compartmental systems in the form of an algorithm. Examples illustrate the various conditions provided in the paper. A preliminary version of the first part of this paper was presented at the IEEE Conference on Decision and Control, CDC 2017, in Melbourne, Australia [25]. In [25] we have investigated only the irreducible case and provided one sufficient condition for the solvability in the reducible case. So, the second part of section 5 (starting from Proposition 17) and the whole section 6, where the final problem solution and the algorithm to obtain it are derived, are the novel contribution of this paper.

2 Preliminaries and problem statement

Given $k, n \in \mathbb{Z}$, with $k \leq n$, the symbol $[k, n]_{\mathbb{Z}}$ denotes the integer set $\{k, k+1, \dots, n\}$, namely $[k, n] \cap \mathbb{Z}$. The semiring of nonnegative real numbers is denoted by $\mathbb{R}_+$. In the sequel, the (i, j)th entry of a matrix A is denoted by $[A]_{i,j}$, while the ith entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_i$. Following [10], we adopt the following terminology and notation. Given a matrix A with entries $[A]_{i,j}$ in $\mathbb{R}_+$, we say that A is a *nonnegative matrix*, if all its entries are nonnegative, namely $[A]_{i,j} \geq 0$ for every i, j, and if so we use the notation $A \geq 0$. If A is a nonnegative matrix, and $A \neq 0$, then A is said to be a *positive matrix* and we adopt the notation $A > 0$. Note that $A > 0$ does not mean that all its entries are positive, but simply that at least one of them is positive and the remaining ones are nonnegative. Notation $A \geq B$ ($A > B$) means $A - B \geq 0$ ($A - B > 0$). The symbols $\leq$ and $<$ are defined accordingly. Also, the same notation is adopted for vectors.

We let $\mathbf{e}_i$ denote the ith vector of the canonical basis in $\mathbb{R}^n$ (where n is always clear from the context), whose entries are all zero except for the ith one that is unitary. The symbol $\mathbf{1}$ denotes a vector with all entries equal to 1 (and whose size is clear from the context). Given a matrix $A \in \mathbb{R}^{n \times m}$ (in particular, a vector), its *nonzero pattern* $\overline{\mathrm{ZP}}(A)$ is the set $\{(i, j) \in [1, n]_{\mathbb{Z}} \times [1, m]_{\mathbb{Z}} : [A]_{i,j} \neq 0\}$. We denote by $S_i \in \mathbb{R}^{(n-1) \times n}$ the selection matrix obtained by removing the ith row in the identity matrix I_n, namely

$$S_i = \left[\begin{array}{c|c|c} I_{i-1} & 0 & 0_{(i-1)\times(n-i)} \\ \hline 0_{(n-i)\times(i-1)} & 0 & I_{n-i} \end{array}\right].$$

The size n will always be clear from the context, namely from the size of the matrix or vector S_i is applied to. For any matrix $A \in \mathbb{R}^{n \times m}$, $S_i A$ denotes the matrix obtained from A by removing the ith row, while for any vector $\mathbf{v} \in \mathbb{R}^n$, $S_i \mathbf{v}$ is the vector obtained from $\mathbf{v}$ by removing the ith entry. A real square matrix A is *Hurwitz* if all its eigenvalues lie in the open left complex halfplane, i.e. for every λ belonging to the spectrum $\sigma(A)$ of A we have $\mathrm{Re}(\lambda) < 0$.

A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. This is equivalent to saying that for every $h \in [1, n]_{\mathbb{Z}}$ the vector $S_h A \mathbf{e}_h$ is nonnegative. For $n \geq 2$, an $n \times n$ nonzero Metzler matrix A is *reducible* [11, 17] if there exists a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ and $A_{2,2}$ are square (nonvacuous) matrices, otherwise it is *irreducible*. In general, given a Metzler matrix A, a permutation matrix Π can be found such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,s} \\ 0 & A_{2,2} & \dots & A_{2,s} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{s,s} \end{bmatrix}, \tag{1}$$

where each diagonal block $A_{i,i}$, of size $n_i \times n_i$, is either scalar ($n_i = 1$) or irreducible. Equation (1) is usually referred to as *Frobenius normal form* of A [12, 17].

If A is an $n \times n$ Metzler matrix, then as proved in [22] it exhibits a real dominant (not necessarily simple) eigenvalue, known as *Frobenius eigenvalue* and denoted by $\lambda_F(A)$. This means that $\lambda_F(A) > \mathrm{Re}(\lambda), \forall\ \lambda \in \sigma(A), \lambda \neq \lambda_F(A)$.

Basic definitions and results about cones may be found, for instance, in [2, 4]. We recall here only those facts that will be used within this paper. A set $\mathcal{K} \subset \mathbb{R}^n$ is said to be a *cone* if $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$; a cone is *convex* if it contains, with any two points, the line segment between them. A convex cone $\mathcal{K}$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of *generating vectors*. This means that a positive integer k and a matrix $W \in \mathbb{R}^{n\times k}$ can be found, such that $\mathcal{K}$ coincides with the set of nonnegative combinations of the columns of W. In this case, we adopt the notation $\mathcal{K} := \mathrm{Cone}(W)$. A convex cone $\mathcal{K}$ is polyhedral if and only if there exists a matrix $C \in \mathbb{R}^{p\times n}$ such that $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n : C\mathbf{x} \geq 0\}$.

A Metzler matrix endowed with the additional property that the entries of each of its columns sum up to a nonpositive number, i.e., $\mathbf{1}^\top A \leq 0^\top$, is called *compartmental matrix* (see [14, 21]). For any such matrix the Frobenius eigenvalue $\lambda_F(A)$ is nonpositive, and if $\lambda_F(A) = 0$ then A is simply stable, i.e., it has the constant mode associated with $\lambda_F(A) = 0$, but no unstable modes.

In this paper we will focus on compartmental models, which are typically used to describe material or energy flows among compartments of a system. Each compartment represents a homogeneous entity within which the entities being modelled are equivalent. An n-dimensional *multi-input linear* compartmental system is a linear state-space model

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is a compartmental matrix, $B \in \mathbb{R}_+^{n\times m}$ is a positive matrix, devoid of zero columns, and $m > 1$. The size n of the state-space model represents the number of compartments, and the ith entry of the state vector, $[\mathbf{x}(t)]_i$, represents the content of the ith compartment at time t. In the sequel we refer to multi-input linear compartmental systems (2) simply as compartmental systems.

Given a compartmental matrix $A \in \mathbb{R}^{n\times n}$ (a compartmental system (2)), we associate with it [6, 20] a *digraph* $\mathcal{D}(A) = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \dots, n\} = [1, n]_{\mathbb{Z}}$ is the set of vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs (or edges). Vertices and compartments are related by a bijective correspondence, and hence in the following they will be treated as equivalent. Given $j, \ell \in \mathcal{V}, j \neq \ell$, there is an arc $(j, \ell) \in \mathcal{E}$ from j to ℓ if and only if $[A]_{\ell,j} > 0$. A sequence $j_1 \to j_2 \to \cdots \to j_k \to j_{k+1}$ is a *path* of length k from j_1 to j_{k+1} provided that $(j_1, j_2), \dots, (j_k, j_{k+1})$ are elements of $\mathcal{E}$. We say that vertex ℓ is *accessible* from j if there exists a path in $\mathcal{D}(A)$ from j to ℓ. Two distinct vertices ℓ and j are said to *communicate* if each of them is accessible from the other, i.e., there is both a path from ℓ to j and a path from j to ℓ. We assume, by definition, that each vertex communicates with itself. It is easy to see that communication between vertices represents an equivalence relation in $\mathcal{V} \times \mathcal{V}$ that allows to partition the set of vertices $\mathcal{V}$ into equivalence classes called *communication classes*, say $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_s$. Class $\mathcal{C}_j$ has access to class $\mathcal{C}_i$ if there is a path from some vertex $k \in \mathcal{C}_j$ to some vertex $h \in \mathcal{C}_i$. Each class $\mathcal{C}_i$ has clearly access to itself. The digraph $\mathcal{D}(A)$ is said to be *strongly connected* if every pair of vertices ℓ and j communicate, and hence it consists of a single communication class. The digraph $\mathcal{D}(A)$ is strongly connected if and only if A is irreducible.

In this standard set-up we introduce an additional specification: we partition the set of vertices $\mathcal{V}$ into two disjoint subsets $\mathcal{V}^0$ and $\mathcal{V}^-$, where

$$\mathcal{V}^0 := \{i \in \mathcal{V} : \mathbf{1}^\top A \mathbf{e}_i = 0\} \qquad \mathcal{V}^- := \{i \in \mathcal{V} : \mathbf{1}^\top A \mathbf{e}_i < 0\}.$$

A vertex i (a compartment i) is said to have *direct outflow to the environment* [3, 16] if it belongs to $\mathcal{V}^-$, while vertices belonging to $\mathcal{V}^0$ have not direct outflow. The vertex i (the ith compartment) is said to be *outflow connected* [3, 16] if there is a path in $\mathcal{D}(A)$ from that vertex to some vertex j (i.e., from the ith compartment to some jth compartment) belonging to $\mathcal{V}^-$.

Example 1. *Consider the following compartmental matrix*

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix}.$$

It is easy to see that there are two communication classes, $\mathcal{C}_1 = [1,2]_{\mathbb{Z}}$ and, $\mathcal{C}_2 = [3,4]_{\mathbb{Z}}$, and class $\mathcal{C}_2$ has access to class $\mathcal{C}_1$ (but the converse is not true). None of the 4 vertices (compartments) is outflow connected. The digraph associated with A is illustrated in Figure 1. We have represented vertices in $\mathcal{V}^0$ with continuous line circles (while for vertices in $\mathcal{V}^-$ we will use dashed line circles). ♣

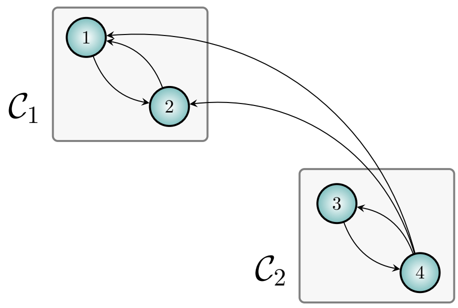

Fig.1 Digraph associated with matrix A of Example 1.

