On the consensus of homogeneous multi-agent systems with arbitrarily switching topology

Maria Elena Valcher [a], Irene Zorzan [b]

[a] *Dip. di Ingegneria dell'Informazione, Univ. di Padova, via Gradenigo 6/B, 35131 Padova, Italy.*

[b] *Dip. di Ingegneria dell'Informazione, Univ. di Padova, via Gradenigo 6/B, 35131 Padova, Italy.*

Abstract

In this paper we investigate the consensus problem under arbitrary switching for homogeneous multi-agent systems with switching communication topology, by assuming that each agent is described by a single-input stabilizable state-space model and that the communication graph is connected at every time instant. Under these assumptions, we construct a common quadratic positive definite Lyapunov function for the switched system describing the evolution of the disagreement vector, thus showing that the agents always reach consensus. In addition, the proof leads to the explicit construction of a constant state-feedback matrix that allows the multi-agent system to achieve consensus.

Key words: Consensus; homogeneous multi-agent system; communication graph; switched system; stabilizability of switched system; quadratic positive definite Lyapunov function.

1 Introduction

Research efforts on multi-agent systems, in general, and consensus problems, in particular, have been quite impressive in the last decade. Originated by some remarkable contributions that still represent the reference points of any paper on the subject [2,12,15,16,24], the research flourished by addressing increasingly more complex set-ups, and hence taking into account higher order (possibly nonlinear) models for the agents, time-varying communication topologies, antagonistic interactions, communication delays, output feedback, packet-loss, etc. (see, e.g.,[5,6,13,16,17,32]).
Even if a significant portion of the research in this area focuses on first and second order systems [10,32], a good number of contributions have investigated the case when the agents' dynamics is described by a generic state-space model. While in the early contributions the communication topology was supposed to be fixed [29,30], more recent papers have explored the case of a time-varying communication topology, possibly switching among a finite set of configurations [21,25,26,28].

All the literature addressing the case of higher order agents with switching communication topologies has been able to relax the connectedness constraint on every communication graph, at the price of imposing additional constraints not only on the switching signal but also on the agents' model. Specifically, in [21] and [31] the consensus problem is solved and an explicit solution is provided, by assuming that the agent's state model is stabilizable, the state matrix A is simply stable, the switching signal describing how the communication topology varies has a minimum dwell-time and ensures that the time-varying communication graph $\mathcal{G}(t)$ is uniformly connected over $[0,+\infty)$. In [14] the state matrix A satisfies some algebraic constraint, the input to state matrix B is of full row rank (a sufficient condition for controllability), switching signals have a dwell time, and the communication topologies are repeatedly jointly rooted. Under these conditions, the consensus problem is solvable and a state feedback matrix is explicitly constructed. In [25], consensus has been investigated, under the assumption that agents are controllable and switching signals have a dwell-time, both in case the communication network over which agents communicated is connected at every time and in case it is frequently connected with a certain period T.

⋆ This paper was not presented at any IFAC meeting. Corresponding author M. E. Valcher. Tel. +39-049-827-7795. Fax +39-049-827-7614.

Email addresses: meme@dei.unipd.it (Maria Elena Valcher), irene.zorzan@studenti.unipd.it (Irene Zorzan).

In most of these contributions, dwell time and controllability have been fundamental requirements in order to design a state feedback controller that ensures a sufficiently rapid convergence. For instance, the proof of Theorem 1 in [25] heavy relies on the possibility to freely allocate the eigenvalues of the matrices $A - \lambda_i BK$, involved in the disagreement dynamics, and on the existence of a dwell time. It is interesting to understand under what conditions multi-agent consensus can be guaranteed corresponding to *every* switching signal and not only corresponding to switching signals with dwell-time.

It is a standard result for switched systems that if all the subsystems are asymptotically stable, then a dwell-time can always be found such that the switched system is asymptotically and hence exponentially stable. However, asymptotic stability of the subsystems alone does not ensure asymptotic stability of the switched system for every switching signal.

In this paper we investigate the consensus problem under arbitrary switching. This strong requirement on the system performances necessarily imposes that the communication network is connected at every time. On the other hand, it turns out that the stabilizability of the agents' dynamics is necessary and sufficient for the problem solvability, just like it happens when we consider a fixed connected communication network [29]. Even more, we provide an explicit solution to the consensus problem. The paper set-up is inspired by the one adopted in [21], but we extend the agents' model decomposition adopted in the previously mentioned reference to the case when the state matrix has also eigenvalues with positive real part. Subsequently, we construct a quadratic positive definite function that ensures the asymptotic stability of the switched system describing the dynamics of the disagreement vector, and thus prove consensus. It is worth remarking that, in general, it is hard or even impossible to construct a common quadratic Lyapunov function for the consensus error system of a multi-agent system with switching topology. When so (see e.g. [26,27,28]), multiple Lyapunov functions have been proposed to stabilize or verify the stability.

It is worth comparing our results with those derived in [7], where a consensus protocol for homogeneous multi-agent systems with arbitrarily switching topologies is proposed, by assuming that the communication graph is connected (and undirected) at every time instant. The set-up adopted in [7] is rather different from the one considered in this paper, since the switching takes place among all possible undirected, connected and *unweighted* communication graphs, but the weights attributed to the graph edges are regarded as control variables that continuously update. So, the switching is not among a finite number of undirected, connected and *weighted* communication graphs, but weights can be adaptively modified. The advantage of this adaptive consensus protocol is that it can be implemented in a completely distributed way by the agents. The con is that the controller significantly increases in size. Indeed, if n is the size of the agents' state and N is the number of the agents, the overall controlled system in [7] has size $2nN + N(N-1)/2$, since the adaptive controller updates both a "protocol state" of size n for each agent, and the distinct $N(N-1)/2$ weights of the edges of the undirected graph at every time instant. In this paper, we will use a static controller and the overall controlled system will have size nN.

