On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions

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Abstract

The aim of this paper is to address consensus and bipartite consensus for a group of homogeneous agents, under the assumption that their mutual interactions can be described by a weighted, signed, connected and structurally balanced communication graph. This amounts to assuming that the agents can be split into two antagonistic groups such that interactions between agents belonging to the same group are cooperative, and hence represented by nonnegative weights, while interactions between agents belonging to opposite groups are antagonistic, and hence represented by nonpositive weights. In this framework, bipartite consensus can always be reached under the stabilizability assumption on the state-space model describing the dynamics of each agent. On the other hand, (nontrivial) standard consensus may be achieved only under very demanding requirements, both on the Laplacian associated with the communication graph and on the agents' description. In particular, consensus may be achieved only if there is a sort of "equilibrium" between the two groups, both in terms of cardinality and in terms of the weights of the "conflicting interactions" amongst agents.

1 Introduction

Mathematical formulation of multi-agents systems and consensus problems has been of interest for a considerable length of time. Some of the pioneering works are reported in [5, 6, 7, 28, 29] and references therein. However, a decade ago, thanks to milestone contributions such as [10, 16, 18, 22, 23, 24, 26], a wide stream of literature on these topics started and flourished. The driving force behind considerable research activity on this topic is the wide variety of areas where the consensus problem lends itself in a natural manner. In applications such as sensor networks, coordination of mobile robots or UAVs, flocking and swarming in animal groups, dynamics of opinion forming, etc., the problem can be formulated as that of a group of agents exchanging information with the objective of reaching a common decision, a consensus, by resorting to distributed algorithms that make use of the information that each agent collects from neighboring agents (see, e.g. [2, 11, 12, 15, 19, 20, 21, 24, 25, 27]. The interested reader is referred to [1] and [23] for a more complete list of references).

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A common assumption in most of the literature about consensus is that agents achieve this goal through collaboration. However, in a number of contexts where the consensus is meaningful, the interactions among agents are not necessarily cooperative. On the contrary, in contexts like markets or social networks [8], for instance, agents may also display non-cooperative or antagonistic interactions with some of their neighboring agents. In social networks, mutual relationships between pairs of individuals may be friendly or hostile, and this may create two antagonistic groups. In economic systems, duopolistic regimes arise quite frequently: all the companies producing a certain product, or providing a certain service, are split into two competing cartels. But this is also the case when modeling two competing teams, as it happens, for instance, in sport disciplines, or in robot competitions like Robocup. Each individual or robot collects information regarding both the team mates and the antagonists, and processes this data in order to take decisions (position, speed, behavior, elevation, ...) that are in agreement with those of their team mates. Game theory provides several examples where players are divided into two competing teams, and antagonistic interactions between the two groups are crucial when modeling the overall system dynamics. Finally, in biology systems, interactions between genes or chemical elements may be cooperative or antagonistic in the form of activators/inhibitors.

In graph theoretic terms, antagonistic interactions can be taken into account by replacing the standard communication graph, characterized by nonnegative weights, with a signed graph [13, 33] displaying both positive and negative weights. Positive arcs correspond to cooperative interactions between agents, while negative arcs describe interactions between antagonistic agents. While there is considerable literature about consensus in cooperating multi-agent systems, research results pertaining to consensus without cooperation are relatively few [3, 4, 9]. In a recent paper [1], Altafini developed the concept of *bipartite consensus* among agents with antagonistic interactions. Specifically, based on definitions and properties of signed weighted (directed or undirected) graphs and of the associated Laplacians (see [13, 17]), he introduced the concept of bipartite consensus (or agreed dissensus). This is the situation when agents are split into two groups such that within each group all the agents converge to a unique decision, but the decisions of the two groups are opposite.

By addressing the classical example of homogeneous agents modeled as simple scalar integrators, he proved that if the signed, weighted and connected communication graph describing the agents' interactions is structurally balanced, then the agents reach bipartite consensus. On the other hand, if the interactions are antagonistic, but not structurally balanced, the only agreement that can be achieved among the agents is the trivial one, where all the agents' states converge to zero.

The main objective of this paper is to extend the results reported in [1], by addressing the general case of N homogeneous agents described by a generic n-dimensional linear state-space model. This represents a natural extension of the case when agents are modeled as simple integrators. In certain situations, agents' status may require more than a single variable for accurate representation. These variables may include, e.g., position and velocity, price and production levels, etc. These decision variables are updated based on the information collected from the neighboring agents, and consensus must be achieved on all of them. Specifically, we establish conditions for consensus and bipartite consensus for a group of N homogeneous agents under the assumption that their mutual interactions can be described by a weighted, signed, connected and structurally balanced communication graph.

We show that, in this set-up, bipartite consensus can always be reached under the fairly weak assumption of stabilizability on the state-space model describing the dynamics of each agent. However, nontrivial consensus to a common decision for the two antagonistic groups can be achieved only under more restrictive requirements, both on the Laplacian associated with the communication graph and on the agents' description. In particular, consensus may be achieved only if there is a sort of "equilibrium" between the two groups, both in terms of cardinality and in terms of the weights of the "conflicting interactions" among agents.

Briefly, Section II introduces the problem formulation and formalizes the consensus and bipartite consensus problems. In addition, basic definitions and results regarding weighted signed graphs and their Laplacians are reviewed, and a new technical result regarding the Laplacian of structurally balanced graphs is presented. Section III investigates the bipartite consensus problem, and it is shown there that, under the structural balance assumption, it is possible to extend to the case of antagonistic interactions and bipartite consensus the results presented in [31, 32] (see, also, [10, 14, 27, 30]) for the consensus of high-order cooperating agents. Section IV explores the consensus problem, by focusing on the case when the common trajectory that the agents converge to is bounded, but not converging to zero. In this section, conditions under which consensus may be achieved are investigated, and an algorithm to design the control law that makes this possible is presented. Finally, it is shown that when the previous conditions are not met, nontrivial consensus can never be achieved.

Notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. For any pair of positive integers k and n with $k \le n$, $[k,n]$ is the set of integers $\{k, k+1, \dots, n\}$. The (i,j)th entry of a matrix A will be denoted by $[A]_{ij}$, the ith entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_i$. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and denoted by $A \ge 0$. The symbol $\mathbf{1}_N$ denotes the N-dimensional vector with all entries equal to 1. The *spectrum* of a square matrix A is denoted by $\sigma(A)$.

2 Consensus and bipartite consensus problems: statements

We consider a multi-agent system consisting of N agents, each of them described by the same single-input continuous-time state-space model. Specifically, $\mathbf{x}_i(t)$, the ith agent state, $i \in [1,N]$, evolves according to the first-order differential equation

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + bu_i(t), \tag{1}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, and $b \in \mathbb{R}^n$. The communication among the N agents is described by an *undirected, weighted and signed, communication graph* [1] $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A}$ is the matrix of the signed weights of $\mathcal{G}$. The (i,j)th entry of $\mathcal{A}$, $[\mathcal{A}]_{ij}$, is nonzero if and only if the arc (j,i) belongs to $\mathcal{E}$, namely information about the status of the jth agent is available to the ith agent. We assume that the interactions between pairs of agents are symmetric and hence $\mathcal{A} = \mathcal{A}^\top$. The interaction between the ith and the jth agents is cooperative if $[\mathcal{A}]_{ij} > 0$ and antagonistic if $[\mathcal{A}]_{ij} < 0$. Also, we assume that $[\mathcal{A}]_{ii} = 0$, for all $i \in [1,N]$. The graph $\mathcal{G}$ is *connected* if, for every pair of vertices j and i, there is path, namely an ordered sequence of

arcs $(j, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, i) \in \mathcal{E}$, connecting them.
The *Laplacian matrix* associated with the adjacency matrix $\mathcal{A}$ is defined as in [1, 13, 17], namely:

$$\mathcal{L} := \mathcal{C} - \mathcal{A}, \tag{2}$$

where $\mathcal{C}$ is the (diagonal) connectivity matrix, whose diagonal entries are the sums of the absolute values of the corresponding row entries of $\mathcal{A}$, namely

$$[\mathcal{C}]_{ii} = \sum_{(j,i)\in\mathcal{E}} |[\mathcal{A}]_{ij}|, \qquad \forall i \in [1, N].$$