In the paper we investigate the following problem:

Positive stabilization problem: Consider the compartmental system (2) and assume that A is compartmental and non-Hurwitz. Under what conditions does there exist $K \in \mathbb{R}^{m\times n}$ such that the state-feedback control law $\mathbf{u}(t) = K\mathbf{x}(t)$ makes the closed-loop system asymptotically stable while preserving the compartmental property, i.e., it makes $A + BK$ compartmental and Hurwitz?

We say that the positive stabilization problem is solvable if there exists $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is compartmental and Hurwitz.

3 Some technical lemmas

To solve the positive stabilization problem, we start by introducing the following technical results that will be useful in the subsequent analysis.

Lemma 2. *[3, 16] A compartmental matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz if and only if all its compartments are outflow connected.*

Lemma 3. *[16, 23] An irreducible compartmental matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz if and only if $\mathbf{1}^\top A < 0^\top$.*

Remark 4. *As an immediate consequence, if the compartmental matrix A is irreducible and non-Hurwitz, then $\mathbf{1}^\top A = 0^\top$.*

Lemma 5. *[24] Let $A \in \mathbb{R}^{n\times n}$ be a reducible compartmental matrix. A is non-Hurwitz if and only if $r \geq 1$ and a permutation matrix Π can be found such that $\Pi^\top A\Pi$ has the following Frobenius form, with either scalar or irreducible diagonal blocks $A_{i,i} \in \mathbb{R}^{n_i\times n_i}, i \in [1,s]_{\mathbb{Z}}$:*

$$\Pi^\top A\Pi = \left[\begin{array}{ccccc|cccc} A_{1,1} & 0 & \dots & \dots & 0 & A_{1,r+1} & \dots & \dots & A_{1,s} \\ 0 & A_{2,2} & 0 & \dots & 0 & A_{2,r+1} & \dots & \dots & A_{2,s} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & & & \vdots \\ 0 & \dots & \dots & 0 & A_{r,r} & A_{r,r+1} & \dots & \dots & A_{r,s} \\ \hline 0 & \dots & \dots & \dots & 0 & A_{r+1,r+1} & \dots & \dots & A_{r+1,s} \\ \vdots & & & & \vdots & 0 & A_{r+2,r+2} & \ddots & A_{r+2,s} \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & 0 & A_{s,s} \end{array}\right], \tag{3}$$
$$\text{with } \ \lambda_F(A_{i,i}) = 0, \forall \ i \in [1,r]_{\mathbb{Z}} \text{ and } \ \lambda_F(A_{i,i}) < 0, \forall \ i \in [r+1,s]_{\mathbb{Z}}.$$

Consequently, $\mathbf{1}^\top A_{i,i} = 0^\top, \forall \ i \in [1,r]_{\mathbb{Z}}$, and $\mathbf{1}^\top A_{i,i} < 0^\top, \forall \ i \in [r+1,s]_{\mathbb{Z}}$.

Remark 6. *It is worth noticing that the Frobenius form (3) immediately identifies the communication classes of the associated digraph $\mathcal{D}(\Pi^\top A\Pi)$. Indeed, each scalar or irreducible diagonal block $A_{i,i}$ identifies one communication class. So, it entails no loss of generality identifying the communication class $\mathcal{C}_i$ with the set of vertices corresponding to the diagonal block $A_{i,i}$. In other words,*

$$\mathcal{C}_i := \left[\sum_{k=1}^{i-1} n_k + 1, \sum_{k=1}^{i} n_k\right]_{\mathbb{Z}}, \qquad i \in [1,s]_{\mathbb{Z}}, \tag{4}$$

where n_k denotes the size of the diagonal block $A_{k,k}$.

We will refer to the classes $\mathcal{C}_i, i \in [1,r]_{\mathbb{Z}}$, as to the *conservative classes* since the corresponding (irreducible) matrix $A_{i,i}$ is not Hurwitz.

4 Problem solution: the case when A is irreducible

In this section we investigate the existence of a matrix $K \in \mathbb{R}^{m\times n}$ solving the positive stabilization problem for the compartmental system (2), under the assumption that A is irreducible. As a first step we introduce the following lemma, which will be fundamental to solve the positive stabilization problem.

Lemma 7. *Consider the compartmental system* (2) *and assume that* A *is irreducible. If there exist* $\mathbf{v} \in \mathbb{R}^m$ *and* $\ell \in [1,n]_{\mathbb{Z}}$ *such that*

$$S_\ell(A\mathbf{e}_\ell + B\mathbf{v}) \geq 0, \tag{5a}$$

$$\mathbf{1}^\top B\mathbf{v} < 0, \tag{5b}$$

then for every $\varepsilon \in (0,1)$ *the feedback matrix* $K = \varepsilon \mathbf{v}\mathbf{e}_\ell^\top \in \mathbb{R}^{m\times n}$ *makes* $A+BK$ *compartmental, irreducible, and Hurwitz.*

Proof. Condition (5a) ensures that for every $\varepsilon \in (0,1)$ the following two conditions hold

$$S_\ell A\mathbf{e}_\ell + \varepsilon S_\ell B\mathbf{v} \geq 0,$$
$$\overline{\mathrm{ZP}}(S_\ell A\mathbf{e}_\ell + \varepsilon S_\ell B\mathbf{v}) \supseteq \overline{\mathrm{ZP}}(S_\ell A\mathbf{e}_\ell).$$

This implies that for every $\varepsilon \in (0,1)$ the matrix $A+\varepsilon B\mathbf{v}\mathbf{e}_\ell^\top$ is Metzler and irreducible. If (5b) holds, then it is also true that for every $\varepsilon \in (0,1)$ one has $\mathbf{1}^\top(A\mathbf{e}_\ell+\varepsilon B\mathbf{v}) < 0$, and hence $\mathbf{1}^\top(A+\varepsilon B\mathbf{v}\mathbf{e}_\ell^\top) < 0^\top$. This ensures that $A+BK$ is compartmental (irreducible) and also Hurwitz (see Lemma 3). □

Remark 8. *We would like to comment on the expression* $K = \varepsilon\mathbf{v}\mathbf{e}_\ell^\top$, *with* ε *arbitrary in* $(0,1)$, *adopted to express in Lemma 7, above, a state-feedback matrix* K *that solves the positive stabilization problem. In order to use condition* $\mathbf{1}^\top(A+BK) < 0^\top$ *to claim that* $A+BK$ *is Hurwitz, one needs to ensure that* $A+BK$ *is irreducible. Since* A *is an irreducible matrix, to guarantee that* $A+BK$ *is irreducible in turn, it is sufficient to ensure that none of the positive off-diagonal entries of* A *has become a zero entry in* $A+BK$. *Specifically, we have to rule out the case when there exists some index* $j \in [1,n]_{\mathbb{Z}}, j \neq \ell$, *such that* $[A]_{j,\ell} > 0$ *while* $[A+BK]_{j,\ell} = 0$. *To this end, if conditions* (5) *hold, it is sufficient to assume that* ε *is positive but smaller than* 1. *Clearly, any specific value of* $\varepsilon \in (0,1)$ *would lead to a specific solution of the positive stabilization problem. On the other hand, it is also clear that the matrices expressed in the form* $K = \varepsilon\mathbf{v}\mathbf{e}_\ell^\top \in \mathbb{R}^{m\times n}$, *with* $\varepsilon \in (0,1)$, *do not provide a complete parametrization of the solutions. For instance, if for a fixed* ℓ *different vectors* $\mathbf{v}_i$ *would be found satisfying* (5), *then every convex combination* $\sum_i \varepsilon_i \mathbf{v}_i$, *with* $\varepsilon_i \in (0,1)$ *for every* i, *would be a solution. However, deriving a complete parametrization of the problem solutions is not one of the goals of this paper. Finally, it is worth mentioning that the need to preserve the irreducibility property of the compartmental system matrix after state-feedback will be clear in the following sections, when we will deal with the reducible case, and we will try to ensure that the irreducibility of the diagonal blocks is preserved when moving from* A *to* $A+BK$.

Proposition 9 below states that when A is irreducible and the positive stabilization problem is solvable, there always exists a solution K with a unique nonzero column.

Proposition 9. *Consider the compartmental system* (2) *and assume that* A *is irreducible and non-Hurwitz. If the positive stabilization problem is solvable, then there exist* $\mathbf{v} \in \mathbb{R}^m$ *and* $\ell \in [1,n]_{\mathbb{Z}}$ *such that for every* $\varepsilon \in (0,1)$ *the matrix* $K := \varepsilon\mathbf{v}\mathbf{e}_\ell^\top \in \mathbb{R}^{m\times n}$ *is a solution.*

Proof. Let $\bar{K} \in \mathbb{R}^{m\times n}$ be a solution of the positive stabilization problem. Let $\ell \in [1,n]_{\mathbb{Z}}$ be such that $0 > \mathbf{1}^\top(A+B\bar{K})\mathbf{e}_\ell = \mathbf{1}^\top A\mathbf{e}_\ell + \mathbf{1}^\top B\bar{K}\mathbf{e}_\ell = \mathbf{1}^\top B\bar{K}\mathbf{e}_\ell$ (such an index exists, otherwise $\lambda_F(A+B\bar{K}) = 0$), and set $\mathbf{v} := \bar{K}\mathbf{e}_\ell$. Clearly, $\mathbf{1}^\top B\mathbf{v} < 0$, and $S_\ell(A\mathbf{e}_\ell + B\mathbf{v}) = S_\ell(A+B\bar{K})\mathbf{e}_\ell \geq 0$. So, by Lemma 7, for every $\varepsilon \in (0,1)$ the feedback matrix $K = \varepsilon\mathbf{v}\mathbf{e}_\ell^\top \in \mathbb{R}^{m\times n}$ makes $A+BK$ irreducible, compartmental and Hurwitz. □

Example 10. *Consider the compartmental system*

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -5 & 0 \\ 0 & 5 & -1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}\mathbf{u}(t).$$