The paper is organized as follows: in section 2 we present some background material on matrices, graphs, and Laplacians. Section 3 presents the problem set-up. In section 4 some preliminary analysis is carried on that allows to simplify the set-up and to reduce the consensus problem to a stabilization problem for a lower-order switched system with autonomous subsystems. A constructive proof of the main result, stating that if the communication network is connected at every time and the agents' model is stabilizable, then consensus can always be achieved, is given in section 5, together with a simple algorithm to explicitly construct a state feedback matrix ensuring consensus.

2 Background material

If p is a positive integer, we denote by $[1,p]$ the finite set $\{1,2,\dots,p\}$. $\mathbf{e}_i$ is the ith *canonical vector* in $\mathbb{R}^N$, where N is always clear from the context. $\mathbf{1}_N$ and $\mathbf{0}_N$ are the N-dimensional vectors with all entries equal to 1 and to 0, respectively. Given $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ the spectrum of A and by $\lambda_{\max}(A) \in \mathbb{R}$ its *spectral abscissa*, defined as $\lambda_{\max}(A) := \max\{\Re(\lambda), \lambda \in \sigma(A)\}$. A is *Hurwitz* if $\lambda_{\max}(A) < 0$. The *Kronecker (or tensor) product* of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is

$$C = [A \otimes B] := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{pm\times qn}.$$

An $n \times n$ matrix A, $n > 1$, is *reducible* if there exists a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices, otherwise it is *irreducible*.

$\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix A with entries in $\mathbb{R}_+$ is a *nonnegative matrix* ($A \geq 0$); if $A \geq 0$ and at least one entry is positive, A is a *positive matrix* ($A > 0$).

A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. For a Metzler matrix, the spectral abscissa is always an eigenvalue (namely the eigenvalue with maximal real part is always real). Given two Metzler matrices A and $\bar{A} \in \mathbb{R}^{n\times n}$, the following monotonicity property holds [19]: if $A \le \bar{A}$, then $\lambda_{\max}(A) \le \lambda_{\max}(\bar{A})$; if in addition $\bar{A}$ is irreducible, then $A < \bar{A}$ implies $\lambda_{\max}(A) < \lambda_{\max}(\bar{A})$.

An *undirected, weighted graph* is a triple [11] $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}_+^{N\times N}$ is the (positive and symmetric) *adjacency matrix* of the weighted graph $\mathcal{G}$. In this paper we assume that $\mathcal{G}$ has no self-loops, namely each diagonal entry $[\mathcal{A}]_{ii}, i \in [1, N]$, is zero. A sequence $j_1 \leftrightarrow j_2 \leftrightarrow j_3 \leftrightarrow \cdots \leftrightarrow j_k \leftrightarrow j_{k+1}$ is a *path* of length k connecting j_1 and j_{k+1} provided that $(j_1, j_2), (j_2, j_3) \dots, (j_k, j_{k+1}) \in \mathcal{E}$. A graph is said to be *connected* if for every pair of distinct vertices $i, j \in \mathcal{V}$ there is a path connecting i and j. This is equivalent to the fact that $\mathcal{A}$ is an irreducible matrix. We define the *Laplacian matrix* $\mathcal{L} \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ as $\mathcal{L} := \mathcal{C} - \mathcal{A}$, where $\mathcal{C} \in \mathbb{R}_+^{N\times N}$ is a diagonal matrix whose ith diagonal entry is the *weighted degree* of vertex i, i.e. $[\mathcal{C}]_{ii} := \sum_{l=1}^N [\mathcal{A}]_{il}$. Accordingly, the Laplacian matrix $\mathcal{L} = \mathcal{L}^\top$ takes the following form:

$$\mathcal{L} = \begin{bmatrix} \ell_{11} & \ell_{12} & \dots & \ell_{1N} \\ \ell_{12} & \ell_{22} & \dots & \ell_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{1N} & \ell_{2N} & \dots & \ell_{NN} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{j=1}^N [\mathcal{A}]_{1j} & -[\mathcal{A}]_{12} & \dots & -[\mathcal{A}]_{1N} \\ -[\mathcal{A}]_{12} & \sum_{j=1}^N [\mathcal{A}]_{2j} & \dots & -[\mathcal{A}]_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -[\mathcal{A}]_{1N} & -[\mathcal{A}]_{2N} & \dots & \sum_{j=1}^N [\mathcal{A}]_{Nj} \end{bmatrix} \in \mathbb{R}^{N\times N}.$$

As all rows of $\mathcal{L}$ sum up to 0, $\mathbf{1}_N$ is always a right eigenvector of $\mathcal{L}$ corresponding to the eigenvalue 0. The following lemma states a useful and well-known result about Laplacian matrices of undirected graphs.

Lemma 1 *[3,16,29] If the undirected, weighted graph $\mathcal{G}$ is connected, then $\mathcal{L}$ is a symmetric positive semidefinite matrix, and 0 is a simple eigenvalue of $\mathcal{L}$. As a consequence, the eigenvalues of $\mathcal{L}$, say $\lambda_i = \lambda_i(\mathcal{L})$, $i \in [1, N]$, are nonnegative and real, and they can always be sorted in non-decreasing order, namely as*

$$0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N. \tag{1}$$

In the following, only undirected, weighted and connected graphs will be considered. Consequently, both the adjacency matrix and the Laplacian matrix will be irreducible matrices.