Therefore

$$[\mathcal{L}]_{ij} = \begin{cases} \sum_{(j,i)\in\mathcal{E}} |[\mathcal{A}]_{ij}|, & \text{if } i = j \\ -[\mathcal{A}]_{ij}, & \text{if } i \neq j. \end{cases} \tag{3}$$

Throughout the paper, we assume that the weighted and signed graph $\mathcal{G}$, describing the interactions among agents, is connected and *structurally balanced*. This latter property means [1, 13, 33] that the set of vertices $\mathcal{V}$ can be partitioned into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that for every $i, j \in \mathcal{V}_p, p \in [1, 2]$, $[\mathcal{A}]_{ij} \geq 0$, while for every $i \in \mathcal{V}_p, j \in \mathcal{V}_q$, $p, q \in [1, 2]$, $p \neq q$, $[\mathcal{A}]_{ij} \leq 0$. This amounts to saying that the agents can be split into two groups, and interactions between pairs of agents belonging to the same group are cooperative, while interactions between pairs of agents belonging to different groups are antagonistic. Therefore, after a suitable reordering of the agents, we can always assume that

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}^\top & \mathcal{A}_{22} \end{bmatrix}, \tag{4}$$

where $k := |\mathcal{V}_1|, N - k := |\mathcal{V}_2|$, $\mathcal{A}_{11} = \mathcal{A}_{11}^\top \in \mathbb{R}_+^{k\times k}$, $\mathcal{A}_{22} = \mathcal{A}_{22}^\top \in \mathbb{R}_+^{(N-k)\times(N-k)}$, while $-\mathcal{A}_{12} \in \mathbb{R}_+^{k\times(N-k)}$. Under this fundamental requirement on the mutual interactions among the agents, we assume that the ith agent adopts the static state-feedback control algorithm:

$$u_i(t) = -K \sum_{(j,i)\in\mathcal{E}} |[\mathcal{A}]_{ij}| \cdot [\mathbf{x}_i(t) - \mathrm{sign}([\mathcal{A}]_{ij})\mathbf{x}_j(t)], \qquad i \in [1, N], \tag{5}$$

where $K \in \mathbb{R}^{1\times n}$ is a feedback matrix to be designed, and $\mathrm{sign}(\cdot)$ is the sign function.

Upon defining the state and input vectors

$$\begin{aligned} \mathbf{x}(t) &:= \begin{bmatrix} \mathbf{x}_1^\top(t) & \mathbf{x}_2^\top(t) & \ldots & \mathbf{x}_N^\top(t) \end{bmatrix}^\top, \\ \mathbf{u}(t) &:= \begin{bmatrix} u_1(t) & u_2(t) & \ldots & u_N(t) \end{bmatrix}^\top, \end{aligned}$$

the overall dynamics of the multi-agent system are henceforth described by the equations

$$\dot{\mathbf{x}}(t) = (I_N \otimes A)\mathbf{x}(t) + (I_N \otimes b)\mathbf{u}(t), \tag{6}$$

$$\mathbf{u}(t) = -(\mathcal{L} \otimes K)\mathbf{x}(t). \tag{7}$$

The aim of this paper is to investigate both the standard (state) consensus and the bipartite (state) consensus problems for the group of agents described by (6)-(7). Specifically, we will first search for conditions ensuring that *bipartite consensus* is possible, namely that a matrix K can be found so that, for every choice of the initial conditions, the agents' states evolve as follows:

$$\lim_{t\to+\infty} \mathbf{x}_i(t) = \begin{cases} \zeta(t), & \forall\ i \in \mathcal{V}_1; \\ -\zeta(t), & \forall\ i \in \mathcal{V}_2, \end{cases}$$

for some function $\zeta(t), t \in \mathbb{R}_+$, that depends on the initial conditions. Subsequently, we will search for conditions ensuring that all the agents converge to the same decision (so, that standard *consensus* among agents is possible), and hence a matrix K can be found such that, for every choice of the initial conditions,

$$\lim_{t\to+\infty} \mathbf{x}_i(t) = \zeta(t), \qquad \forall\ i \in \mathcal{V},$$

for some function $\zeta(t), t \in \mathbb{R}_+$, that depends on the initial conditions.

Before proceeding, it is convenient to recall from [1] some fundamental results about the Laplacian of a weighted, signed, undirected graph $\mathcal{G}$, in general, and about the Laplacian of a structurally balanced graph, in particular. First of all, the Laplacian matrix $\mathcal{L}$, defined as in (2), is a symmetric and positive semidefinite matrix, whose nonnegative real eigenvalues can always be sorted in such a way that

$$0 \le \lambda_1(\mathcal{L}) \le \lambda_2(\mathcal{L}) \le \ldots \le \lambda_N(\mathcal{L}).$$

Note that, $\mathcal{L}$ being symmetric and hence diagonalizable, there exists a basis of $\mathbb{R}^N$ consisting of eigenvectors of $\mathcal{L}$. Clearly, one can always find (at least) one (left or right) eigenvector $\mathbf{v}$ of $\mathcal{L}$ whose entries sum up to a nonzero value (i.e., $\sum_{i=1}^N [\mathbf{v}]_i \neq 0$)[1]. When so, we can assume without loss of generality that $\sum_{i=1}^N [\mathbf{v}]_i = 1$.

We also recall the concept of *gauge transformation* [1] (also known as change of orthant order). We denote by $\mathcal{D}$ the set of all $N \times N$ diagonal matrices whose diagonal entries are either 1 or -1. Two $N \times N$ matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be *equivalent through a gauge transformation*, if there exists $D \in \mathcal{D}$ such that $\mathcal{A}_2 = D\mathcal{A}_1 D$. In the sequel, we will make use of the following fundamental result.

Proposition 1 [1] *Given a connected, weighted and signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, let $\mathcal{L}$ denote the Laplacian matrix associated with $\mathcal{A}$. The following facts are equivalent:*

1. *$\mathcal{G}$ is structurally balanced;*
2. *$\mathcal{A}$ is equivalent, through a gauge transformation, to a nonnegative matrix;*
3. *$\lambda_1(\mathcal{L}) = 0$.*

Note that, when $\mathcal{G}$ is structurally balanced and connected, 0 is a simple eigenvalue of $\mathcal{L}$. Next we establish a key technical result that will be used in subsequent sections.

[1] Indeed, condition $\sum_{i=1}^N [\mathbf{v}]_i = 0$ is equivalent to $\mathbf{v}^\top \mathbf{1}_N = 0$. So, if every eigenvector $\mathbf{v}$ would be orthogonal to $\mathbf{1}_N$, the vector space generated by all the eigenvectors would be a vector subspace of $\langle \mathbf{1}_N \rangle^\perp$, and hence the eigenvectors could not generate $\mathbb{R}^N$, a contradiction.

Lemma 1 *Given a connected, weighted and signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with $\mathcal{A}$ having at least one negative entry, let $\mathcal{L}$ denote the Laplacian matrix associated with $\mathcal{A}$. If $\mathcal{G}$ is structurally balanced, and $\mathcal{V}_1$ and $\mathcal{V}_2$ denote the two disjoint sets of vertices that partition the set $\mathcal{V}$, then*

i) at least two of the row sums $\sum_{j=1}^N [\mathcal{L}]_{ij}, i \in [1, N]$, are nonzero, and hence positive;

ii) if all row sums $\sum_{j=1}^N [\mathcal{L}]_{ij}, i \in [1, N]$, are identical, then the two sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ have the same cardinality.