Notice that A is an irreducible and non-Hurwitz matrix. Let $\mathbf{b}_i$ denote the ith column of B, $i = 1,2$. We preliminarily observe that positive stabilization from a single input is not possible. Indeed, if a solution using only the ith input would be possible, then as shown in [24] a row vector $\mathbf{k}_i^\top \in \mathbb{R}^{1\times 3}$, with $\mathbf{k}_i < 0$, could be found such that $A+\mathbf{b}_i\mathbf{k}_i^\top$ is compartmental and Hurwitz. However, the nonzero patterns of A and B clearly show that this is possible neither for $i = 1$ nor for $i = 2$. In general, the necessary and sufficient condition derived in [24] requires to verify that there exists $i \in [1,2]_{\mathbb{Z}}$ and $\ell \in [1,3]_{\mathbb{Z}}$ such that condition $\overline{\mathrm{ZP}}(S_\ell\mathbf{b}_i) \subseteq \overline{\mathrm{ZP}}(S_\ell A\mathbf{e}_\ell)$ holds. But it is easily seen that this is impossible. Focusing now on the state-feedback from both inputs, it is easy to verify that any feedback matrix of the form

$$K = \begin{bmatrix} k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R},$$

does not solve the positive stabilization problem. However, for every $\varepsilon \in (0,1)$ the two feedback matrices

$$K = \varepsilon\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \text{ and } K = \varepsilon\begin{bmatrix} 0 & 0 & -0.5 \\ 0 & 0 & 0.25 \end{bmatrix},$$

make $A+BK$ compartmental and Hurwitz. ♣

The previous example shows that, even if a solution $K \in \mathbb{R}^{m\times n}$ with a unique nonzero column can be found, not all columns (i.e., not all indices $\ell \in [1,n]_{\mathbb{Z}}$) play an equivalent role. The following proposition provides, for a fixed $\ell \in [1,n]_{\mathbb{Z}}$, equivalent conditions for the existence of a vector $\mathbf{k} \in \mathbb{R}^m$ such that $K := \mathbf{k}\mathbf{e}_\ell^\top$ is a solution.

Proposition 11. *Consider the compartmental system (2) and assume that A is irreducible and non-Hurwitz. Let ℓ be a fixed index in $[1,n]_{\mathbb{Z}}$ and introduce the set $L := \{j \in [1,n]_{\mathbb{Z}},\ j \neq \ell : [A]_{j,\ell} = 0\}$. Let B_L be the matrix obtained by selecting only the rows of B indexed in L, and introduce the cone $\mathcal{K}_L := \{\mathbf{y} \in \mathbb{R}^m : B_L\mathbf{y} \geq 0\}$. Being $\mathcal{K}_L$ a convex polyhedral cone, there exists $W := \begin{bmatrix} \mathbf{w}_1 & \dots & \mathbf{w}_N \end{bmatrix} \in \mathbb{R}^{m\times N}$ such that $\mathcal{K}_L = \mathrm{Cone}(W)$. The following facts are equivalent:*

(i) There exists $\mathbf{v} \in \mathbb{R}^m$ such that conditions (5) hold;

(ii) The N-dimensional row vector $\mathbf{1}^\top BW$ has at least one negative entry;

(iii) There exists $i \in [1,N]_{\mathbb{Z}}$ such that the vector $\mathbf{w}_i = W\mathbf{e}_i$ satisfies

$$\mathbf{1}^\top B\mathbf{w}_i < 0 \tag{6a}$$
$$[B\mathbf{w}_i]_j \geq 0,\ \forall j \in L; \tag{6b}$$

(iv) There exists $\mathbf{k} \in \mathbb{R}^m$ such that $K = \mathbf{k}\mathbf{e}_\ell^\top \in \mathbb{R}^{m\times n}$ is a solution to the positive stabilization problem.

Proof. (i) ⇒ (ii) Clearly, any vector $\mathbf{v} \in \mathbb{R}^m$ satisfying condition (5a) is such that $B_L\mathbf{v} \geq 0$. This means that $\mathbf{v} \in \mathcal{K}_L = \text{Cone}(W)$, and hence there exists $\mathbf{u} \in \mathbb{R}^N_+$, such that $\mathbf{v} = W\mathbf{u}$. Then, condition (5b) can be rewritten as $0 > \mathbf{1}^\top B\mathbf{v} = \mathbf{1}^\top BW\mathbf{u}$, and this implies that the vector $\mathbf{1}^\top BW$ has at least one negative entry, namely (ii) holds.

(ii) ⇒ (iii) Let $i \in [1,N]_{\mathbb{Z}}$ be such that $0 > [\mathbf{1}^\top BW]_i = \mathbf{1}^\top BW\mathbf{e}_i = \mathbf{1}^\top B\mathbf{w}_i$. Clearly, $\mathbf{w}_i$ satisfies condition (6a). Moreover, since $\mathbf{w}_i \in \text{Cone}(W) = \mathcal{K}_L$, we have $B_L\mathbf{w}_i \geq 0$, namely $[B\mathbf{w}_i]_j \geq 0$ for every $j \in L$.

(iii) ⇒ (i) We want to prove that if (iii) holds, there exists $\varepsilon > 0$ such that $\mathbf{v} := \varepsilon\mathbf{w}_i$ satisfies (i). Clearly, $\mathbf{1}^\top B\mathbf{v} = \varepsilon\mathbf{1}^\top B\mathbf{w}_i < 0$ for every $\varepsilon > 0$, namely $\mathbf{v}$ satisfies (5b) for every $\varepsilon > 0$. Moreover, since $\mathbf{v} \in \mathcal{K}_L$, we have $B_L\mathbf{v} \geq 0$ for every $\varepsilon > 0$, namely $[B\mathbf{v}]_j \geq 0$ for every $j \in L$, $\varepsilon > 0$. Consequently, when $L \cup \{\ell\} = [1,n]_{\mathbb{Z}}$ the vector $\mathbf{v}$ satisfies (5a) for every $\varepsilon > 0$. On the other hand, when $L \cup \{\ell\} \subsetneq [1,n]_{\mathbb{Z}}$, for every $j \notin L$, $j \neq \ell$, we have $[A]_{j,\ell} > 0$, and hence there always exists $\varepsilon_j > 0$ such that $[A]_{j,\ell} + \varepsilon_j\mathbf{e}_j^\top B\mathbf{w}_i \geq 0$. So, if we choose $\varepsilon := \min_{j\notin(L\cup\{\ell\})}\varepsilon_j$, the vector $\mathbf{v}$ satisfies condition (5a).

(i) ⇒ (iv) Follows from Lemma 7, by assuming $\mathbf{k} = \varepsilon\mathbf{v}$, with ε arbitrary in $(0,1)$.

(iv) ⇒ (i) Set $\mathbf{v} := \mathbf{k}$. Since $A + BK = A + B\mathbf{v}\mathbf{e}_\ell^\top$ is compartmental and Hurwitz, the first property ensures that (5a) holds. Meanwhile the Hurwitz property of the compartmental matrix $A + B\mathbf{v}\mathbf{e}_\ell^\top$ implies that $\mathbf{1}^\top(A + B\mathbf{v}\mathbf{e}_\ell^\top) < 0^\top$ and since $\mathbf{1}^\top A = 0^\top$ this implies that (5b) holds. □

Proposition 11 provides, under the assumption that A is an irreducible compartmental and non-Hurwitz matrix, necessary and sufficient conditions for the existence of a solution taking the form $K = \mathbf{k}\mathbf{e}_\ell^\top$, $\exists\mathbf{k} \in \mathbb{R}^m$, where ℓ is a fixed index in $[1,n]_{\mathbb{Z}}$. On the other hand, by Proposition 9, if the positive stabilization problem is solvable, then there always exists a solution K taking that form for some index $\ell \in [1,n]_{\mathbb{Z}}$. This immediately leads to the following necessary and sufficient condition for positive stabilization.

Theorem 12. *Consider the compartmental system (2) and assume that A is irreducible and non-Hurwitz. The positive stabilization problem is solvable if and only if there exist a vector $\mathbf{v} \in \mathbb{R}^m$ and an index $\ell \in [1,n]_{\mathbb{Z}}$ such that conditions (5) (or any of the equivalent conditions of Proposition 11) hold.*

Example 13. *Consider the compartmental system*

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) = \begin{bmatrix} -1-\alpha & 1 & 0 & 0 \\ 1 & -1-\alpha & 0 & \alpha \\ 0 & 0 & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -2\alpha \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{u}(t),$$

where $\alpha > 0$ is an arbitrary positive scalar. Note that for every $\alpha > 0$ the pair (A, B) is controllable in the standard sense of Linear System Theory, namely $\mathrm{rank}[B \;\; AB \;\; A^2B \;\; A^3B] = 4$. *Also, notice that A is irreducible and non-Hurwitz (as $\mathbf{1}^\top A = 0^\top$). One can verify that for every $\ell \in [1,4]_{\mathbb{Z}}$ there is no vector $\mathbf{v} \in \mathbb{R}^2$ such that conditions (5) are satisfied. Indeed, set $\mathbf{v} = [v_1 \;\; v_2]^\top$. Condition (5b) means that $\mathbf{1}^\top B\mathbf{v} < 0$, namely $5v_1 + 5v_2 = 5(v_1 + v_2) < 0$. So, the two entries of $\mathbf{v}$ sum up to a negative number. This necessarily means that the first and the third entries of the vector $B\mathbf{v}$ are negative. As a result, there is no $\ell \in [1,4]_{\mathbb{Z}}$ such that condition (5a) holds, since the vector $S_\ell(A\mathbf{e}_\ell + B\mathbf{v})$ has at least one negative entry. So, by Theorem 12, the positive* stabilization *problem for the pair (A, B) is not solvable (even if the pair (A, B) is controllable).* ♣

Example 14 (Room temperature regulation)**.** *Consider the thermal system of Figure 2. It consists of three rooms, two of them (room 2 and room 3) directly connected to the air-conditioning system. Let α, β and γ be the thermal transmission coefficients between the adjacent rooms* $(1,2)$, $(1,3)$, *and* $(2,3)$, *respectively. We can assume that α, β and γ are arbitrary positive real numbers. Denote by $x_i, i \in [1,3]_{\mathbb{Z}}$, the (positive) difference between the temperature in the ith room and the desired temperature x_d. If we assume that the system is thermally isolated from the external environment, the time evolution of the temperatures in the three rooms is described by the following compartmental model:*