3 Problem statement

Consider N agents, each of them described by the same n-dimensional, continuous-time, single-input system:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t), \qquad t \in \mathbb{R}_+, \tag{2}$$

where $\mathbf{x}_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$ are the state vector and the input of the ith agent, respectively, $A \in \mathbb{R}^{n\times n}$ is a non-Hurwitz matrix, and $B \in \mathbb{R}^n$. We assume that the pair (A, B) is stabilizable. Consider a set of p communication topologies describing the interactions among the agents, each configuration being described by an undirected, weighted and connected communication graph $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k, \mathcal{A}_k)$, $k \in [1, p]$, with $\mathcal{V} = \{1, \dots, N\}$, $\mathcal{E}_k \subseteq \mathcal{V} \times \mathcal{V}$, $\mathcal{A}_k = \mathcal{A}_k^\top \in \mathbb{R}_+^{N\times N}$ irreducible and such that (s.t.) $[\mathcal{A}_k]_{ii} = 0$ for every $i \in [1, N]$. Assume that mutual interactions among agents vary with time, switching among the p possible configurations. Specifically, let $\sigma\colon \mathbb{R}_+ \to [1, p]$ denote a right continuous switching function describing at every time instant which of the p communication topologies is active. No further assumption is introduced on σ, in particular, we do not impose that each switching signal has some dwell-time $\tau = \tau(\sigma) > 0$ [8]. Assume that each ith agent adopts the following DeGroot type control law [21,29]:

$$u_i(t) = K \sum_{j=1}^N [\mathcal{A}_{\sigma(t)}]_{ij} [\mathbf{x}_j(t) - \mathbf{x}_i(t)].$$

where $K \in \mathbb{R}^{1\times n}$ is a feedback matrix to be designed. If we denote by $\mathbf{x}(t) \in \mathbb{R}^{nN}$ and $\mathbf{u}(t) \in \mathbb{R}^N$ the state vector and the input vector, respectively, of the multi-agent system, i.e.

$$\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1^\top(t) & \dots & \mathbf{x}_N^\top(t) \end{bmatrix}^\top \quad \mathbf{u}(t) := \begin{bmatrix} u_1(t) & \dots & u_N(t) \end{bmatrix}^\top$$

the dynamics of the overall multi-agent system with switching communication topology is described by:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (I_N \otimes A)\mathbf{x}(t) + (I_N \otimes B)\mathbf{u}(t) \\ \mathbf{u}(t) &= -(\mathcal{L}_{\sigma(t)} \otimes K)\mathbf{x}(t), \end{aligned}$$

or equivalently by:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= [(I_N \otimes A) - (I_N \otimes B)(\mathcal{L}_{\sigma(t)} \otimes K)]\mathbf{x}(t) \\ &= [(I_N \otimes A) - (\mathcal{L}_{\sigma(t)} \otimes BK)]\mathbf{x}(t), \end{aligned} \tag{3}$$

where we made use of the elementary properties of the Kronecker product. The previous system (3) is a continuous-time, switched system, switching among p autonomous and linear subsystems:

$$\dot{\mathbf{x}}(t) = [(I_N \otimes A) - (\mathcal{L}_k \otimes BK)]\mathbf{x}(t), \qquad k \in [1, p]. \tag{4}$$

The *consensus problem under switching topology* can be stated as follows: determine a feedback matrix $K \in \mathbb{R}^{1\times n}$ such that, for every initial condition and every switching function [1], the overall system (3) asymptotically reaches (nontrivial) consensus, by this meaning that $\lim_{t\to+\infty} \mathbf{x}_i(t) - \mathbf{x}_j(t) = 0$ for every $i, j \in [1, N]$, without imposing the convergence to zero of the agents' evolutions.

4 Preliminary analysis

In this section we make some considerations both on the agent's state space representation and on the dynamics of the overall multi-agent system that allow us to reduce the dimensionality of the problem and to restate it in a more convenient form. Consider the ith agent's description (2) and assume that the pair (A, B) takes the following form

$$A = \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix} \quad B = \begin{bmatrix} B_u \\ B_s \end{bmatrix}, \tag{5}$$

where $A_u \in \mathbb{R}^{d\times d}$ is a matrix with all its eigenvalues in the closed right half-plane $\{s \in \mathbb{C} : \Re(s) \geq 0\}$, $A_s \in \mathbb{R}^{(n-d)\times(n-d)}$ is a Hurwitz matrix, $B_u \in \mathbb{R}^d$ and $B_s \in \mathbb{R}^{(n-d)}$. Notice that this assumption entails no loss of generality since we can always reduce ourselves to this situation by resorting to a suitable coordinate transformation. Also, this set-up extends the one proposed in [21], where A_u was supposed to be an anti-symmetric matrix, namely a matrix with all its eigenvalues on the imaginary axis. Partition the ith agent's state vector $\mathbf{x}_i(t)$ in a way consistent with A and B, namely as

$$\mathbf{x}_i(t) = \begin{bmatrix} \mathbf{x}_i^u(t) \\ \mathbf{x}_i^s(t) \end{bmatrix},$$

where $\mathbf{x}_i^u(t) \in \mathbb{R}^d$ and $\mathbf{x}_i^s(t) \in \mathbb{R}^{(n-d)}$. Define $\mathbf{x}_C^u(t) := \frac{1}{N}\sum_{i=1}^N \mathbf{x}_i^u(t)$, $\mathbf{x}_C^s(t) := \frac{1}{N}\sum_{i=1}^N \mathbf{x}_i^s(t)$, $\boldsymbol{\delta}_i^u(t) := \mathbf{x}_i^u(t) - \mathbf{x}_1^u(t)$, $\boldsymbol{\delta}_i^s(t) := \mathbf{x}_i^s(t) - \mathbf{x}_1^s(t)$, $i = 2, \dots, N$. Note that