Proof. We first observe that the signed weight matrix $\mathcal{A}$ can be expressed as $\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-$, where $\mathcal{A}_+$ and $\mathcal{A}_-$ are the matrices that represent the positive and the negative entries of $\mathcal{A}$, respectively. Accordingly, if we denote by $\mathcal{L}_+$ and $\mathcal{L}_-$ the Laplacians corresponding to the matrices $\mathcal{A}_+$ and $\mathcal{A}_-$, respectively, then $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$. Since all row sums of $\mathcal{L}_+$ are zero, the row sums of $\mathcal{L}$ ordinately coincide with the row sums of the nonnegative matrix $\mathcal{L}_-$. Moreover, $\mathcal{L}_- \mathbf{1}_N = -2\mathcal{A}_- \mathbf{1}_N$. By assumption, at least one element $[\mathcal{A}]_{ik}$ is negative (and hence, by symmetry, $[\mathcal{A}]_{ki} < 0$), therefore condition i) holds (for the ith and kth rows).
To prove ii), we need to show that if all the row sums of $\mathcal{A}_-$ take the same value, say $\alpha < 0$, then $|\mathcal{V}_1| = |\mathcal{V}_2|$. By the structural balance assumption, the matrix $\mathcal{A}_-$ can be reduced, by means of permutations, to the following form

$$\mathcal{A}_- = \begin{bmatrix} 0 & \mathcal{A}_{12} \\ \mathcal{A}_{12}^\top & 0 \end{bmatrix}, \qquad -\mathcal{A}_{12} \in \mathbb{R}_+^{k \times (N-k)}, \mathcal{A}_{12} \neq 0.$$

Next, assume that $k = |\mathcal{V}_1| \neq |\mathcal{V}_2| = N - k$. If both conditions

$$\begin{cases} \mathcal{A}_{12} \mathbf{1}_{N-k} &= \alpha \mathbf{1}_k, \\ \mathbf{1}_k^\top \mathcal{A}_{12} &= \alpha \mathbf{1}_{N-k}^\top, \end{cases}$$

were true, the fact that $\alpha \neq 0$ would lead to the contradiction

$$\alpha(N-k) = \left(\mathbf{1}_k^\top \mathcal{A}_{12}\right) \mathbf{1}_{N-k} \neq \mathbf{1}_k^\top \left(\mathcal{A}_{12} \mathbf{1}_{N-k}\right) = \alpha k.$$

Hence, $k = N - k$. ∎

Remark 1 *It is worthwhile noticing that, in the special case when all row sums of $\mathcal{L}$ take the same value, say $\hat{\lambda} \in \mathbb{R}$, namely $\mathcal{L}\mathbf{1}_N = \hat{\lambda}\mathbf{1}_N$, then there is a special relationship between the Laplacian $\mathcal{L}$ and the unsigned Laplacian $\mathcal{L}^u$ of $\mathcal{A}$, this latter defined as*

$$[\mathcal{L}^u]_{ij} = \begin{cases} \sum_{(j,i) \in \mathcal{E}} [\mathcal{A}]_{ij}, & \text{if } i = j \\ -[\mathcal{A}]_{ij}, & \text{if } i \neq j. \end{cases}$$

Indeed, by following the same lines as in the proof of Lemma 1, it is easily seen that

$$\mathcal{L}^u = \mathcal{L} - 2\mathcal{C}_-,$$

where $\mathcal{C}_-$ is the (diagonal) connectivity matrix of $\mathcal{A}_-$. On the other hand, as $\mathcal{L}^u \mathbf{1}_N = 0$, condition $\mathcal{L}\mathbf{1}_N = \hat{\lambda}\mathbf{1}_N$ is equivalent to

$$\mathcal{C}_- \mathbf{1}_N = \frac{\hat{\lambda}}{2} \mathbf{1}_N,$$

and since $\mathcal{C}_-$ is diagonal, this means $\mathcal{C}_- = \frac{\hat{\lambda}}{2} I$. As a result,

$$\mathcal{L}^u = \mathcal{L} - \hat{\lambda} I,$$

which ensures that

$$\sigma(\mathcal{L}) = (\lambda_1(\mathcal{L}), \lambda_2(\mathcal{L}), \dots, \lambda_N(\mathcal{L})) = (\mu_1 + \hat{\lambda}, \mu_2 + \hat{\lambda}, \dots, \mu_N + \hat{\lambda}),$$

where $(\mu_1, \mu_2, \dots, \mu_N)$ is the spectrum of $\mathcal{L}^u$.

3 Bipartite consensus

As a first step we want to derive necessary and sufficient conditions for bipartite consensus of non-cooperating multi-agent systems described as in (6), under the assumption that the feedback control algorithm is described as in (7) and the communication graph $\mathcal{G}$ is structurally balanced.

It will be shown that the results regarding consensus of cooperating multi-agent systems described by (6), first reported in [32] (see, also, [31, 14] for additional information), can be extended to the case of non-cooperating agents and bipartite consensus under the assumption that the control algorithm is described as in (7), and provided that the communication graph $\mathcal{G}$ is structurally balanced.

Theorem 1 *Consider the multi-agent system (6) with control algorithm (7), where $K \in \mathbb{R}^{1 \times N}$ is a matrix to be determined and the communication graph $\mathcal{G}$ is connected and structurally balanced. The agents asymptotically reach bipartite consensus if and only if the $N-1$ polynomials*

$$\psi_i(s) := \det(sI_n - A) + \lambda_i(\mathcal{L}) \cdot [K \mathrm{adj}(sI_n - A) b] = \det(sI_n - A + \lambda_i(\mathcal{L}) b, K), \qquad i \in [2, N], \quad (8)$$

where $\mathrm{adj}((sI_n - A)$ denotes the adjoint matrix of $sI_n - A$, are Hurwitz.

Proof. As $\mathcal{G}$ is structurally balanced, we can assume without any loss of generality that $\mathcal{V}_1 = [1, k]$ and $\mathcal{V}_2 = [k+1, N]$, and hence $\mathcal{A}$ is described as in (4), where $\mathcal{A}_{11} \in \mathbb{R}_+^{k \times k}$, $\mathcal{A}_{22} \in \mathbb{R}_+^{(N-k) \times (N-k)}$, while $-\mathcal{A}_{12} \in \mathbb{R}_+^{k \times N}$. It is easily seen that the gauge transformation

$$D = \begin{bmatrix} I_k & 0 \\ 0 & -I_{N-k} \end{bmatrix} \in \mathcal{D}$$

is such that $D\mathcal{A}D$ is a nonnegative matrix. Let $\mathbf{p} := [\, p_1 \quad p_2 \quad \dots \quad p_N \,]^\top \in \mathbb{R}^N$ be a left eigenvector of $\mathcal{L}$ corresponding to $\lambda_1(\mathcal{L}) = 0$. Since $D\mathbf{p}$ is the left eigenvector of the $D\mathcal{L}D$, corresponding to the zero eigenvalue, and $D\mathcal{L}D$ is the Laplacian matrix associated with the nonnegative matrix $D\mathcal{A}D$, we can assume

$$p_i = \begin{cases} \frac{1}{N}, & i \in [1, k]; \\ -\frac{1}{N}, & i \in [k+1, N]. \end{cases}$$

Introduce the nonsingular matrix

$$S := \left[\begin{array}{c|ccc|ccc} p_1 & p_2 & \dots & p_k & p_{k+1} & \dots & p_N \\ \hline -\mathbf{1}_{k-1} & & I_{k-1} & & & 0 & \\ \hline -\mathbf{1}_{N-k} & & 0 & & & -I_{N-k} & \end{array}\right]$$

and consider the coordinate transformation

$$\begin{aligned} \left[\begin{array}{c} \zeta(t) \\ \hline \delta_2(t) \\ \vdots \\ \delta_k(t) \\ \hline \delta_{k+1} \\ \vdots \\ \delta_N(t) \end{array}\right] &:= \left[\begin{array}{c|ccc|ccc} p_1 I_n & p_2 I_n & \dots & p_k I_n & p_{k+1} I_n & \dots & p_N I_n \\ \hline -\mathbf{1}_{k-1} \otimes I_n & & I_{n(k-1)} & & & 0 & \\ \hline -\mathbf{1}_{N-k} \otimes I_n & & 0 & & & -I_{n(N-k)} & \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1(t) \\ \hline \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_k(t) \\ \hline \mathbf{x}_{k+1} \\ \vdots \\ \mathbf{x}_N(t) \end{array}\right] \\ &= (S \otimes I_n)\mathbf{x}(t). \end{aligned}$$