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) = \begin{bmatrix} -(\alpha+\beta) & \alpha & \beta \\ \alpha & -(\alpha+\gamma) & \gamma \\ \beta & \gamma & -(\gamma+\beta) \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t).$$

Notice that A is irreducible and non-Hurwitz (as $\mathbf{1}_3^\top A = 0^\top$). We want to determine, if possible, a state-feedback control law for the air-conditioning system that allows to regulate all temperatures by making use only of the temperature of room 1. It is easy to verify that for the fixed index $\ell = 1$, the vector $\mathbf{v} = \begin{bmatrix} -\alpha & -\beta \end{bmatrix}^\top$ satisfies conditions (5). Then, it follows from Lemma 7 that for every $\varepsilon \in (0,1)$ the matrix $K = \varepsilon \mathbf{v}\mathbf{e}_1^\top$ makes $A + BK$ compartmental, irreducible, and Hurwitz and hence solves the positive stabilization problem. Note that also $K = \mathbf{v}\mathbf{e}_1^\top$ solves the positive stabilization problem, but in this case $A + BK$ is reducible. ♣

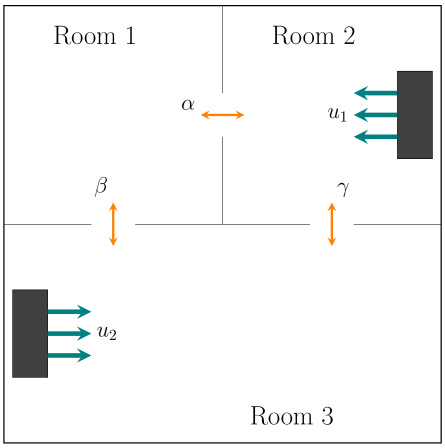

Fig. 2 A simple 3-room thermal system.

5 Sufficient conditions for the problem solvability when A is reducible

We now consider the case when A is reducible and provide sufficient conditions for the solvability of the positive stabilization problem. In the rest of the paper it will be convenient to introduce the following non-restrictive

Assumptions. The compartmental matrix $A \in \mathbb{R}^{n\times n}$ is non-Hurwitz, in the Frobenius normal form (3), with scalar or irreducible diagonal blocks $A_{i,i} \in \mathbb{R}^{n_i\times n_i}$, and the index $r \in [1,s]_{\mathbb{Z}}, r \geq 1$, is such that $\lambda_F(A_{i,i}) = 0$ for every $i \in [1,r]_{\mathbb{Z}}$, and $\lambda_F(A_{i,i}) < 0$ for every $i \in [r+1,s]_{\mathbb{Z}}$. Accordingly, we assume that the input-to-state matrix B is partitioned in a way consistent with the block-partition (3) of A, namely as

$$B = \begin{bmatrix} B_1^\top & B_2^\top & \dots & B_s^\top \end{bmatrix}^\top \tag{7}$$

where $B_i \in \mathbb{R}_+^{n_i\times m}$. Also, we let the classes $\mathcal{C}_i, i \in [1,s]_{\mathbb{Z}}$, be defined as in (4).

The following result extends Lemma 7 to the case of a compartmental, reducible, non-Hurwitz matrix A.

Proposition 15. *Consider the compartmental system (2), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ satisfying the previous Assumptions. If for every $i \in [1,r]_{\mathbb{Z}}$ there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \mathcal{C}_i$ such that*

$$S_{\ell_i}(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i) \geq 0, \tag{8a}$$

$$\mathbf{1}^\top B\mathbf{v}_i < 0, \tag{8b}$$

then for every $\varepsilon \in (0,1)$ the feedback matrix

$$K := \varepsilon \sum_{i=1}^{r} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top \tag{9}$$

solves the positive stabilization problem.

Proof. We want to prove that for every $\varepsilon \in (0,1)$ the matrix K given in (9) makes $A + BK$ compartmental and Hurwitz. To this end, let us first partition the state-feedback matrix K in a way consistent with the block-partition of A and B, namely as

$$K = \begin{bmatrix} K_1 & K_2 & \dots & K_s \end{bmatrix}, \tag{10}$$

where $K_i \in \mathbb{R}^{m\times n_i}$, for every $i \in [1,s]_{\mathbb{Z}}$. By assumption, for every $i \in [1,r]_{\mathbb{Z}}$, there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \mathcal{C}_i$ such that (8) hold. Consequently, for every $\varepsilon \in (0,1)$ the following three conditions hold true:

$$S_{\ell_i}A\mathbf{e}_{\ell_i} + \varepsilon S_{\ell_i}B\mathbf{v}_i \geq 0, \tag{11a}$$

$$\mathbf{1}^\top(A\mathbf{e}_{\ell_i} + \varepsilon B\mathbf{v}_i) < 0, \tag{11b}$$

$$\overline{\mathrm{ZP}}(S_{h_i}A_{i,i}\mathbf{e}_{h_i} + \varepsilon S_{h_i}B_i\mathbf{v}_i) \supseteq \overline{\mathrm{ZP}}(S_{h_i}A_{i,i}\mathbf{e}_{h_i}), \tag{11c}$$

where $h_i := \ell_i - \sum_{k=1}^{i-1} n_k \in [1,n_i]_{\mathbb{Z}}$. We preliminarily notice that every hth column of $A + BK$, $h \in [1,n]_{\mathbb{Z}} \setminus \{\ell_1,\dots,\ell_r\}$, coincides with the hth column of A and hence $S_h(A+BK)\mathbf{e}_h \geq 0$ and

$\mathbf{1}^\top(A+BK)\mathbf{e}_h \le 0$. On the other hand, for every $h \in \{\ell_1, \dots, \ell_r\}$ condition (11a) ensures that $S_h(A+BK)\mathbf{e}_h \ge 0$, while condition (11b) guarantees that $\mathbf{1}^\top(A+BK)\mathbf{e}_h < 0$. Consequently, $A+BK$ is Metzler and compartmental. Finally, (11c) implies that $\forall\ \varepsilon \in (0,1)$ the matrix $K_i = \varepsilon \mathbf{v}_i \mathbf{e}_{h_i}^\top \in \mathbb{R}^{m \times n_i}$ (see (10)) makes the compartmental matrix $A_{i,i} + B_i K_i$ irreducible (and Hurwitz).

To prove that $A+BK$ is Hurwitz, we will prove that in the closed-loop system every compartment is outflow connected (i.e., for every vertex p of the graph $\mathcal{D}(A+BK) := (\mathcal{V}^0 \cup \mathcal{V}^-, \mathcal{E})$ there exists a vertex q, possibly coinciding with p, such that there is a path from p to q and $q \in \mathcal{V}^-$, namely $\mathbf{1}^\top(A+BK)\mathbf{e}_q < 0$). By Lemma 2 this fact ensures that $A+BK$ is Hurwitz. In order to prove the outflow connectedness of all compartments (vertices) of the closed-loop system, we need to partition its compartments in a convenient way. Specifically, we group its compartments according to the communication classes introduced in (3) for the open-loop system, namely the classes $\mathcal{C}_i, i \in [1,s]_{\mathbb{Z}}$, defined in (4). Notice that these are not necessarily the communication classes of $\mathcal{D}(A+BK)$. Indeed, the matrices A and $A+BK$ in general have different nonzero patterns (the off-diagonal blocks $B_j K_i$, $j \ne i$, are nonnegative, and not necessarily zero), namely the interconnection topology among compartments in the closed-loop system is potentially different from the one characterizing the open-loop system. However, this is not a problem, as our goal is to prove that all the compartments of $A+BK$ are outflow connected, and not to determine the communication classes of $\mathcal{D}(A+BK)$[1]. To this aim, we make the following considerations:

1) Every $\mathcal{C}_i$, with $i \in [1,r]_{\mathbb{Z}}$, is such that (a) there is a path from any compartment of $\mathcal{C}_i$ to any other compartment of $\mathcal{C}_i$ (since $A_{i,i} + B_i K_i$ is irreducible); and (b) there is a compartment in $\mathcal{C}_i$ (the ℓ_ith one) that has outflow to the environment (since $\mathbf{1}^\top(A+BK)\mathbf{e}_{\ell_i} < 0$).

2) Every $\mathcal{C}_i$, with $i \in [r+1,s]_{\mathbb{Z}}$, still exhibits property (a) since the ith diagonal block of $A+BK$ coincides with the irreducible matrix $A_{i,i}$. We now prove by induction that either $\mathcal{C}_i$ satisfies property (b), or it has access to $\mathcal{C}_j$, $j < i$, for which (b) holds. Start from $i = r+1$, namely from $\mathcal{C}_{r+1}$, and note that each $(j, r+1)$th block of $A+BK$ coincides with the original block in A, namely $A_{j,r+1}$. Two cases may occur: either there exists $\ell_{r+1} \in \mathcal{C}_{r+1}$ such that

$$\mathbf{1}^\top(A+BK)\mathbf{e}_{\ell_{r+1}} = \sum_{k=1}^{r+1} \mathbf{1}^\top A_{k,r+1} \mathbf{e}_{\ell_{r+1}} < 0,$$

and hence $\mathcal{C}_{r+1}$ exhibits property (b); or $\sum_{k=1}^{r+1} \mathbf{1}^\top A_{k,r+1} = 0^\top$, and, if this is the case, the Hurwitz property of $A_{r+1,r+1}$ implies that there exists $j \in [1,r]_{\mathbb{Z}}$ such that $A_{j,r+1} > 0$, namely the class $\mathcal{C}_{r+1}$ has access to some $\mathcal{C}_j, j \in [1,r]_{\mathbb{Z}}$, that, by part 1), satisfies property (b).
Now, let $\bar{k} \in [r+1, s-1]_{\mathbb{Z}}$ and suppose that, for every $k \in [r+1, \bar{k}]_{\mathbb{Z}}$, $\mathcal{C}_k$ either satisfies property (b) or it has access to $\mathcal{C}_j$, $j < k$, that satisfies property (b). By applying to $\mathcal{C}_{\bar{k}+1}$ the same reasoning adopted for $\mathcal{C}_{r+1}$ we can claim that either $\mathcal{C}_{\bar{k}+1}$ exhibits property (b) or

[1]For the sake of completeness, we observe that vertices communicating in $\mathcal{D}(A)$, and hence belonging to the same communication class, also communicate in $\mathcal{D}(A+BK)$, and hence belong to the same communication class also in $\mathcal{D}(A+BK)$. However, the term BK may introduce additional arcs connecting vertices of different classes of $\mathcal{D}(A)$. As a result, each communication class of $\mathcal{D}(A+BK)$ either coincides with a class of $\mathcal{D}(A)$ or with the union of some classes of $\mathcal{D}(A)$.

it has access to $\mathcal{C}_j$, $j < \bar{k}+1$, for which (b) holds. So, every compartment in every $\mathcal{C}_i$, with $i \in [r+1, s]_{\mathbb{Z}}$, is outflow connected.