$$\mathbf{x}_C(t) := \begin{bmatrix} \mathbf{x}_C^u(t) \\ \mathbf{x}_C^s(t) \end{bmatrix} \in \mathbb{R}^n \text{ and } \boldsymbol{\delta}(t) := \begin{bmatrix} \boldsymbol{\delta}_2^u(t) \\ \vdots \\ \boldsymbol{\delta}_N^u(t) \\ \boldsymbol{\delta}_2^s(t) \\ \vdots \\ \boldsymbol{\delta}_N^s(t) \end{bmatrix} \in \mathbb{R}^{n(N-1)}$$

are the *center* of the agents [21] and the *disagreement vector from the* 1*st agent* at time t, respectively, and asymptotic consensus of the multi-agent system (3) is equivalent to the convergence to zero of $\boldsymbol{\delta}(t)$.
Partition the feedback matrix K in a way consistent with A and B, namely as $K = [K_u\ K_s]$, with $K_u \in \mathbb{R}^{1\times d}$ and $K_s \in \mathbb{R}^{1\times(n-d)}$. It is a matter of simple computation to see that if we choose K of the form $K = [K_u\ 0]$, the overall dynamics is described by equation (6), where $\tilde{\mathcal{L}}_{\sigma(t)}$ is expressed in terms of $\mathcal{L}_{\sigma(t)}$ as

$$\tilde{\mathcal{L}}_{\sigma(t)} = \begin{bmatrix} -\mathbf{1}_{N-1} & I_{N-1} \end{bmatrix} \mathcal{L}_{\sigma(t)} \begin{bmatrix} \mathbf{0}_{N-1}^\top \\ I_{N-1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}. \tag{7}$$

By arguments very similar to the ones adopted by Su and Huang in Lemma 2 of [21], Proposition 1 immediately follows.

Proposition 1 *The feedback matrix $K = [K_u\ 0]$, with $K_u \in \mathbb{R}^{1\times d}$, solves the consensus problem under switching topologies if and only if K_u makes the switched system*

$$\dot{\boldsymbol{\delta}}^u(t) = \left[(I_{N-1} \otimes A_u) - \left(\tilde{\mathcal{L}}_{\sigma(t)} \otimes B_u K_u\right)\right] \boldsymbol{\delta}^u(t), \tag{8}$$

with $\boldsymbol{\delta}^u(t)^\top := [\boldsymbol{\delta}_2^u(t)^\top \ \dots \ \boldsymbol{\delta}_N^u(t)^\top] \in \mathbb{R}^{d(N-1)}$, asymptotically (and hence exponentially) stable under arbitrary switching.

Before providing a constructive proof of the solvability of the consensus problem under switching topology, we state a technical result that will be useful in the following section.

Lemma 2 *Consider an undirected, weighted and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with $\mathcal{V} = \{1, \dots, N\}$, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$, $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}_+^{N\times N}$. Partition the Laplacian $\mathcal{L}$ as*

$$\mathcal{L} = \begin{bmatrix} \ell_{11} & b^\top \\ b & C \end{bmatrix}, \tag{9}$$

where $b \in \mathbb{R}^{N-1}$ and $C = C^\top \in \mathbb{R}^{(N-1)\times(N-1)}$. Define $\tilde{\mathcal{L}}$ as in (7), namely as

$$\tilde{\mathcal{L}} := \begin{bmatrix} -\mathbf{1}_{N-1} & I_{N-1} \end{bmatrix} \mathcal{L} \begin{bmatrix} \mathbf{0}_{N-1}^\top \\ I_{N-1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}.$$

Then, the following properties hold:

i) C is a positive definite matrix, i.e., $C = C^\top \succ 0$;
ii) all eigenvalues of $\tilde{\mathcal{L}}$ are real and positive. Specifically, $\sigma(\tilde{\mathcal{L}}) = \sigma(\mathcal{L}) \setminus \{0\}$;

[1] Note that if we want to ensure that the consensus problem is solvable for every switching signal, then it must be solvable corresponding to every constant switching signal, and this motivates the assumption that each graph $\mathcal{G}_k$ is connected.

$$\left[\begin{array}{c}\dot{\mathbf{x}}_C^u(t)\\ \hline \dot{\boldsymbol{\delta}}_2^u(t)\\ \vdots\\ \dot{\boldsymbol{\delta}}_N^u(t)\\ \hline \dot{\mathbf{x}}_C^s(t)\\ \hline \dot{\boldsymbol{\delta}}_2^s(t)\\ \vdots\\ \dot{\boldsymbol{\delta}}_N^s(t)\end{array}\right] = \left[\begin{array}{c|c|c|c} A_u & 0 & 0 & 0\\ \hline 0 & (I_{N-1}\otimes A_u) - (\tilde{\mathcal{L}}_{\sigma(t)}\otimes B_u K_u) & 0 & 0\\ \hline 0 & 0 & A_s & 0\\ \hline 0 & -\tilde{\mathcal{L}}_{\sigma(t)}\otimes B_s K_u & 0 & I_{N-1}\otimes A_s\end{array}\right] \left[\begin{array}{c}\mathbf{x}_C^u(t)\\ \hline \boldsymbol{\delta}_2^u(t)\\ \vdots\\ \boldsymbol{\delta}_N^u(t)\\ \hline \mathbf{x}_C^s(t)\\ \hline \boldsymbol{\delta}_2^s(t)\\ \vdots\\ \boldsymbol{\delta}_N^s(t)\end{array}\right], \tag{6}$$

iii) $\tilde{\mathcal{L}} = \tilde{P}C$, *with* $\tilde{P} := I_{N-1} + \mathbf{1}_{N-1}\mathbf{1}_{N-1}^\top \in \mathbb{R}^{(N-1)\times(N-1)}$.