Note that $\delta_i(t) = \mathbf{x}_i(t) - \mathbf{x}_1(t)$, for $i \in [2, k]$, while $\delta_i(t) = -\mathbf{x}_i(t) - \mathbf{x}_1(t)$, for $i \in [k+1, N]$. So, convergence to zero of all δ_i's corresponds to bipartite consensus. It is a matter of simple computation to see that

$$S\mathcal{L}S^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{L}_2 \end{bmatrix}$$

and hence the $N-1$ eigenvalues of $\mathcal{L}_2$ are $0 < \lambda_2(\mathcal{L}) \leq \ldots \leq \lambda_N(\mathcal{L})$. Also, upon setting

$$\hat{\mathbf{u}}(t) := \left[\begin{array}{c} \hat{u}_1(t) \\ \hline \hat{u}_2(t) \\ \vdots \\ \hat{u}_N(t) \end{array}\right] = S\mathbf{u}(t),$$

the multi-agent system dynamics are described, with respect to the new state and input coordinates, by

$$\dot{\zeta}(t) = A\zeta(t) + b\hat{u}_1(t), \tag{9}$$

$$\begin{bmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{bmatrix} = (I_{N-1} \otimes A) \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix} + (I_{N-1} \otimes b) \begin{bmatrix} \hat{u}_2(t) \\ \vdots \\ \hat{u}_N(t) \end{bmatrix}, \tag{10}$$

$$\hat{u}_1(t) = 0, \tag{11}$$

$$\begin{bmatrix} \hat{u}_2(t) \\ \vdots \\ \hat{u}_N(t) \end{bmatrix} = -(\mathcal{L}_2 \otimes K) \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix}. \tag{12}$$

These equations can be equivalently rewritten as

$$\dot{\zeta}(t) = A\zeta(t), \tag{13}$$

$$\begin{bmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{bmatrix} = [(I_{N-1} \otimes A) - (I_{N-1} \otimes b)(\mathcal{L}_2 \otimes K)] \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix} \tag{14}$$

Due to the way the variables $\delta_i(t)$'s are defined, it is clear that bipartite consensus is achieved if and only if the system (14) is asymptotically stable. This is the case if and only if the characteristic polynomial of $[(I_{N-1} \otimes A) + (I_{N-1} \otimes b)(\mathcal{L}_2 \otimes K)]$ is Hurwitz. By following the same reasoning adopted in [32], it can be shown that

$$\det\Big(sI_{n(N-1)} - [(I_{N-1} \otimes A) - (I_{N-1} \otimes b)(\mathcal{L}_2 \otimes K)]\Big) = \prod_{i=2}^{N}\Big(\det(sI_n - A) + \lambda_i(\mathcal{L}) \cdot [K\text{adj}(sI_n - A)b]\Big),$$

and hence bipartite consensus is achieved if and only if all the polynomials $\psi_i(s), i \in [2, N]$, are Hurwitz. ■

Remark 2 *By referring to the notation adopted within the proof, it is easy to see that, when the polynomials $\psi_i(s), i \in [2, N]$, in (8) are Hurwitz, and hence the new state variables $\delta_i(t), i \in [2, N]$, converge to zero, then*

$$\begin{aligned} \zeta(t) &= \sum_{i=1}^{N} p_i \mathbf{x}_i(t) = \sum_{i=1}^{k} p_i(\mathbf{x}_1(t) + \delta_i(t)) + \sum_{i=k+1}^{N} p_i(-\mathbf{x}_1(t) - \delta_i(t)) \\ &= \left[\sum_{i=1}^{k} p_i - \sum_{i=k+1}^{N} p_i\right] \mathbf{x}_1(t) + \Big(\sum_{i=1}^{k} p_i\delta_i(t) - \sum_{i=k+1}^{N} p_i\delta_i(t)\Big) \\ &= \mathbf{x}_1(t) + \Big(\sum_{i=1}^{k} p_i\delta_i(t) - \sum_{i=k+1}^{N} p_i\delta_i(t)\Big) \end{aligned}$$

tends to $\mathbf{x}_1(t)$, as t tends to $+\infty$, and hence describes the behavior of the first group of agents, while $-\zeta(t)$ describes the behavior of the second group. As for standard consensus, the evolution of $\zeta(t)$ depends on the initial conditions and on the properties of the state matrix A, in the agents' description. Indeed, if A is Hurwitz, the only bipartite consensus we achieve is the trivial one to the zero trajectory, while if A is simply stable or not stable, corresponding to certain initial conditions, $\zeta(t)$ may be bounded or even diverging.

Remark 3 *The Hurwitz property of the polynomials $\psi_i(s), i \in [2, N]$, is the same condition derived in [32] (see also [31]) for the solvability of the standard consensus problem, in case of cooperation. So, the assumption on the communication graph does not change the solvability condition but only the outcome.*

Next, we show that the stabilizability of the pair (A, b) is a necessary and sufficient condition for the solvability of the bipartite consensus problem. The result is analogous to the one obtained for the standard consensus problem, since it reduces to the problem of imposing that the real polynomials $\psi_i(s), i \in [2, N]$, are all Hurwitz, a result obtained in [31] (see also [10]). Specifically, upon defining the polynomial

$$\psi(s, \lambda) := \det(sI_n - A) + \lambda K\text{adj}(sI_n - A)b, \qquad \lambda \in \mathbb{R},$$

our goal is to show that, under the stabilizability assumption on (A, b), it is always possible to choose the polynomial $K\text{adj}(sI_n - A)b$, of degree smaller than $n = \deg\det(sI_n - A)$, in such a way that $\psi(s, \lambda)$ is Hurwitz for every λ belonging to some half-line $[\hat{\lambda}, +\infty) \subsetneq [0, +\infty)$ including all the positive eigenvalues of $\mathcal{L}$.

For completeness, we provide a constructive proof of the above statement. The proof is completely independent from the one given in [31] (based on the work of Tuna [30]) and is essential in demonstrating why the analogous algebraic conditions for reaching standard consensus are more difficult to achieve.

Theorem 2 *Consider the multi-agent system (6) with control algorithm (7), and suppose that the communication graph $\mathcal{G}$ is connected and structurally balanced. There exists $K \in \mathbb{R}^{1\times N}$ such that agents asymptotically reach bipartite consensus if and only if the pair (A, b) is stabilizable.*

Proof. By the previous theorem, we have to show that the stabilizability of the pair (A, b) is a necessary and sufficient condition for the existence of a matrix K such that all the polynomials $\psi_i(s), i \in [2, N]$, in (8) are Hurwitz.

[Necessity] If the pair (A, b) is not stabilizable, there exists a similarity transformation T such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad Tb = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}, \tag{15}$$

where the pair $(A_{11}, b_1) \in \mathbb{R}^{r\times r} \times \mathbb{R}^r$ is controllable, and the $(n-r)$-dimensional matrix A_{22} is not Hurwitz. Consequently, for every choice of K both $\det(sI_n - A)$ and $K\text{adj}(sI_n - A)b$ are multiple of $\det(sI_{n-r} - A_{22})$. Therefore all $\psi_i(s)$ are multiple of $\det(sI_{n-r} - A_{22})$ and hence are not Hurwitz.

Conversely, assume that the pair (A, b) is stabilizable, and hence there exists a similarity transformation T such that TAT^{-1} and Tb are described as in (15), with $\det(sI_{n-r} - A_{22})$ Hurwitz and

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{r-1} \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{16}$$

If we accordingly partition K as $K = [\,K_1 \quad K_2\,]$, with $K_1 = [\,k_0 \quad k_1 \quad \dots \quad k_{r-1}\,]$, then

$$\begin{aligned} \det(sI_n - A) &= \det(sI_r - A_{11})\det(sI_{n-r} - A_{22}) \\ K\text{adj}(sI_n - A)b &= [k_{r-1}s^{r-1} + k_{r-2}s^{r-2} + \ldots + k_1 s + k_0]\det(sI_{n-r} - A_{22}). \end{aligned}$$