To conclude, every compartment of the closed-loop system $A + BK$ is outflow connected and hence $A + BK$ is also Hurwitz. This means that the feedback matrix K defined in (9) solves the positive stabilization problem for every $\varepsilon \in (0, 1)$. $\square$

Example 16. *Consider the following compartmental system*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{u}(t) = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathbf{u}(t).$$

Clearly, $A_{1,1}$ is irreducible and non-Hurwitz, while $A_{2,2}$ is irreducible and Hurwitz. So, $r = 1$ and $s = 2$. It is easy to verify that the vector $\mathbf{v}_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^\top$ satisfies conditions (8) for $\ell_1 = 1$. This means that the sufficient condition of Proposition 15 is satisfied. It is a matter of simple computation to see that, for every $\varepsilon \in (0, 1)$, the feedback matrix

$$K = \varepsilon \mathbf{v}_1 \mathbf{e}_1^\top = \varepsilon \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

makes $A + BK$ compartmental and Hurwitz. ♣

The previous positive stabilizability condition requires the existence of r vectors $\mathbf{v}_i, i \in [1, r]_{\mathbb{Z}}$, satisfying conditions (8). If such vectors cannot be found, the positive stabilization problem may still be solvable as described by the following result.

Proposition 17. *Consider the compartmental system (2), with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ satisfying the previous Assumptions. Introduce the following sets:*

$$E_0 := \left\{ i \in [1, s]_{\mathbb{Z}} : \exists \ell_i \in \mathcal{C}_i \text{ such that } \mathbf{1}^\top A \mathbf{e}_{\ell_i} < 0 \right\} \subseteq [r+1, s]_{\mathbb{Z}}$$
$$E_1 := \left\{ i \in ([1, s]_{\mathbb{Z}} \smallsetminus E_0) : \exists \mathbf{v}_i \in \mathbb{R}^m, \ \ell_i \in \mathcal{C}_i \text{ such that } S_{\ell_i}(A \mathbf{e}_{\ell_i} + B \mathbf{v}_i) \geq 0 \text{ and } \mathbf{1}^\top B \mathbf{v}_i < 0 \right\}.$$

If for every $i \in E_2 := ([1, r]_{\mathbb{Z}} \smallsetminus E_1)$ there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \mathcal{C}_i$ such that[2]

$$S_{\ell_i}(A \mathbf{e}_{\ell_i} + B \mathbf{v}_i) \geq 0, \tag{12a}$$
$$\mathbf{1}^\top B \mathbf{v}_i = 0, \tag{12b}$$
$$B_d \mathbf{v}_i > 0, \quad \exists d \in (E_0 \cup E_1), \tag{12c}$$

then, for every $\varepsilon \in (0, 1)$, the feedback matrix

$$K := \varepsilon \sum_{i \in (E_1 \cup E_2)} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top = \begin{bmatrix} K_1 & K_2 & \dots & K_s \end{bmatrix} \tag{13}$$

solves the positive stabilization problem.

[2]Note that if $E_0 \cup E_1 = \varnothing$, condition (12c) does not hold, and hence this sufficient condition cannot be applied. As we will see in the following, if $E_0 \cup E_1 = \varnothing$ the positive stabilization problem cannot be solved.

Proof. We want to prove that for every $\varepsilon \in (0,1)$ the matrix K given in (13) makes $A + BK$ compartmental and Hurwitz. To this end, preliminarily notice that:
(1) Condition $K_i = 0$ holds for every $i \notin (E_1 \cup E_2)$; (2) for every $h \in [1,n]_{\mathbb{Z}} \setminus \{\ell_i, i \in E_1 \cup E_2\}$ the hth column of $A + BK$ coincides with the hth column of A. Consequently,

$$S_h(A + BK)\mathbf{e}_h \geq 0, \qquad \text{and} \qquad \mathbf{1}^\top(A + BK)\mathbf{e}_h \leq 0. \tag{14}$$

To prove that $A + BK$ is compartmental, we need to prove that inequalities (14) hold true also for every $h \in \{\ell_i, i \in E_1 \cup E_2\}$.

Consider first the indices belonging to E_1. For every $i \in E_1$, there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \mathcal{C}_i$ such that conditions (8) hold. Define $h_i := \ell_i - \sum_{k=1}^{i-1} n_k \in [1, n_i]_{\mathbb{Z}}$. By proceeding as in the proof of Proposition 15, we can claim that for every $\varepsilon \in (0,1)$ and every $h \in \{\ell_i, i \in E_1\}$ conditions (14) hold true and, in addition, the matrix $K_i = \varepsilon \mathbf{v}_i \mathbf{e}_{h_i}^\top \in \mathbb{R}^{m \times n_i}$ makes $A_{i,i} + B_i K_i$ (compartmental and) irreducible.

Consider, now, the set $E_2 \subseteq [1,r]_{\mathbb{Z}}$. For every $i \in E_2$ there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \mathcal{C}_i$ such that conditions (12) hold. Define also in this case $h_i := \ell_i - \sum_{k=1}^{i-1} n_k$. We observe that for every $\varepsilon \in (0,1)$ the following three conditions hold true:

$$S_{\ell_i} A \mathbf{e}_{\ell_i} + \varepsilon S_{\ell_i} B \mathbf{v}_i \geq 0, \tag{15a}$$

$$\mathbf{1}^\top (A\mathbf{e}_{\ell_i} + \varepsilon B \mathbf{v}_i) = 0. \tag{15b}$$

$$\overline{\mathrm{ZP}}(S_{h_i} A_{i,i} \mathbf{e}_{h_i} + \varepsilon S_{h_i} B_i \mathbf{v}_i) \supseteq \overline{\mathrm{ZP}}(S_{h_i} A_{i,i} \mathbf{e}_{h_i}). \tag{15c}$$

By following the same reasoning as in the proof of Proposition 15, we can claim that $\forall\ \varepsilon \in (0,1)$ and every $h \in \{\ell_i, i \in E_2\}$ the inequalities (14) hold (and therefore, at this stage, we can claim that $A + BK$ is compartmental), and the matrix $K_i = \varepsilon \mathbf{v}_i \mathbf{e}_{h_i}^\top \in \mathbb{R}^{m \times n_i}$ makes $A_{i,i} + B_i K_i$ (compartmental and) irreducible. Moreover, there exists $d \in (E_0 \cup E_1)$ such that $A_{d,i} + B_d K_i = 0 + B_d K_i > 0$, where we used the fact that $i \in E_2 \subseteq [1,r]_{\mathbb{Z}}$ implies $A_{d,i} = 0$.

To prove that $A + BK$ is Hurwitz, we will prove that in the closed-loop system every compartment is outflow connected. To this aim, we partition (as we did in the proof of Proposition 15) the compartments of the closed-loop system $A + BK$ according to the partition into communication classes $\mathcal{C}_i, i \in [1,s]_{\mathbb{Z}}$, of the open-loop system. Again, this is only a convenient partition of the compartments of the closed-loop system that does not necessarily coincide with the partition into communication classes.

We want to show that, for every $i \in [1,s]_{\mathbb{Z}}$, every compartment in $\mathcal{C}_i$ is outflow connected, considering all possible cases:

- Case $i \in E_0$: Clearly, $i \in E_0$ if and only if there exists in $\mathcal{C}_i$ a compartment, the ℓ_ith one, that has outflow to the environment in the open-loop system. On the other hand, condition $K_i = 0$ ensures that $A_{i,i} + B_i K_i = A_{i,i}$ is irreducible, and $A_{j,i} + B_j K_i = A_{j,i}$ for every $j \in [1,s]_{\mathbb{Z}}, j \neq i$. Therefore every compartment in $\mathcal{C}_i$ is outflow connected also in the closed-loop system $A + BK$.

- Case $i \in E_1$: Every $\mathcal{C}_i$, $i \in E_1$, is such that (a) there is a path from any compartment of $\mathcal{C}_i$ to any other compartment of $\mathcal{C}_i$ (since the diagonal blocks $A_{i,i} + B_i K_i$ are irreducible); and (b) there is a compartment, the ℓ_ith one, that has outflow to the environment (see (11b)).

- Case $i \in E_2$: Every $\mathcal{C}_i$ with $i \in E_2$ is such that (a) there is a path from any compartment of $\mathcal{C}_i$ to any other compartment of $\mathcal{C}_i$ (again because the blocks $A_{i,i} + B_i K_i$ are irreducible); and (b) there is a path from a compartment in $\mathcal{C}_i$ to a compartment in $\mathcal{C}_d$ with $d \in (E_0 \cup E_1)$, namely $\mathcal{C}_i$ has access to $\mathcal{C}_d$ (since $B_d K_i > 0$). So, even the compartments in $\mathcal{C}_i$ with $i \in E_2$ are outflow connected.