Proof. i) The connectedness assumption on $\mathcal{G}$ implies that $\mathcal{L}$ is an irreducible matrix and that $-b$ is a positive vector. Hence, $-\mathcal{L}$ is an irreducible Metzler matrix and the following inequality holds

$$-\mathcal{L} = \begin{bmatrix} -\ell_{11} & -b^\top \\ -b & -C \end{bmatrix} > \begin{bmatrix} -\ell_{11} & 0 \\ 0 & -C \end{bmatrix} =: \mathcal{M}.$$

The irreducibility of $-\mathcal{L}$ and the monotonicity of the spectral abscissa imply that $0 = \lambda_{\max}(-\mathcal{L}) > \lambda_{\max}(\mathcal{M}) \geq \lambda_{\max}(-C)$. Hence, $-C = -C^\top$ is a negative definite matrix, namely $C = C^\top$ is positive definite.

ii) Denote by S the $N \times N$ nonsingular matrix

$$S := \begin{bmatrix} 1 & \mathbf{0}_{N-1}^\top \\ -\mathbf{1}_{N-1} & I_{N-1} \end{bmatrix},$$

and notice that its inverse takes the following form

$$S^{-1} = \begin{bmatrix} 1 & \mathbf{0}_{N-1}^\top \\ \mathbf{1}_{N-1} & I_{N-1} \end{bmatrix}.$$

From $\mathcal{L}\mathbf{1}_N = 0$ it immediately follows that

$$S\mathcal{L}S^{-1} = \begin{bmatrix} 0 & b^\top \\ \mathbf{0}_{N-1} & \tilde{\mathcal{L}} \end{bmatrix}.$$

This implies that $\sigma(\mathcal{L}) = \sigma(\tilde{\mathcal{L}}) \cup \{0\}$. The connectedness assumption on $\mathcal{G}$ implies that all the eigenvalues in $\sigma(\tilde{\mathcal{L}})$ are real and positive.

iii) Since $\mathcal{L}$ is partitioned as in (9), then $\tilde{\mathcal{L}}$ can be rewritten as $\tilde{\mathcal{L}} = -\mathbf{1}_{N-1}b^\top + C$. On the other hand, recalling that $\mathcal{L}\mathbf{1}_N = 0$, we have $b + C\mathbf{1}_{N-1} = 0$, namely $b = -C\mathbf{1}_{N-1} = -C^\top\mathbf{1}_{N-1}$. This in turn implies that $\tilde{\mathcal{L}}$ can be expressed as $\tilde{\mathcal{L}} = (I_{N-1} + \mathbf{1}_{N-1}\mathbf{1}_{N-1}^\top)C = \tilde{P}C$, where $\tilde{P} = I_{N-1} + \mathbf{1}_{N-1}\mathbf{1}_{N-1}^\top$. ∎

5 A constructive proof of the solvability of the consensus problem under switching topology

In the previous section, by resorting to the technique first proposed in [21] for marginally stable agents and switching signals with dwell time, we have reduced the problem of finding a solution $K = [K_u\ K_s]$ for the consensus problem, to the problem of determining $K_u \in \mathbb{R}^{1\times d}$ that makes the switched system (8) asymptotically stable under arbitrary switching [2] . This represents a standard stabilization problem for switched systems with linear and autonomous subsystems [8,22,23]. Clearly, a necessary condition for the switched system (8) to be asymptotically stable under arbitrary switching is the asymptotic stability of all the subsystem matrices, namely the fact that for every $k \in [1,p]$ the matrix $I_{N-1}\otimes A_u - \tilde{\mathcal{L}}_k \otimes B_u K_u$ is Hurwitz. A classical result derived in the context of consensus problems [2,29,30] states that any such matrix is Hurwitz if and only if $A_u - \lambda_i(\tilde{\mathcal{L}}_k)B_u K_u \in \mathbb{R}^{d\times d}$ is Hurwitz for every $i \in [2,N]$, where $\lambda_i(\tilde{\mathcal{L}}_k)$ denotes the ith eigenvalue of the matrix $\tilde{\mathcal{L}}_k$. This requires the pair (A_u, B_u) to be stabilizable, and since A_u has all eigenvalues in the closed right half-plane, this amounts to saying that the pair (A_u, B_u) in (5) must be reachable. The stabilizability assumption on the pair (A, B), we introduced in the initial set-up, is in fact equivalent to the reachability of the pair (A_u, B_u).

On the other hand, the Hurwitz property of all the subsystem matrices is only a necessary condition for a switched system to be asymptotically stable under arbitrary switching [8]. Hence, in order to prove that the switched system (8) is asymptotically stable under arbitrary switching (and hence a solution to the consensus problem under switching topology exists), we will resort to a standard tool for analysing the stability of

[2] Confining our attention to state-feedback matrices $K = [K_u\ K_s]$ with $K_s = 0$ may seem a restrictive way to solve the problem. As it will be clear later, the conditions on the pair (A, B) and on the graphs $\mathcal{G}_{\sigma(t)}$ that ensure the existence of a solution with that structure are the same ones that ensure the existence of a generic solution K.

a switched system: the existence of common quadratic Lyapunov functions.

A quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$, where $P = P^\top \succ 0, P \in \mathbb{R}^{(N-1)d\times(N-1)d}$, is a positive definite matrix, is said to be a *common quadratic Lyapunov function* for the p subsystems

$$\dot{\boldsymbol{\delta}}^u(t) = M_k\ \boldsymbol{\delta}^u(t), \qquad k \in [1,p], \tag{10}$$

where $M_k := I_{N-1}\otimes A_u - \tilde{\mathcal{L}}_k \otimes B_u K_u \in \mathbb{R}^{(N-1)d\times(N-1)d}$, if $\dot{V}_k(\mathbf{x}) := \mathbf{x}^\top[M_k^\top P + PM_k]\mathbf{x} < 0$ for every $k \in [1,p]$ and every $\mathbf{x} \neq 0$. It is well known [8,9,18] that the existence of a common quadratic Lyapunov function guarantees the asymptotic stability of a switched system.

We are now in a position to provide a constructive proof of the solvability of the consensus problem under arbitrarily switching topology.

Proposition 2 *Consider the switched system* (8) *and assume that the pair* (A_u, B_u) *is reachable. Assume without loss of generality (w.l.o.g.) that* (A_u, B_u) *is in controllable canonical form, i.e.,*

$$A_u = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & 0 & 1 \\ -\beta_0 & \dots & -\beta_{d-2} & -\beta_{d-1} \end{bmatrix}, \qquad B_u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \mathbf{e}_d.$$

Let $\psi(s) := s^{d-1} + \alpha_{d-2}s^{d-2} + \cdots + \alpha_1 s + \alpha_0$ *be an arbitrary Hurwitz polynomial of degree* $d-1$*. Assume that the feedback matrix* $K_u \in \mathbb{R}^{1\times d}$ *takes the following form*

$$K_u = k_d \bar{K}_u, \quad \text{with} \quad \bar{K}_u = \begin{bmatrix} \alpha_0 & \dots & \alpha_{d-2} & 1 \end{bmatrix}, \tag{11}$$

with $k_d > 0$*. Then, there always exists* $k_d > 0$ *s.t. the switched system* (8) *is asymptotically stable under arbitrary switching.*

Proof. We want to prove that $k_d > 0$ sufficiently large always exists s.t. the subsystem matrices M_k of the switched system (8) admit a common quadratic Lyapunov function. To this aim, we introduce a coordinate transformation on the matrices M_k. Define the nonsingular matrix $T \in \mathbb{R}^{d\times d}$ as

$$T = \left[\begin{array}{c} V_1 \\ \hline \bar{K}_u \end{array}\right] = \left[\begin{array}{ccc|c} & I_{d-1} & & \mathbf{0}_{d-1} \\ \hline \alpha_0 & \dots & \alpha_{d-2} & 1 \end{array}\right],$$

where $V_1 \in \mathbb{R}^{(d-1)\times d}$. It is easy to see that its inverse takes the following form

$$T^{-1} = \left[\begin{array}{c|c} V_2 & v \end{array}\right] = \left[\begin{array}{ccc|c} & I_{d-1} & & \mathbf{0}_{d-1} \\ -\alpha_0 & \dots & -\alpha_{d-2} & 1 \end{array}\right],$$

where $v \in \mathbb{R}^d$ and $V_2 \in \mathbb{R}^{d\times(d-1)}$. Now, define the $d(N-1)\times d(N-1)$ nonsingular matrices

$$\tilde{T} = \left[\begin{array}{c} I_{N-1}\otimes V_1 \\ \hline I_{N-1}\otimes \bar{K}_u \end{array}\right] = \left[\begin{array}{ccc} V_1 & & \\ & \ddots & \\ & & V_1 \\ \hline \bar{K}_u & & \\ & \ddots & \\ & & \bar{K}_u \end{array}\right]$$

$$\tilde{T}^{-1} = \left[\begin{array}{c|c} I_{N-1}\otimes V_2 & I_{N-1}\otimes v \end{array}\right] = \left[\begin{array}{ccc|ccc} V_2 & & & v & & \\ & \ddots & & & \ddots & \\ & & V_2 & & & v \end{array}\right].$$

It is a matter of simple computation to see that for every $k \in [1,p]$ we have:

$$\begin{aligned} \mathbb{N}_k := \tilde{T}(\tilde{\mathcal{L}}_k \otimes B_u K_u)\tilde{T}^{-1} \qquad & \in \mathbb{R}^{d(N-1)\times d(N-1)} \\ = \begin{bmatrix} \tilde{\mathcal{L}}_k \otimes (V_1 B_u K_u V_2) & \tilde{\mathcal{L}}_k \otimes (V_1 B_u K_u v) \\ \tilde{\mathcal{L}}_k \otimes (\bar{K}_u B_u K_u V_2) & \tilde{\mathcal{L}}_k \otimes (\bar{K}_u B_u K_u v) \end{bmatrix} & \\ = \begin{bmatrix} 0_{(d-1)(N-1)\times(d-1)(N-1)} & 0 \\ 0 & k_d \tilde{\mathcal{L}}_k \end{bmatrix} & \end{aligned} \tag{12}$$

where we exploited the fact that $V_1 B_u = V_1 \mathbf{e}_d = 0$, $K_u V_2 = k_d \bar{K}_u V_2 = 0$, and $\bar{K}_u B_u = 1$ and $K_u v = k_d$. On the other hand, it is easy to verify that $\mathbb{A} := \tilde{T}(I_{N-1}\otimes A_u)\tilde{T}^{-1} \in \mathbb{R}^{d(N-1)\times d(N-1)}$ takes the form