Clearly if $\det(sI_r - A_{11})$ is also Hurwitz, we can trivially make all the polynomials $\psi_i(s), i \in [2, N]$, Hurwitz by choosing $K = 0$. On the other hand, assume that

$$\det(sI_r - A_{11}) = [s^r + a_{r-1}s^{r-1} + \ldots + a_1 s + a_0] = d_-(s)d_{0+}(s),$$

where $d_{0+}(s) \in \mathbb{R}[s]$ is a monic polynomial, with degree $\ell \geq 1$, whose roots have nonnegative real part, while $d_-(s)$ is monic, Hurwitz and has degree $r - \ell$. Since all k_i's are arbitrary, we can choose them in such a way that $k_{r-1}s^{r-1}+k_{r-2}s^{r-2}+\ldots+k_1s+k_0$ is a multiple of $d_-(s)$. In this way

$$k_{r-1}s^{r-1}+k_{r-2}s^{r-2}+\ldots+k_1s+k_0=d_-(s)[\beta_{r-\ell-1}s^{r-\ell-1}+\ldots+\beta_1s+\beta_0],$$

where β_i, $i=0,\ldots,r-\ell-1$, are arbitrary. To prove that for a suitable choice of the matrix K, all $\psi_i(s), i\in[2,N]$, can become Hurwitz, is equivalent to show that it is possible to choose $\beta_0,\beta_1,\ldots,\beta_{r-\ell-1}\in\mathbb{R}$ in such a way that the $N-1$ polynomials

$$p_i(s):=d_{0+}(s)+\lambda_i(\mathcal{L})[\beta_{r-\ell-1}s^{r-\ell-1}+\ldots+\beta_1s+\beta_0], \qquad i\in[2,N],$$

are Hurwitz. Introduce any monic Hurwitz polynomial $\hat\beta(s)\in\mathbb{R}[s]$ of degree $r-\ell-1$. By referring to the positive root locus of the function $\dfrac{\hat\beta(s)}{d_{0+}(s)}$, namely to the set of zeros of the expression

$$d_{0+}(s)+\gamma\hat\beta(s), \qquad \gamma\in\mathbb{R}_+,$$

it is clear that all the root locus branches start from the $r-\ell$ unstable zeros of $d_{0+}(s)$ and end in the open left half plane: $r-\ell-1$ end at the finite zeros of $\hat\beta(s)$, while the last branch goes to $-\infty$ along the negative real halfline. This proves that there exists $\bar\gamma>0$ such that for every $\gamma\geq\bar\gamma$ the polynomial $d_{0+}(s)+\gamma\hat\beta(s)$ is Hurwitz. But then, by choosing

$$\beta_{r-\ell-1}s^{r-\ell-1}+\ldots+\beta_1s+\beta_0=\frac{\bar\gamma}{\lambda_2(\mathcal{L})}\hat\beta(s),$$

we can ensure that all polynomials $p_i(s), i\in[2,N]$, and hence all polynomials $\psi_i(s), i\in[2,N]$, are Hurwitz. This proves that if (A,b) is stabilizable, then some matrix K can be found such that all $\psi_i(s), i\in[2,N]$, are Hurwitz. ∎

Example 1 *Consider the system of $N=3$ agents, each of them described by the 3-dimensional ($n=3$) (controllable, and hence stabilizable) state-space model*

$$\dot{\mathbf{x}}_i(t)=A\mathbf{x}_i(t)+bu_i(t)=\begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix}\mathbf{x}_i(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u_i(t), \qquad i\in[1,3].$$

Note that $\det(sI_3-A)=(s-1)^3$ *and* $\mathrm{adj}(sI_3-A)b=\begin{bmatrix}1 & s & s^2\end{bmatrix}^\top$. *We assume that the signed and weighted communication matrix $\mathcal{A}$ is*

$$\mathcal{A}=\begin{bmatrix}0&-1&-2\\-1&0&3\\-2&3&0\end{bmatrix},$$

which means that the corresponding graph is structurally balanced with $\mathcal{V}_1=\{1\}$ and $\mathcal{V}_2=\{2,3\}$. The associated Laplacian is

$$\mathcal{L}=\begin{bmatrix}3&1&2\\1&4&-3\\2&-3&5\end{bmatrix},$$

and its eigenvalues are $\lambda_1(\mathcal{L}) = 0,\ \lambda_2(\mathcal{L}) = 6 - \sqrt{3},\ \lambda_3(\mathcal{L}) = 6 + \sqrt{3}$. *The matrix*

$$S = \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

is such that

$$S\mathcal{L}S^{-1} = \begin{bmatrix} 0 & \\ & \mathcal{L}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -2 & 7 \end{bmatrix}.$$

Introduce the state and input transformation:

$$\begin{bmatrix} \zeta(t) \\ \delta_2(t) \\ \delta_3(t) \end{bmatrix} = (S \otimes I_3)\mathbf{x}(t), \qquad \hat{\mathbf{u}}(t) = S\mathbf{u}(t).$$

The state dynamics with respect to the new basis and the new input are described as in (13)-(14) for $N = 3$. *As* $\det(sI_3 - A) = (s-1)^3$ *has all roots in the closed right half-plane, we assume* $\hat{\beta}(s) = (s+1)^2 = s^2 + 2s + 1$. *It is easily seen (for instance, through Routh criterion), that*

$$\det(sI_3 - A) + \gamma\hat{\beta}(s) = (s-1)^3 + \gamma(s+1)^2,$$

is Hurwitz for every $\gamma > 1 + \sqrt{5}$. *Choose, then, for simplicity,* $\bar{\gamma} = 6 - \sqrt{3} = \lambda_2(\mathcal{L}) > 1 + \sqrt{5}$ *and* $K\mathrm{adj}(sI_2 - A)b = \hat{\beta}(s)$. *Then*

$$\psi_i(s) = \det(sI_3 - A) + \lambda_i(\mathcal{L})\hat{\beta}(s) = (s-1)^3 + \lambda_i(\mathcal{L})(s+1)^2, \qquad i = 2, 3,$$

are two Hurwitz polynomials. The choice $K\mathrm{adj}(sI_2 - A)b = \hat{\beta}(s) = s^2 + 2s + 1$ *corresponds to setting* $K = [\,1 \quad 2 \quad 1\,]$, *and it is just a matter of straightforward computation to verify that the matrix* $[(I_2 \otimes A) - (I_2 \otimes b)(\mathcal{L}_2 \otimes K)]$ *is Hurwitz.*

4 Consensus

In this section, we investigate whether for multi-agent systems described in (6), with a weighted, signed, connected and structurally balanced communication graph $\mathcal{G}$, the control algorithm (7) may lead to a common decision for some suitable choice of K. This is equivalent to investigating whether it is possible to impose that for every index $i \in [1, N]$ we have $\lim_{t\to+\infty} \mathbf{x}_i(t) = \zeta(t)$, for some function $\zeta(t)$. In the following, to avoid considering either trivial or undesirable cases[2], we will search for conditions ensuring that the state space evolution of the agents is always bounded, but not always converging to zero. In this way, we rule out the case when all the agents dynamics converge to zero, independently of the initial conditions, but also the case when all agents may have diverging trajectories.

[2]In addition, notice that the trivial case, when all the agents' states asymptotically converge to zero, has already been regarded as a special case of the bipartite consensus case.

To investigate this problem, we need to distinguish between the following two cases: A) the case when all row sums of $\mathcal{L}$ take the same value (if so, we know from Lemma 1 that the two sets of the partition, $\mathcal{V}_1$ and $\mathcal{V}_2$, have the same cardinality, say $k = N/2$); and B) the case when not all row sums of $\mathcal{L}$ are the same.