- Case $i \notin \cup_{i=0}^2 E_i$: We first note that $[1, r]_{\mathbb{Z}} \subseteq E_1 \cup E_2$. Therefore any $i \notin \cup_{i=0}^2 E_i$ necessarily belongs to $[r+1, s]_{\mathbb{Z}}$ and satisfies the following conditions: the matrix $A_{i,i} + B_i K_i = A_{i,i}$ is a compartmental, irreducible, Hurwitz matrix and $\sum_{k=1}^{i} \mathbf{1}^\top A_{k,i} = 0^\top$. This implies that there exists $j \in [1, i-1]_{\mathbb{Z}}$ such that $A_{j,i} > 0$, namely $\mathcal{C}_i$ has access to $\mathcal{C}_j$, $j \in [1, i-1]_{\mathbb{Z}}$. Set $\bar{k} := \min\{i \notin (E_0 \cup E_1 \cup E_2)\}$. By proceeding as in the proof of Proposition 15, we can show that (a) there is a path from any compartment of $\mathcal{C}_{\bar{k}}$ to any other compartment of $\mathcal{C}_{\bar{k}}$, (b) there is a path from a compartment in $\mathcal{C}_{\bar{k}}$ to a compartment in $\mathcal{C}_i$ with $i \in (E_0 \cup E_1 \cup E_2)$. Consequently, all the compartments in $\mathcal{C}_{\bar{k}}$ are outflow connected. By proceeding recursively, conditions (a) and (b) prove to be true for all the remaining $i \notin (E_0 \cup E_1 \cup E_2)$.

To conclude, every compartment of the closed-loop system $A + BK$ is outflow connected, and hence $A + BK$ is also Hurwitz. This means that the feedback matrix K defined in (13) solves the positive stabilization problem. ☐

Remark 18. *Note that every solution K of the positive stabilization problem has* at least *r non-zero columns. This is due to the fact that the original digraph $\mathcal{D}(A)$ has r conservative (i.e., non-Hurwitz) communication classes, namely communication classes that have no access to any other class nor to the environment. So, every state-feedback matrix K that solves the positive stabilization problem must be designed in such a way that in $\mathcal{D}(A+BK)$ these classes communicate either with the environment or with other classes. However, the case may occur that a solution K with r non-zero columns might not exist even if the positive stabilization problem is solvable, as shown in the following example.*

Example 19. *Consider the same compartmental system introduced in Example 1 (see Figure 1)*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{u}(t) = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathbf{u}(t).$$

Clearly, $s = 2$, $A_{1,1}$ is irreducible and non-Hurwitz, while $A_{2,2}$ is irreducible and Hurwitz. So, $r = 1$. In this case we have that neither class has a compartment with direct outflow to the environment, and hence $E_0 = \emptyset$. It is easy to verify that $1 \notin E_1$. On the other hand, for $i = 2$ we see that the vector $\mathbf{v}_2 = \begin{bmatrix} 0 & -1/2 \end{bmatrix}^\top$ satisfies conditions (8) for $\ell_2 = 4$. Therefore $E_1 = \{2\}$. Finally, for $i = 2$ we observe that vector $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ satisfies conditions (12) for $\ell_1 = 1$, and hence $E_2 = \{1\}$. As a result, the sufficient condition of Proposition 17 is satisfied, and it is easy to verify that for every $\varepsilon \in (0,1)$ the feedback matrix

$$K = \varepsilon(\mathbf{v}_1\mathbf{e}_1^\top + \mathbf{v}_2\mathbf{e}_4^\top) = \varepsilon\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1/2 \end{bmatrix}$$

makes $A+BK$ compartmental and Hurwitz. Indeed, the digraph associated with any such $A+BK$ is illustrated in Figure 3. Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ are not the communication classes of $A+BK$, which is an irreducible matrix and hence its digraph is strongly connected. On the other hand, the 4th compartment (4th vertex) in $\mathcal{D}(A+BK)$ has direct outflow to the environment (this has been highlighted using a dashed line for its border) and this ensures that all the compartments are outflow connected. Notice that in this case the sufficient condition of Proposition 15 does not hold. ♣

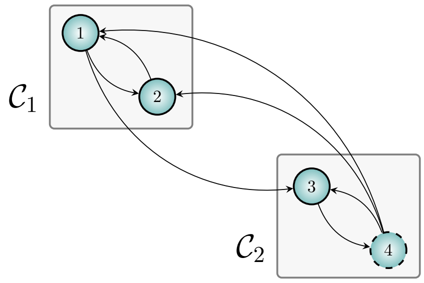

Fig. 3 Digraph associated with matrix $A+BK$ of Example 19.

6 Necessary and sufficient conditions for the problem solvability

The reasoning behind the proofs of Proposition 15 and Proposition 17 can be further exploited to determine a necessary and sufficient condition for the solvability of the positive stabilization problem. We need to define the concept of *distance from the environment* of the sets $\mathcal{C}_i$, $i \in [1,s]_{\mathbb{Z}}$, which provide a partition of the n compartments, namely of the set $[1,n]_{\mathbb{Z}}$, and for which the following property holds: For every $p,q \in \mathcal{C}_i$ there is a path $p \to q_1 \to \cdots \to q_k \to q$ with $q_d \in \mathcal{C}_i, d \in [1,k]_{\mathbb{Z}}$.

We say that $\mathcal{C}_i$ has *direct outflow to the environment*, or *distance $\delta = 0$ from the environment*, if there exists a compartment in $\mathcal{C}_i$ with direct outflow to the environment. This ensures that, given any compartment p of $\mathcal{C}_i$, either it has direct outflow to the environment, or there is a path $p \to q_1 \to \cdots \to q_k \to q_{k+1}$ with $q_d \in \mathcal{C}_i, d \in [1,k+1]_{\mathbb{Z}}$, to a compartment q_{k+1} that has direct outflow to the environment.

If the set $\mathcal{C}_i$ has no direct outflow to the environment, we say that it has *distance $\delta \geq 1$ from the environment* if there exists an arc from a compartment p of $\mathcal{C}_i$ to some compartment $q \in \mathcal{C}_j$, where $\mathcal{C}_j$ is a set having distance $\delta - 1$ from the environment, and no $\delta' < \delta$ can be found for which the previous property holds.

We say that $\mathcal{C}_i$ has *infinite distance from the environment* if none of the compartments of $\mathcal{C}_i$ is outflow connected.

The concept of distance from the environment allows us to restate the positive stabilization problem in slightly different terms. Specifically, we can claim that the positive stabilization problem is solvable if and only if there exists $K \in \mathbb{R}^{m\times n}$ such that in the closed-loop system (namely, for the closed-loop matrix $A+BK$) each set $\mathcal{C}_i$, $i \in [1,s]_{\mathbb{Z}}$, has finite distance from the environment[3].

[3]Note that we find it convenient to refer to the sets $\mathcal{C}_i, i \in [1,s]_{\mathbb{Z}}$, that represent the communication classes of

We want now to comment on the sufficient conditions provided in Propositions 15 and 17, in terms of the previously defined concepts. In Proposition 15 we require that for each conservative communication class of the original matrix A, namely for each $\mathcal{C}_i, i \in [1,r]_{\mathbb{Z}}$, a state-feedback can be found such that the set $\mathcal{C}_i$ has direct outflow to the environment in the resulting feedback system. If this is the case, then also the other sets $\mathcal{C}_i, i \in [r+1,s]_{\mathbb{Z}}$, will have finite distance from the environment, either because they have, in turn, direct outflow to the environment or because they have access (through a path of finite length) to other sets $\mathcal{C}_j, j < i$, that have direct outflow to the environment.

In Proposition 17 we move a step further. We denote by E_0 the set of indices of the classes $\mathcal{C}_i$ with distance $\delta = 0$ from the environment for the original matrix A. The set E_1, on the other hand, includes the indices of all classes $\mathcal{C}_i$ that are not directly connected to the environment but can acquire this property as a result of a state-feedback acting on one of its compartments. So, if for each conservative class of the original system, $\mathcal{C}_i, i \in [1,r]_{\mathbb{Z}}$, whose index i does not belong to E_1, we can find a state feedback that connects $\mathcal{C}_i$ with some class $\mathcal{C}_j, j \in E_0 \cup E_1$, then each such class will have distance $\delta = 1$ from the environment, and all the remaining classes $\mathcal{C}_j, j \in [r+1,s]_{\mathbb{Z}} \setminus (E_0 \cup E_1)$, will have finite distance from the environment in turn, thus ensuring that $A + BK$ is stable.

The following algorithm extends the idea of Proposition 17 as follows. At the initial step, it considers all the sets $\mathcal{C}_i$ that either have direct outflow to the environment ($i \in E_0$) or can gain direct outflow to the environment by means of a state-feedback ($i \in E_1$). If in this way we have considered all the sets ($E_0 \cup E_1 = [1,s]_{\mathbb{Z}}$), then a solution is immediately provided[4]. If there are sets whose indices do not belong to $E_0 \cup E_1$, namely the set of indices of the "remaining sets" $R^{(0)}$ is not empty, then we consider first the sets $\mathcal{C}_i, i \in R^{(0)}$, that have access to some set $\mathcal{C}_j, j \in E_0 \cup E_1$, (and hence distance $\delta = 1$ from the environment). We let $N^{(1)}$ denote the set of indices of these sets, and consider now the sets $\mathcal{C}_i, i \in R^{(0)} \setminus N^{(1)}$, for which the access to some set $\mathcal{C}_j, j \in E_0 \cup E_1$, can be obtained by means of a state-feedback. We let $E_2^{(1)}$ denote the set of such indices. The union set $N^{(1)} \cup E_2^{(1)}$ represents the set of indices of all sets $\mathcal{C}_i$ that have distance $\delta = 1$ from the environment either because of the structure of A or as a result of a state-feedback. Subsequently we update the distance δ to the value 2, update the index set of the remaining $\mathcal{C}_i$'s (now $R^{(1)}$) by subtracting $N^{(1)}$ and $E_2^{(1)}$ from $R^{(0)}$, and determine the sets $N^{(2)}$ and $E_2^{(2)}$. If at some step δ we have emptied the index set $R^{(\delta)}$ of the remaining sets, then the positive stabilization problem is solvable and a solution is explicitly proposed. If, on the other hand, at some step we have not decreased the cardinality of $R^{(\delta)}$ then the problem is not solvable.

We now formalize the previously described algorithm for a compartmental system (2), with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ satisfying the previous Assumptions.