$$\begin{aligned} \mathbb{A} &= \begin{bmatrix} I_{N-1}\otimes (V_1 A_u V_2) & I_{N-1}\otimes (V_1 A_u v) \\ I_{N-1}\otimes (\bar{K}_u A_u V_2) & I_{N-1}\otimes (\bar{K}_u A_u v) \end{bmatrix} \\ &= \begin{bmatrix} I_{N-1}\otimes \tilde{A}_u & I_{N-1}\otimes \mathbf{e}_{d-1} \\ I_{N-1}\otimes (\bar{K}_u A_u V_2) & (\bar{K}_u A_u v) I_{N-1} \end{bmatrix}, \end{aligned} \tag{13}$$

where we set $\tilde{A}_u := V_1 A_u V_2$ and we used the fact that $\bar{K}_u A_u v$ is a scalar. Direct calculation leads to

$$\tilde{A}_u = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & 0 & 1 \\ -\alpha_0 & \dots & -\alpha_{d-3} & -\alpha_{d-2} \end{bmatrix} \in \mathbb{R}^{(d-1)\times(d-1)}.$$

With respect to the new coordinates $\mathbf{d}(t) := \tilde{T}\boldsymbol{\delta}^u(t)$, the switched system (8) is now described by the equation

$$\dot{\mathbf{d}}(t) = (\mathbb{A} - \mathbb{N}_{\sigma(t)})\mathbf{d}(t), \tag{14}$$

and clearly system (8) is asymptotically stable under arbitrary switching if and only if the switched system (14) is. Since $\tilde{A}_u$ is a companion matrix and $\psi(s) = \det(sI_{d-1} - \tilde{A}_u)$ is by hypothesis a Hurwitz polynomial, $\tilde{A}_u$ is Hurwitz. This ensures that there exists a $(d-1) \times (d-1)$ positive definite matrix $P_1 = P_1^\top \succ 0$ s.t.

$$\tilde{A}_u^\top P_1 + P_1\tilde{A}_u = -I_{d-1}. \tag{15}$$

Now, observe that $\tilde{P} := I_{N-1} + \mathbf{1}_{N-1}\mathbf{1}_{N-1}^\top$ (see Lemma 2) is a symmetric and positive definite matrix, and so is its inverse $\tilde{P}^{-1} = I_{N-1} - \frac{1}{N}\mathbf{1}_{N-1}\mathbf{1}_{N-1}^\top$. So, introduce the positive definite matrix

$$P := \begin{bmatrix} I_{N-1} \otimes P_1 & 0 \\ 0 & \tilde{P}^{-1} \end{bmatrix} \in \mathbb{R}^{d(N-1)\times d(N-1)}, \tag{16}$$

and consider the quadratic Lyapunov function $V(\mathbf{d}) = \mathbf{d}^\top P\mathbf{d}$. For every kth subsystem of the switched system (14), the derivative $\dot{V}_k(\mathbf{d}) = \mathbf{d}^\top[(\mathbb{A} - \mathbb{N}_k)^\top P + P(\mathbb{A} - \mathbb{N}_k)]\mathbf{d}$, $k \in [1,p]$, can be written as $\dot{V}_k(\mathbf{d}) = \mathbf{d}^\top \Psi_k \mathbf{d}$, with

$$\Psi_k := \begin{bmatrix} I_{N-1} \otimes (-I_{d-1}) & \Omega \\ \Omega^\top & W - k_d(C_k + C_k^\top) \end{bmatrix},$$

where we made use of (13), of Lemma 2, statement iii) (i.e., $\tilde{\mathcal{L}}_k = \tilde{P}C_k$ for every $k \in [1,p]$), and we assumed

$$\Omega := I_{N-1} \otimes (P_1\mathbf{e}_{d-1}) + \tilde{P}^{-1} \otimes (\bar{K}_u A_u V_2)^\top,$$
$$W := 2(\bar{K}_u A_u v)\tilde{P}^{-1}.$$

$\dot{V}_k(\mathbf{d})$ is negative definite for every $k \in [1,p]$, namely the symmetric matrix Ψ_k is negative definite for every $k \in [1,p]$, if and only if [1]

$$W + \Omega^\top\Omega - k_d(C_k + C_k^\top) \prec 0, \qquad \forall\, k \in [1,p].$$

From Lemma 2, statement i), for every $k \in [1,p]$ we have $C_k = C_k^\top \succ 0$, and hence $\bar{k}_d > 0$ can be found such that for every $k_d > \bar{k}_d$ condition $W + \Omega^\top\Omega - 2k_dC_k \prec 0$ holds for $k \in [1,p]$. Specifically, this is true if we assume

$$\bar{k}_d := \frac{\max_{\mathbf{z}:\mathbf{z}^\top\mathbf{z}=1} \mathbf{z}^\top(W + \Omega^\top\Omega)\mathbf{z}}{\min_{k\in[1,p]} \min_{\mathbf{z}:\mathbf{z}^\top\mathbf{z}=1} 2\mathbf{z}^\top C_k\mathbf{z}} = \frac{\lambda_{\max}(W + \Omega^\top\Omega)}{2\min_{k\in[1,p]} \lambda_{\min}(C_k)},$$

where $\lambda_{\max}(S)$ and $\lambda_{\min}(S)$ denote the largest and smallest (real) eigenvalues of the symmetric matrix S, and we made use of standard results about quadratic forms [4]. This means that $k_d > 0$ sufficiently large can always be found s.t. $V(\mathbf{d}) = \mathbf{d}^\top P\mathbf{d}$, with $P = P^\top \succ 0$ defined as in (16), is a common quadratic Lyapunov function for the p subsystems

$$\dot{\mathbf{d}}(t) = (\mathbb{A} - \mathbb{N}_k)\mathbf{d}(t), \qquad k \in [1,p],$$

of the switched system (14). This in turn ensures [9,18] that system (14), and hence system (8), are asymptotically stable under arbitrary switching. ∎

From Proposition 1 and Proposition 2, the following Theorem directly follows.