4.1 All row sums of $\mathcal{L}$ take the same value

Suppose that $\exists\ \hat{\lambda} \in \mathbb{R}$ such that $\mathcal{L}\mathbf{1}_N = \hat{\lambda}\mathbf{1}_N$. Clearly, by Lemma 1, $\hat{\lambda} > 0$ and obviously $\hat{\lambda} \in \sigma(\mathcal{L})$. The vector $\mathbf{p} := [\,1/N \quad 1/N \quad \dots \quad 1/N\,]^\top \in \mathbb{R}^N$ is hence a left eigenvector of $\mathcal{L}$ corresponding to $\hat{\lambda}$. We introduce the nonsingular matrix

$$S := \left[\begin{array}{c|ccc} 1/N & 1/N & \dots & 1/N \\ \hline -\mathbf{1}_{N-1} & & I_{N-1} & \end{array}\right].$$

Consider the coordinate transformation

$$\left[\begin{array}{c} \zeta(t) \\ \hline \delta_2(t) \\ \vdots \\ \delta_N(t) \end{array}\right] := \left[\begin{array}{c|ccc} 1/NI_n & 1/NI_n & \dots & 1/NI_n \\ \hline -\mathbf{1}_{N-1}\otimes I_n & & I_{n(N-1)} & \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1(t) \\ \hline \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_N(t) \end{array}\right] = (S\otimes I_n)\mathbf{x}(t).$$

It is a matter of simple computation to see that in this case, due to the fact that $\sum_{j=1}^N [\mathcal{L}]_{ij} = \hat{\lambda}, \forall\ i \in [1, N]$,

$$S\mathcal{L}S^{-1} = \begin{bmatrix} \hat{\lambda} & 0 \\ 0 & \mathcal{L}_2 \end{bmatrix},$$

for some matrix $\mathcal{L}_2$, whose eigenvalues are the $N-1$ (possibly not distinct) eigenvalues of $\mathcal{L}$, obtained by removing $\hat{\lambda}$ from the N-tuple $(\lambda_1(\mathcal{L}), \lambda_2(\mathcal{L}), \dots, \lambda_N(\mathcal{L}))$. Note that there is no guarantee that $\hat{\lambda}$ is a simple eigenvalue of $\mathcal{L}$ and hence $\hat{\lambda}$ can belong also to $\sigma(\mathcal{L}_2)$. By following the same steps as in the proof of Theorem 1 (but in this case $\delta_i(t) = \mathbf{x}_i(t) - \mathbf{x}_1(t)$ for all $i \in [2, N]$), we can obtain the following description:

$$\begin{aligned} \dot{\zeta}(t) &= (A - \hat{\lambda}bK)\zeta(t), \\ \begin{bmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{bmatrix} &= [(I_{N-1}\otimes A) - (I_{N-1}\otimes b)(\mathcal{L}_2\otimes K)] \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix}. \end{aligned}$$

Therefore, consensus to some common bounded trajectory $\zeta(t)$ is achieved if and only if the following conditions hold:

i) $\hat{\lambda}$ is a simple eigenvalue of $\mathcal{L}$;

ii) all polynomials $\psi_i(s) = \det(sI_n - A) + \lambda_i(\mathcal{L})[K\mathrm{adj}(sI_n - A)b]$, $\lambda_i \in \sigma(\mathcal{L}), \lambda_i \neq \hat{\lambda}$, are Hurwitz;

iii) $A - \hat{\lambda}bK$ is simply (but not asymptotically) stable.

Remark 4 *It is worth noticing that while the conditions for bipartite consensus given in Theorem 2 pertain only to the properties of the agents' description, the possibility of reaching a consensus among agents with antagonistic interactions depends not only on the model adopted for the agents, but also on the algebraic properties of the associated Laplacian.*

We illustrate the above distinction by means of two examples.

Example 2 *Consider the system of four agents ($N = 4$), each of them described by the 3-dimensional ($n = 3$) state-space model*

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + bu_i(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_i(t), \qquad i \in [1, 4].$$

Note that $\det(sI_3 - A) = (s+1)^3$. *We assume that the signed communication matrix $\mathcal{A}$ is*

$$\mathcal{A} = \begin{bmatrix} 0 & 2 & -1 & -1 \\ 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 3 \\ -1 & -1 & 3 & 0 \end{bmatrix},$$

which means that the corresponding graph is structurally balanced with two antagonistic groups of agents of the same cardinality $\mathcal{V}_1 = \{1, 2\}$ and $\mathcal{V}_2 = \{3, 4\}$. The associated Laplacian is

$$\mathcal{L} = \begin{bmatrix} 4 & -2 & 1 & 1 \\ -2 & 4 & 1 & 1 \\ 1 & 1 & 5 & -3 \\ 1 & 1 & -3 & 5 \end{bmatrix},$$

and its eigenvalues are $\lambda_1(\mathcal{L}) = 0,\ \lambda_2(\mathcal{L}) = 4,\ \lambda_3(\mathcal{L}) = 6,\ \lambda_4(\mathcal{L}) = 8$. We note that $\mathcal{L}\mathbf{1}_4 = 4\mathbf{1}_4$, and hence all row sums are the same and they are equal to $\hat{\lambda} = 4$. Assume that the agents adopt the feedback control law (7), with $K = [\,12 \quad 1 \quad 1\,]$. The consensus is possible if and only if conditions i)-iii) above hold. Clearly, $\hat{\lambda} = 4$ is a simple eigenvalue of $\mathcal{L}$, and hence condition i) holds. If we consider the polynomial $\psi(s, \lambda) = \det(sI_n - A) + \lambda[K\mathrm{adj}(sI_n - A)b]$, as λ varies over the nonnegative real numbers, it is easily seen, by resorting to the Routh criterion, for instance, that $\psi(s, \lambda)$ is Hurwitz for every $\lambda \in [0, 2) \cup (4, +\infty]$. This implies, in particular, that $\psi(s, \lambda)$ is Hurwitz for $\lambda \in \{0, 6, 8\}$. So, condition ii) holds. Finally, for $\lambda = \hat{\lambda} = 4$, $\psi(s, \lambda) = (s+7)(s^2+7)$ which proves that $A - \hat{\lambda}bK$ is simply (but not asymptotically) stable. So, for certain sets of initial conditions, there is convergence of the state trajectories of all agents to a sinusoidal trajectory.

Example 3 *Consider a system of $N = 4$ agents, with signed communication matrix*

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{bmatrix},$$

which ensures that the corresponding graph is structurally balanced with $\mathcal{V}_1 = \{1,2\}$ and $\mathcal{V}_2 = \{3,4\}$. The associated Laplacian is

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & 1 & 1 \\ -1 & 3 & -1 & -1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix},$$

and its eigenvalues are $\lambda_1(\mathcal{L}) = 0,\ \lambda_2(\mathcal{L}) = \lambda_3(\mathcal{L}) = \lambda_4(\mathcal{L}) = 4$. Moreover all row sums are equal to 4. In this case it is clear that, independently of the agents' state-space description (A, b), no consensus can be reached except possibly for the one that leads all the agents to the zero state, since condition i) is not satisfied.

At this point the following natural question arises: assuming that condition i) holds, namely that $\hat{\lambda} > 0$ is a simple eigenvalue of $\mathcal{L}$, under what conditions some feedback matrix K can be found so that conditions ii) and iii) hold? It is worthwhile noticing that this problem is more complicated than the analogous one we addressed for the bipartite consensus in Section III, because in this case, to satisfy ii) and iii), a matrix K must be found such that $\psi(s, \lambda)$ is an Hurwitz polynomial except on a (possibly infinite) interval having $\hat{\lambda}$ on one of its extremes. We need to address the following two situations:

1. If $\hat{\lambda} = \lambda_N(\mathcal{L})$, namely $\hat{\lambda}$ is the largest of the eigenvalues of $\mathcal{L}$, we have to find K such that $\psi(s, \lambda)$ is Hurwitz in $[0, \hat{\lambda})$[3] and it has a simple zero at 0 (or a pair of simple imaginary conjugate zeros) for $\lambda = \hat{\lambda}$.
2. If, on the other hand, $\hat{\lambda}$ is not the largest eigenvalue of $\mathcal{L}$, we have to find K such that[4] $\psi(s, \lambda)$ is Hurwitz in $[0, \hat{\lambda}) \cup (\hat{\lambda} + \Delta, +\infty)$, and is not Hurwitz in $[\hat{\lambda}, \hat{\lambda} + \Delta]$, where $\Delta > 0$ is chosen in such a way that no eigenvalues of $\mathcal{L}$ lie in $[\hat{\lambda}, \hat{\lambda} + \Delta]$ except for $\hat{\lambda}$.

It is clear that the latter case imposes strong constraints on the polynomial $\psi(s, \lambda)$ that, in general, cannot be achieved for an arbitrary choice of $\mathcal{L}$, and hence for an arbitrary choice of the eigenvalues $\lambda_i \in \sigma(\mathcal{L})$, unless the degree of $\psi(s, \lambda)$ is sufficiently large. More precisely, as we know that the characteristic polynomial of the non-controllable part of the pair (A, b) is a common divisor of $\det(sI_n - A)$ and $K\mathrm{adj}(sI_n - A)b$ for every choice of K, and hence also a divisor of $\psi(s, \lambda)$ for every $\lambda \in \mathbb{R}_+$, in order to be able to meet the previous constraints in any possible situation we have to assure that the part of the polynomial $\psi(s, \lambda)$ we can actually "shape" according to the needs has at least degree 3. The preceding discussion leads to the following result.