Algorithm 1:

A0. Define the sets E_0 and E_1 as in Proposition 17. To every $i \in E_1$ we associate the pair $(\mathbf{v}_i, \ell_i)$, where $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \mathcal{C}_i$ are such that $S_{\ell_i} A \mathbf{e}_{\ell_i} + S_{\ell_i} B \mathbf{v}_i \geq 0$ and $\mathbf{1}^\top B \mathbf{v}_i < 0$.

---

the original matrix A, but the result would hold true for any other partition of the set of compartments $[1,n]_{\mathbb{Z}}$, provided that all the compartments in each $\mathcal{C}_i$ communicate with each other.

[4] This represents a special case of Proposition 17: the case when, by means of a state-feedback, we can ensure that all sets $\mathcal{C}_i, i \in [1,s]_{\mathbb{Z}}$, have direct outflow to the environment, i.e., distance $\delta = 0$.

Initialize $\delta = 0$, $\tilde{E}_2^{(0)} = E_2^{(0)} = \varnothing$ and $R^{(0)} = [1, s]_{\mathbb{Z}} \setminus \left(E_0 \cup E_1 \cup \tilde{E}_2^{(0)}\right)$.

A1. If $R^{(\delta)} = \varnothing$, then the positive stabilization problem is solvable and for every $\varepsilon \in (0, 1)$ the feedback matrix

$$K := \varepsilon \sum_{i \in E_1 \cup E_2^{(0)} \cup \dots \cup E_2^{(\delta)}} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top$$

is a solution. STOP.

If $R^{(\delta)} \neq \varnothing$, go to A2.

A2. Define $N^{(\delta+1)} = \left\{i \in R^{(\delta)} \colon A_{d_i, i} \neq 0, \exists\, d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \dots \cup \tilde{E}_2^{(\delta)}\right\} \subseteq [r+1, s]_{\mathbb{Z}}$.

Set $E_2^{(\delta+1)} = \left\{i \in R^{(\delta)} \setminus N^{(\delta+1)} \colon \exists \mathbf{v}_i \in \mathbb{R}^m,\ \ell_i \in \mathcal{C}_i \text{ such that conditions (16) hold}\right\}$,

$$S_{\ell_i} A \mathbf{e}_{\ell_i} + S_{\ell_i} B \mathbf{v}_i \geq 0, \tag{16a}$$

$$\mathbf{1}^\top B \mathbf{v}_i = 0, \tag{16b}$$

$$B_{d_i} \mathbf{v}_i > 0, \quad \exists d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \dots \cup \tilde{E}_2^{(\delta)}. \tag{16c}$$

To every $i \in E_2^{(\delta+1)}$ we associate the pair $(\mathbf{v}_i, \ell_i)$, where $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \mathcal{C}_i$ satisfy conditions (16). Set $\tilde{E}_2^{(\delta+1)} = E_2^{(\delta+1)} \cup N^{(\delta+1)}$.

A3. Set $R^{(\delta+1)} = R^{(\delta)} \setminus \tilde{E}_2^{(\delta+1)}$. If $R^{(\delta+1)} = R^{(\delta)}$, then the positive stabilization problem is not solvable. STOP.

Otherwise, update $\delta = \delta + 1$ and repeat from A1.

Remark 20. *There are various ways to make Algorithm 1 more efficient. For instance, at step A2, when determining $E_2^{(\delta+1)}$, one could include in the set of classes for which we have already guaranteed a finite distance from the environment also the classes indexed in $N^{(\delta+1)}$, namely one could check condition (16c) for $d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \dots \cup \tilde{E}_2^{(\delta)} \cup N^{(\delta+1)}$. However, in doing so, δ would no longer represent the distance of the class from the environment.*

It is also worthwhile noticing that condition (16a) ensures that $B_{d_i} \mathbf{v}_i \geq 0, \forall d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \dots \cup \tilde{E}_2^{(\delta)}$. So, the purpose of (16c) is simply to determine for which indices the inequality is strict. This needs not to be checked for every $d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \dots \cup \tilde{E}_2^{(\delta)}$ at every step δ. Indeed, for $\delta = 0$ it must be tested only for $d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} = E_0 \cup E_1$, while for $\delta > 0$ it can be tested only for $d_i \in \tilde{E}_2^{(\delta)}$. Indeed, if condition (16c) would have been true for some $d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \dots \cup \tilde{E}_2^{(\delta-1)}$, then i would already be in $E_2^{(\delta)}$ and hence it would not be an element of $R^{(\delta)}$.

Finally, it is also clear that for each value of i, conditions (16) need to be satisfied only for one value of ℓ_i. So, in the best case one needs to solve this LP only for one index per class, while in the worst case for every index in the class. Strategies based on the evaluation of the zero entries of the columns of A first and the analysis of the corresponding rows in B then, as highlighted in Proposition 11, may optimize the search for the index ℓ_i.

It is clear, by the way the algorithm has been conceived, that it always comes to an end in no more than $s-1$ steps (namely, δ cannot be greater than $s-1$). To prove that the positive stabilization problem is solvable if and only if Algorithm 1 ends with $R^{(\delta)} = \emptyset$ for some $\delta \in \mathbb{Z}_+$, we need the following result.

Lemma 21. *Consider the compartmental system (2), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ satisfying the previous Assumptions. If the positive stabilization problem is solvable, then there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \mathcal{C}_i, i \in [1,s]_{\mathbb{Z}}$, such that for every $\varepsilon \in (0,1)$ the matrix*

$$K^* = \varepsilon \sum_{i=1}^{s} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top = \begin{bmatrix} K_1^* & \dots & K_s^* \end{bmatrix} \tag{17}$$

makes $A + BK^*$ compartmental and Hurwitz, and for every $i \in [1,s]_{\mathbb{Z}}$ the diagonal blocks $A_{i,i} + B_i K_i^*$ are irreducible.

Proof. Let $K \in \mathbb{R}^{m\times n}$ be an arbitrary solution to the positive stabilization problem. Define the matrix $K' = \begin{bmatrix} K_1' & \dots & K_s' \end{bmatrix} \in \mathbb{R}^{m\times n}$, column by column, as follows:

$$K'\mathbf{e}_h := \begin{cases} 0, & \text{if } \mathbf{1}^\top A\mathbf{e}_h < 0, \\ \varepsilon K\mathbf{e}_h, & \text{if } \mathbf{1}^\top A\mathbf{e}_h = 0, \end{cases}$$

where ε is arbitrary in $(0,1)$. We first observe that for every $h \in [1,n]_{\mathbb{Z}}$ (and for every $\varepsilon \in (0,1)$) $S_h(A+BK')\mathbf{e}_h \geq 0$ and

$$\overline{\mathrm{ZP}}(S_h(A+BK')\mathbf{e}_h) \supseteq \overline{\mathrm{ZP}}(S_h A\mathbf{e}_h). \tag{18}$$

This implies, in particular, that for every $i \in [1,s]_{\mathbb{Z}}$ and every $h_i \in [1,n_i]_{\mathbb{Z}}$

$$\overline{\mathrm{ZP}}(S_{h_i}(A_{i,i} + B_i K_i')\mathbf{e}_{h_i}) \supseteq \overline{\mathrm{ZP}}(S_{h_i} A_{i,i}\mathbf{e}_{h_i}).$$

This ensures that for every $\varepsilon \in (0,1)$ the matrix K' makes $A+BK'$ Metzler with all diagonal blocks $A_{i,i} + B_i K_i'$ irreducible. To prove that $A+BK'$ is still compartmental and Hurwitz, it is sufficient to note that:
(1) For every $h \in [1,n]_{\mathbb{Z}}$

$$\mathbf{1}^\top (A+BK')\mathbf{e}_h = \begin{cases} < 0, & \text{if } \mathbf{1}^\top A\mathbf{e}_h < 0, \\ \leq 0, & \text{if } \mathbf{1}^\top A\mathbf{e}_h = 0. \end{cases}$$

This ensures that $A+BK'$ is compartmental and the set of compartments of $A+BK'$ with direct outflow to the environment includes the set of compartments of $A+BK$ with direct outflow to the environment.
(2) By the way K' has been defined, if the hth compartment of $A+BK'$, $h \in [1,n]_{\mathbb{Z}}$, does not have direct outflow to the environment then h is such that $\mathbf{1}^\top A\mathbf{e}_h = 0$ and hence $K'\mathbf{e}_h = \varepsilon K\mathbf{e}_h$. Therefore $(A+BK')\mathbf{e}_h = (A+\varepsilon BK)\mathbf{e}_h$, and condition $\varepsilon \in (0,1)$ ensures that all the arcs from the hth compartment that appear in the digraph associated with $A+BK$ also appear in the digraph associated with $A+BK'$.

So, the Hurwitz property of $A+BK$ ensures that all the compartments in $A+BK$ are outflow connected, and this property is preserved in $A+BK'$, thus ensuring that $A+BK'$ is Hurwitz, too.

We initialize the matrix K^* by assuming $K^* := K'$. We then proceed as follows:

- Set $D_0 := \{i \in [1,s]_{\mathbb{Z}} : \mathbf{1}^\top(A+BK')\mathbf{e}_{\ell_i} < 0, \exists\ \ell_i \in \mathcal{C}_i\}$. This is the set of indices of the classes in $A+BK^* = A+BK'$ that have direct outflow to the environment. For every $i \in D_0$, we select one such ℓ_i and impose that for every $h \in \mathcal{C}_i, h \neq \ell_i$, $K^*\mathbf{e}_h = 0$. Note that after this change D_0 still represents the set of indices of the classes in $A+BK^*$, for the updated K^*, that have direct outflow to the environment.

- Set $D_1 := \{i \in [1,s]_{\mathbb{Z}} \setminus D_0 : A_{j,i} + B_j K_i' \neq 0, \exists\ j \in D_0\}$. If $i \in D_1$, this means that there exist $j \in D_0, \ell_i \in \mathcal{C}_i$ and $q \in \mathcal{C}_j$ such that $[A+BK']_{q,\ell_i} \neq 0$. For every $i \in D_1$, we set $K^*\mathbf{e}_h = 0$ for every $h \in \mathcal{C}_i, h \neq \ell_i$. Also in this case D_1 represents the set of indices of classes in $A+BK^*$ that have distance one from the environment, both before and after having updated K^*.