Proposition 2 *Consider the multi-agent system (6) with control algorithm (7). Suppose that the communication graph $\mathcal{G}$ is connected and structurally balanced with respect to the partition in two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ of equal cardinality. Suppose, also, that the row sums of $\mathcal{L}$ all equal to $\hat{\lambda} > 0$, and $\hat{\lambda}$ is a simple eigenvalue of $\mathcal{L}$. If the following conditions hold true:*

[3] Recall that 0 is always an eigenvalue of $\mathcal{L}$. Note, also, that this is not the only possible solution, as we could introduce more complicated conditions on the behavior of $\psi(s, \lambda)$, as λ varies in $\mathbb{R}_+$, but it is the simplest.

[4] Also in this case the algebraic conditions imposed on $\psi(s, \lambda)$ could be slightly different. For instance we could replace the interval $[\hat{\lambda}, \hat{\lambda} + \Delta]$ with $[\hat{\lambda} - \Delta, \hat{\lambda}]$, but this would not affect the solvability conditions.

a) A is Hurwitz;

b) the dimension of the controllable subspace of the pair (A, b) is at least 3;

then there exists $K \in \mathbb{R}^{1\times N}$ such that the N agents asymptotically reach consensus.

Proof. Choose any $\Delta > 0$ such that no eigenvalue of $\mathcal{L}$ lie in $[\hat\lambda, \hat\lambda + \Delta]$, except for $\hat\lambda$ itself. By assumptions a) and b), and following the same process and notation as in the proof of Theorem 2, we can factorize the Hurwitz polynomial $\det(sI_n - A)$ as $\det(sI_n - A) = d_-(s)d(s)$, where both $d_-(s)$ and $d(s)$ are Hurwitz monic polynomials, the characteristic polynomial $\det(sI_{n-r} - A_{22})$ of the non-controllable part of (A, b) (see the decomposition (15)) is a factor of $d_-(s)$, while $\deg d(s) = 3$.

Since for every choice of a polynomial $p(s)$ of degree $n-1$, which is a multiple of $\det(sI_{n-r} - A_{22})$, we can always find a matrix K such that $K\mathrm{adj}(sI_n - A)b = p(s)$, we want to show that by choosing $p(s) = d_-(s)n(s)$ for a suitable second order Hurwitz polynomial $n(s)$, we can always ensure that

$$\psi(s, \lambda) = d_-(s)[d(s) + \lambda n(s)]$$

is Hurwitz for $\lambda \in [0, \hat\lambda) \cup (\hat\lambda + \Delta, +\infty)$, and is not Hurwitz for $\lambda \in [\hat\lambda, \hat\lambda + \Delta]$. Clearly, this amounts to showing that a second order Hurwitz polynomial $n(s)$ can be found such that the third order monic polynomial $d(s) + \lambda n(s)$ satisfies the previous constraints. Assume, w.l.o.g., $d(s) = s^3 + d_2 s^2 + d_1 s + d_0$, and note that, by the Routh criterion, the Hurwitz property of this polynomial is equivalent to the following conditions on its coefficients:

$$d_0, d_1, d_2 > 0 \qquad \text{and} \qquad d_1 d_2 - d_0 > 0. \tag{17}$$

We want to prove that, by choosing

$$n(s) = s^2 + \frac{d_1 d_2 - d_0}{\hat\lambda(\hat\lambda + \Delta)} s + \left(d_1 + \frac{(d_2 + 2\hat\lambda + \Delta)(d_1 d_2 - d_0)}{\hat\lambda(\hat\lambda + \Delta)} \right),$$

we can ensure that $d(s) + \lambda n(s)$ is Hurwitz for $\lambda \in [0, \hat\lambda) \cup (\hat\lambda + \Delta, +\infty)$, and is not Hurwitz for $\lambda \in [\hat\lambda, \hat\lambda + \Delta]$. Note that by the assumptions on $\hat\lambda, \Delta$ and (17), the second order polynomial $n(s)$ is necessarily Hurwitz, as all its coefficients are positive. By applying the Routh criterion to $d(s) + \lambda n(s)$ we obtain

$$\begin{array}{c|cc}
3 & 1 & d_1 + \lambda \dfrac{d_1 d_2 - d_0}{\hat\lambda(\hat\lambda + \Delta)} \\[2ex]
2 & d_2 + \lambda & d_0 + \lambda \left(d_1 + \dfrac{(d_2 + 2\hat\lambda + \Delta)(d_1 d_2 - d_0)}{\hat\lambda(\hat\lambda + \Delta)} \right) \\[2ex]
1 & \dfrac{d_1 d_2 - d_0}{\hat\lambda(\hat\lambda + \Delta)} \cdot \dfrac{(\lambda - \hat\lambda)(\lambda - \hat\lambda - \Delta)}{d_2 + \lambda} & 0 \\[2ex]
0 & d_0 + \lambda \left(d_1 + \dfrac{(d_2 + 2\hat\lambda + \Delta)(d_1 d_2 - d_0)}{\hat\lambda(\hat\lambda + \Delta)} \right) & 0
\end{array}$$

So, it is easy to see that the coefficients in the first column of the table are all positive for $\lambda \in [0, \hat{\lambda}) \cup (\hat{\lambda} + \Delta, +\infty)$, and the coefficient in row "1" annihilates for $\lambda = \hat{\lambda}$ and $\lambda = \hat{\lambda} + \Delta$, and is negative in $(\hat{\lambda}, \hat{\lambda} + \Delta)$. ■

Remark 5 *From the previous proof it is clear that if the degree of $d(s)$ were lower than 3, a polynomial $n(s)$ of degree smaller than $\deg d(s)$ such that $d(s) + \lambda n(s)$ is Hurwitz for $\lambda \in [0, \hat{\lambda}) \cup (\hat{\lambda} + \Delta, +\infty)$, and is not Hurwitz for $\lambda \in [\hat{\lambda}, \hat{\lambda} + \Delta]$ would not necessarily exist. It is also clear that when $\hat{\lambda}$ is not the largest eigenvalue of $\mathcal{L}$, $\psi(s, \hat{\lambda})$ has necessarily two imaginary conjugate zeros (and not a zero in the origin), and hence the consensus is achieved on a periodic trajectory. On the other hand, when $\hat{\lambda} = \lambda_n(\mathcal{L})$, the problem of finding $n(s)$ of degree smaller than $\deg d(s)$ such that $d(s)+\lambda n(s)$ is Hurwitz for $\lambda \in [0, \hat{\lambda})$, and is not Hurwitz for $\lambda \in [\hat{\lambda}, +\infty)$, can be trivially solved even for $\deg d(s) = 1$, and hence with a controllable subspace of dimension 1. This result is quite immediate and hence we omit the proof, but to better clarify it we provide the following example.*

Example 4 *Consider the system of $N = 4$ agents, each of them described by the 2-dimensional ($n = 2$) state-space model*

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + bu_i(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_i(t), \qquad i \in [1, 4].$$

Note that A is Hurwitz and the pair (A, b) has 1-dimensional controllable subspace. The characteristic polynomial of the non-controllable part is $\det(sI - A_{22}) = (s + 1)$. We assume that the signed communication matrix $\mathcal{A}$ is

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & -2 & -2 \\ 1 & 0 & -2 & -2 \\ -2 & -2 & 0 & 1 \\ -2 & -2 & 1 & 0 \end{bmatrix},$$

which means that the corresponding graph is structurally balanced with two antagonistic groups of agents of the same cardinality $\mathcal{V}_1 = \{1, 2\}$ and $\mathcal{V}_2 = \{3, 4\}$. The associated Laplacian is

$$\mathcal{L} = \begin{bmatrix} 5 & -1 & 2 & 2 \\ -1 & 5 & 2 & 2 \\ 2 & 2 & 5 & -1 \\ 2 & 2 & -1 & 5 \end{bmatrix},$$

and its eigenvalues are $\lambda_1(\mathcal{L}) = 0,\ \lambda_2(\mathcal{L}) = 6,\ \lambda_3(\mathcal{L}) = 6,\ \lambda_4(\mathcal{L}) = 8$. We note that $\mathcal{L}\mathbf{1}_4 = 8\mathbf{1}_4$, and hence all row sums are the same and they are equal to $\hat{\lambda} = 8 = \lambda_4(\mathcal{L})$. In this case we have to choose K in such a way that $K\mathrm{adj}(sI_2 - A)b = (s + 1)n_0$ makes the polynomial $\psi(s, \lambda) = (s+1)[s+1+\lambda n_0]$ Hurwitz for $\lambda < 8$ and simply stable for $\lambda = \hat{\lambda} = 8$. To this end, it is sufficient to choose $K = n_0 = -1/8$ and the result is obtained.

4.2 Not all row sums of $\mathcal{L}$ take the same value

We first observe that a necessary condition for the overall state $\mathbf{x}(t)$ to converge to some bounded trajectory $\mathbf{z}_{tot}(t) := [\,\zeta(t)^\top \quad \zeta(t)^\top \quad \dots \quad \zeta(t)^\top\,]^\top$ is that there exist a vector $\mathbf{v}_{tot} := [\,\mathbf{v}^\top \quad \mathbf{v}^\top \quad \dots \quad \mathbf{v}^\top\,]^\top$, $\mathbf{v} \neq 0$, and some $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) = 0$, such that $[(I_N \otimes A) - (I_N \otimes b)(\mathcal{L} \otimes K)]\,\mathbf{v}_{tot} = \alpha \mathbf{v}_{tot}$. This condition can be easily seen to be equivalent to the set of conditions:

$$\left\{ \begin{array}{rcl} (A - \sum_{j=1}^N [\mathcal{L}]_{1j} bK)\mathbf{v} & = & \alpha \mathbf{v} \\ (A - \sum_{j=1}^N [\mathcal{L}]_{2j} bK)\mathbf{v} & = & \alpha \mathbf{v} \\ & \vdots & \\ (A - \sum_{j=1}^N [\mathcal{L}]_{Nj} bK)\mathbf{v} & = & \alpha \mathbf{v} \end{array} \right. \tag{18}$$

If not all row sums of $\mathcal{L}$ take the same value, the above set of conditions is satisfied if and only if $A\mathbf{v} = \alpha\mathbf{v}$ and $K\mathbf{v} = 0$. But this implies that $(A - \lambda bK)\mathbf{v} = \alpha\mathbf{v}$ for every $\lambda \in \mathbb{R}$, and hence that $\psi(s, \lambda) = \det(sI_n - A) + \lambda K \mathrm{adj}(sI_n - A)b$ has a zero in α for every choice of λ.

Now we select a left eigenvector $\mathbf{p} := [\,p_1 \quad p_2 \quad \dots \quad p_N\,]^\top \in \mathbb{R}^N$ of $\mathcal{L}$ corresponding to any eigenvalue $\hat{\lambda} \in \sigma(\mathcal{L})$ and endowed with the property that $\sum_{j=1}^N p_j \neq 0$. As noted earlier, such an eigenvector always exists, and we can always assume $\sum_{j=1}^N p_j = 1$. Introduce the matrix

$$S := \left[\begin{array}{c|c} p_1 & p_2 \quad \dots \quad p_N \\ \hline -\mathbf{1}_{N-1} & I_{N-1} \end{array} \right],$$

and consider the coordinate transformation

$$\left[\begin{array}{c} \zeta(t) \\ \hline \delta_2(t) \\ \vdots \\ \delta_N(t) \end{array} \right] := \left[\begin{array}{c|c} p_1 I_n & p_2 I_n \quad \dots \quad p_N I_n \\ \hline -\mathbf{1}_{N-1} \otimes I_n & I_{n(N-1)} \end{array} \right] \left[\begin{array}{c} \mathbf{x}_1(t) \\ \hline \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_N(t) \end{array} \right] = (S \otimes I_n)\mathbf{x}(t).$$

It is a matter of simple computation to see that in this case

$$S\mathcal{L}S^{-1} = \left[\begin{array}{c|c} \hat{\lambda} & 0 \\ \hline \sum_{j=1}^N [\mathcal{L}]_{2j} - \sum_{j=1}^N [\mathcal{L}]_{1j} & \\ \vdots & \mathcal{L}_2 \\ \sum_{j=1}^N [\mathcal{L}]_{Nj} - \sum_{j=1}^N [\mathcal{L}]_{1j} & \end{array} \right]$$

and the eigenvalues of $\mathcal{L}_2$ are the remaining $N-1$ eigenvalues of $\mathcal{L}$ (note that, as before, $\hat{\lambda}$ can also be an eigenvalue of $\mathcal{L}_2$, in case its multiplicity as eigenvalue of $\mathcal{L}$ is greater than 1). By proceeding as in the previous two cases, we obtain the following description:

$$\dot{\zeta}(t) = (A - \hat{\lambda} bK)\zeta(t),$$

$$\begin{bmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{bmatrix} = [(I_{N-1} \otimes A) - (I_{N-1} \otimes b)(\mathcal{L}_2 \otimes K)] \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix} + \begin{bmatrix} (\sum_{j=1}^N [\mathcal{L}]_{2j} - \sum_{j=1}^N [\mathcal{L}]_{1j})bK \\ \vdots \\ (\sum_{j=1}^N [\mathcal{L}]_{Nj} - \sum_{j=1}^N [\mathcal{L}]_{1j})bK \end{bmatrix} \zeta(t)$$

For every choice of $\hat{\lambda}$, the characteristic polynomial of $(I_{N-1} \otimes A) - (I_{N-1} \otimes b)(\mathcal{L}_2 \otimes K)$ is the product of $N-1$ polynomials $\psi_i(s)$, each of them having (at least) one not asymptotically stable zero in α. This prevents consensus among agents.

By putting together the results of these two subsections, we can formalize the following result.

Theorem 3 *Consider the multi-agent system (6) with control algorithm (7), and assume that the communication graph $\mathcal{G}$ is connected and structurally balanced, with respect to the partition in two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$. The consensus problem is solvable only if the following conditions are satisfied:*

i) the two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ have the same cardinality;

ii) all row sums of the Laplacian $\mathcal{L}$ are equal to $\hat{\lambda} > 0$, and $\hat{\lambda}$ is a simple eigenvalue of $\mathcal{L}$;

iii) A is Hurwitz.

If i)-iii) hold, then a necessary and sufficient condition for the consensus problem to be solvable is that there exists $K \in \mathbb{R}^{1\times N}$ such that $\psi(s, \hat{\lambda})$ is simply stable, while $\psi(s, \lambda_i)$ is Hurwitz for every $\lambda_i \in \sigma(\mathcal{L}), \lambda_i \neq \hat{\lambda}$. Such a matrix K always exists if and only if either one of the following cases apply: (1) $\hat{\lambda}$ is the largest eigenvalue of $\mathcal{L}$ and $b \neq 0$; or (2) $\hat{\lambda}$ is not the largest eigenvalue of $\mathcal{L}$ and the dimension of the controllable subspace of (A, b) is at least three.

5 Conclusions

In this paper we have investigated the bipartite consensus problem and the standard consensus problem for a group of N homogeneous agents, each of them described by a generic n-dimensional state-space model. Under the assumption that the communication graph describing the cooperative/antagonistic relationships among agents is structurally balanced, we have seen that bipartite consensus can be achieved if and only if the state-space model describing each agent is stabilizable. In addition, the common agents' dynamics is regulated by the state matrix A involved in the agents' description. In fact, the common agents' dynamics converge to zero, to a bounded or diverging common state trajectory depending on the spectrum of A (and on the agents' initial conditions). These results turn out to be analogous to the ones derived for standard consensus in case of cooperative interactions [31].

In the second part of the paper we have investigated the possibility of achieving consensus to a common bounded trajectory even in case of antagonistic interactions, provided that there is structural balance in the agents' communications. In this case conditions allowing for a nontrivial agreement between the two groups are very stringent, and require in particular that the two teams have the same cardinality.

An interesting open problem is that of investigating what kind of nontrivial agreements may be reached in case of antagonistic interactions, when the structural balance assumption is not satisfied.

Acknowledgments

We are indebted with the Reviewers for their fruitful comments and, in particular, with one of them for suggesting Remark 1.

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