- By proceeding in this way, we determine all the sets D_δ and we set to zero the columns of K^* accordingly. Since $A+BK'$ is Hurwitz all the sets $\mathcal{C}_i$ have finite distance from the environment and this property is preserved in all the subsequent modifications that lead to the final K^*. By the way we have proceeded, K^* has the structure (17), and the nonzero vectors $\mathbf{v}_i$ coincide with $K\mathbf{e}_{\ell_i}$.

□

Theorem 22. *Consider the compartmental system* (2), *with* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times m}$ *satisfying the previous Assumptions. The positive stabilization problem is solvable if and only if Algorithm 1 applied to the pair* (A,B) *ends with* $R^{(\delta)} = \emptyset$ *for some* $\delta \in \mathbb{Z}_+$. *When so, the feedback matrix* K *generated by Algorithm 1 (see step A1) represents a solution.*

Proof. [Sufficiency] The proof trivially follows from the fact that if Algorithm 1 ends with $R^{(\delta)} = \emptyset$, then the final closed-loop matrix $A+BK$ is compartmental, and for every compartment $h \in [1,n]_{\mathbb{Z}}$ of the matrix $A+BK$ we have been able to either guarantee that it belongs to a set that has direct outflow to the environment (this is the case if $h \in \mathcal{C}_i$, for some $i \in E_0 \cup E_1$) or to ensure that it belongs to a set $\mathcal{C}_i$ that has finite distance from the environment (this is the case if $i \in \tilde{E}_2^{(\delta)}$ for some $\delta \geq 1$). Therefore in the closed-loop system, all the compartments are outflow connected, and this ensures the Hurwitz property.

[Necessity] By Lemma 21, if the positive stabilization problem is solvable, then there exists a solution taking the structure (17). Algorithm 1 constructs, among all solutions (17), one in which each set $\mathcal{C}_i, i \in [1,s]_{\mathbb{Z}}$, has in the resulting closed-loop system $A+BK$ the minimum possible distance from the environment. So, if Algorithm 1 does not end with a solution, a solution (17) does not exist. □

Example 23. *Consider the following compartmental system*

$$\dot{\mathbf{x}}(t) = \left[\begin{array}{ccc|cc|c} -2 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right]\mathbf{x}(t) + \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 1 \\ 0 & 0 \\ \hline 1 & 0 \end{array}\right]\mathbf{u}(t) = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ 0 & A_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,3} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}\mathbf{u}(t).$$

Notice that A is in Frobenius normal form (3), with diagonal block $A_{1,1}$ irreducible and non-Hurwitz, and diagonal blocks $A_{2,2}$ and $A_{3,3}$ irreducible and Hurwitz. So, $s = 3$ and $r = 1$. We apply Algorithm 1 to the pair (A, B) and determine, if possible, a solution to the positive stabilization problem.

[$\delta = 0$] *We set $E_0 = \varnothing$, $E_1 = \{2\}$, and we associate the pair $(\mathbf{v}_2, \ell_2) := \left(\begin{bmatrix}1/4 & -1/2\end{bmatrix}^\top, 4\right)$ to $i = 2$.*
We initialize $\tilde{E}_2^{(0)} = E_2^{(0)} = \varnothing$ and $R^{(0)} = \{1, 3\}$.
We define $N^{(1)} = \{3\}$, $E_2^{(1)} = \varnothing$, and $\tilde{E}_2^{(1)} = E_2^{(1)} \cup N^{(1)} = \{3\}$.
We set $R^{(1)} = R^{(0)} \setminus \tilde{E}_2^{(1)} = \{1\}$, and since $R^{(1)} \subset R^{(0)}$ we update δ.

[$\delta = 1$] *We define $N^{(2)} = \varnothing$ and $E_2^{(2)} = \{1\}$. We associate the pair $(\mathbf{v}_1, \ell_1) := \left(\begin{bmatrix}1 & -1\end{bmatrix}^\top, 1\right)$ to $i = 1$.*
We set $\tilde{E}_2^{(2)} = E_2^{(2)} \cup N^{(2)} = \{1\}$.
We set $R^{(2)} = R^{(1)} \setminus \tilde{E}_2^{(2)} = \varnothing$. Since $R^{(2)} = \varnothing$ the positive stabilization problem is solvable and for every $\varepsilon \in (0, 1)$ the feedback matrix

$$K := \varepsilon \sum_{i \in E_1 \cup E_2^{(1)} \cup E_2^{(2)}} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ -1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}$$

is a solution. ♣

Example 24. *Consider the following compartmental system*

$$\dot{\mathbf{x}}(t) = \left[\begin{array}{cc|cc|cc|c} -1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right] \mathbf{x}(t) + \left[\begin{array}{cc} 0 & 2 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \\ \hline 1 & 1 \\ 0 & 0 \\ \hline 1 & 0 \end{array}\right] \mathbf{u}(t)$$

$$= \begin{bmatrix} A_{1,1} & 0 & A_{1,3} & A_{1,4} \\ 0 & A_{2,2} & A_{2,3} & 0 \\ 0 & 0 & A_{3,3} & A_{3,4} \\ 0 & 0 & 0 & A_{4,4} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \mathbf{u}(t).$$

Notice that A is in Frobenius normal form (3), with diagonal blocks $A_{1,1}$ and $A_{2,2}$ irreducible and non-Hurwitz, and diagonal blocks $A_{3,3}$ and $A_{4,4}$ irreducible and Hurwitz. So, $s = 4$ and $r = 2$. We apply Algorithm 1 to the pair (A, B) and determine, if possible, a solution to the positive stabilization problem.

[$\delta = 0$] *We set $E_0 = \varnothing$, $E_1 = \{3\}$, and we associate the pair $(\mathbf{v}_3, \ell_3) := \left(\begin{bmatrix}1/4 & -1/2\end{bmatrix}^\top, 5\right)$ to $i = 3$.*
We initialize $\tilde{E}_2^{(0)} = E_2^{(0)} = \varnothing$ and $R^{(0)} = \{1, 2, 4\}$.
We define $N^{(1)} = \{4\}$, $E_2^{(1)} = \varnothing$, and $\tilde{E}_2^{(1)} = E_2^{(1)} \cup N^{(1)} = \{4\}$.
We set $R^{(1)} = R^{(0)} \setminus \tilde{E}_2^{(1)} = \{1, 2\}$, and since $R^{(1)} \subset R^{(0)}$ we update δ.

[$\delta = 1$] *We define* $N^{(2)} = \varnothing$ *and* $E_2^{(2)} = \{1\}$. *We associate the pair* $(\mathbf{v}_1, \ell_1) := \left(\begin{bmatrix} 1 & -1 \end{bmatrix}^\top, 1\right)$ *to* $i = 1$.
We set $\tilde{E}_2^{(2)} = E_2^{(2)} \cup N^{(2)} = \{1\}$.
We set $R^{(2)} = R^{(1)} \setminus \tilde{E}_2^{(2)} = \{2\}$, *and since* $R^{(2)} \subset R^{(1)}$ *we update* δ.

[$\delta = 2$] *We define* $N^{(3)} = \varnothing$, $E_2^{(3)} = \varnothing$ *and* $\tilde{E}_2^{(3)} = E_2^{(3)} \cup N^{(3)} = \varnothing$.
We set $R^{(3)} = R^{(2)} \setminus \tilde{E}_2^{(3)} = \{2\}$. *Since* $R^{(3)} = R^{(2)}$ *we stop and conclude that the positive stabilization problem is not solvable.* ♣

7 Concluding remarks

To conclude the paper, we would like to briefly comment on the performance of Algorithm 1 in terms of number of inequalities to be solved (note that at each step the number of unknown variables is always m, so Algorithm 1 is always linear in the number of inputs). Algorithm 1 aims at minimizing the maximum distance that a communication class can have from the environment for the resulting matrix $A + BK$. To optimize this parameter, the Algorithm 1 may repeatedly inspect the vertices of the same class $\mathcal{C}_i$ every time the step A2 of the Algorithm 1 is iterated, until the class index i is included in $\tilde{E}_2^{(\delta+1)}$ for some $\delta \geq 0$. This requires evaluating n inequalities in m unknown variables (the entries of $\mathbf{v}_i$) to verify (16a) and (16b) for the indices $\ell_i \in \mathcal{C}_i$. Under this viewpoint, this is not the most efficient way to solve the problem (modulo the improvements already mentioned in Remark 20).

If the goal is that of reducing as much as possible the number of nonzero columns of K, Algorithm 1 can be significantly modified, to adopt a rather different strategy. Specifically, one can first determine all the classes $\mathcal{C}_i$ that have finite distance from the environment already for the matrix A. Then, every time a state-feedback control action is designed, consisting of a single nonzero column, that ensures that all vertices belonging to some set $\mathcal{C}_i$ are outflow connected to the environment in $A + BK$, one can determine all the sets $\mathcal{C}_j$ that have access to $\mathcal{C}_i$, and they need not to be considered in the following steps. Similarly, if we are able to design a state-feedback action that connects a new set $\mathcal{C}_i$ to one of the previously considered sets, then, again, all sets $\mathcal{C}_j$ having access to $\mathcal{C}_i$ necessarily have finite distance from the environment. Under this perspective, the number of LPs that need to be solved can be significantly reduced and replaced by the elementary verification that the block matrices $A_{i,j}$, representing edges between classes, are nonzero.

In any event, and differently from other techniques proposed for state-feedback stabilization of positive systems as, for instance the one in [18], Algorithm 1 deeply relies on the assumption that the partition of the digraph $\mathcal{D}(A)$ into communication classes is known, and if this is the case the number of LPs that need to be solved can be minimized. But the computational complexity is always influenced by the digraph structure. In the worst case, one could possibly solve the LPs

$$S_{\ell_i}(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i) \geq 0 \qquad \mathbf{1}^\top B\mathbf{v}_i \leq 0$$

for every $i \in [1, s]_{\mathbb{Z}}$ and every $\ell_i \in \mathcal{C}_i$ (namely, n families of n inequalities in m unknown variables) and when $\mathbf{1}^\top B\mathbf{v}_i = 0$ memorize the values of indices d_i for which condition (16c) holds. This would make the computational burden analogous to the one of the LP proposed in [18].

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