Theorem 1 *Assume that the pair (A,B) is stabilizable and described as in (5). Assume also, w.l.o.g., that (A_u,B_u) is in controllable canonical form. Let $\psi(s) := s^{d-1} + \alpha_{d-2}s^{d-2} + \cdots + \alpha_1 s + \alpha_0$ be an arbitrary Hurwitz polynomial of degree $d-1$ and let $K_u \in \mathbb{R}^{1\times d}$ be described as in (11). Then there exists $k_d > 0$ s.t. the switched system (8) is asymptotically stable under arbitrary switching, and hence $K = [K_u\ 0]$ solves the consensus problem under switching topology.*

To conclude, we propose an extremely simple algorithm that summarizes how to determine a feedback matrix $K \in \mathbb{R}^{1\times n}$ that solves the consensus problem under switching topology, by assuming that (A,B) is an arbitrary stabilizable pair.

Algorithm:

A0. Select an arbitrary Hurwitz monic polynomial of degree $d-1$: $\psi(s) := s^{d-1} + \alpha_{d-2}s^{d-2} + \cdots + \alpha_1 s + \alpha_0$.
A1. Determine $Q_1 \in \mathbb{R}^{n\times n}$ s.t. the pair $(Q_1AQ_1^{-1}, Q_1B)$ takes the form (5), with A_s Hurwitz and A_u having all eigenvalues in the closed right half-plane.
A2. Determine $Q_2 \in \mathbb{R}^{d\times d}$ s.t. the pair $(Q_2A_uQ_2^{-1}, Q_2B)$ is in controllable canonical form.
A3. Compute the matrices $\tilde{A}_u, \mathbb{A}$ and $\tilde{\mathcal{L}}_k, k \in [1,p]$. Choose $P_1 = P_1^\top \succ 0$ such that (15) holds.
A4. Define P as in (16) and choose $k_d > 0$ so that $(\mathbb{A} - \mathbb{N}_k)^\top P + P(\mathbb{A} - \mathbb{N}_k) \prec 0$ for every $k \in [1,p]$. By referring to the notation adopted within the previous

proof, this is surely the case if[3]

$$k_d > \frac{\lambda_{\max}(W + \Omega^\top \Omega)}{2 \min_{k \in [1,p]} \lambda_{\min}(C_k)}.$$

Then $K := k_d \begin{bmatrix} \boldsymbol{\alpha} Q_2 & 0 \end{bmatrix} Q_1$, where

$$\boldsymbol{\alpha} := \begin{bmatrix} \alpha_0 & \dots & \alpha_{d-2} & 1 \end{bmatrix},$$

solves the consensus problem with switching topology.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[2] J.A. Fax and R.M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Trans. Aut. Contr.*, 49, no. 9:1465–1476, 2004.

[3] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathem. J.*, 23:298–305, 1973.

[4] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge Univ. Press, Cambridge (GB), 1985.

[5] X. Li, M.Z.Q. Chen, H. Su, and C. Li. Consensus networks with switching topology and time-delays over finite fields. *Automatica*, 68:39–43, 2015.

[6] Z. Li, Z. Duan, and F.L. Lewis. Distributed robust consensus control of multi-agent systems with heterogeneous matching uncertainties. *Automatica*, 50 (3):883–889, 2014.

[7] Z. Li, W. Ren, X. Liu and L. Xie. Distributed consensus of linear multi-agent systems with adaptive dynamic protocols. *Automatica*, 49 (7):1986–1995, 2013.

[8] D. Liberzon. *Switching in Systems and Control.* Birkhauser, Boston (MA), 2003.

[9] H. Lin and P.J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Trans. Aut. Contr.*, 54 (2):308–322, 2009.

[10] Z. Meng, G. Shi, K.H. Johansson, M. Cao, and Y. Hong. Behaviors of networks with antagonistic interactions and switching topologies. *arXiv:1402.2766v2*, 19 Apr, 2016.

[11] B. Mohar. The Laplacian spectrum of graphs. *Graph Theory, Combinatorics, and Applications*, 2:871–898, 1991.

[12] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proc. of the IEEE*, 95 (1):215–233, 2007.

[13] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Aut. Contr.*, 49 (9):1520 –1533, 2004.

[3] It is worth noticing that since $C_k = C_k^\top \succ 0$ is an $(N-1) \times (N-1)$ submatrix of the Laplacian $\mathcal{L}_k$, by the "Interlacing theorem of the eigenvalues of a symmetric matrix" (see [4]), we can claim that $0 < \lambda_{\min}(C_k) \leq \lambda_2(\mathcal{L}_k)$, namely the smallest (and positive) eigenvalue of C_k is upper bounded by the smallest positive eigenvalue of the Laplacian $\mathcal{L}_k$. In general, not much more can be said about $\lambda_{\min}(C_k)$ without entering into the details of the structure of the graph and subgraph associated with $\mathcal{L}_k$ and C_k, respectively.

[14] J. Qin and C. Yu. Exponential consensus of general linear multi-agent systems under directed dynamic topology. *Automatica*, 50:2327–2333, 2014.

[15] W. Ren and E.M. Atkins. Distributed multi-vehicle coordinated control via local information exchange. *Int. J. Robust Nonlinear Contr.*, 17 (10-11):1002–1033, 2007.

[16] W. Ren and R.W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Aut. Contr.*, 50 (5):655–661, 2005.

[17] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45 (11):2557–2562, 2009.

[18] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49:545–592, 2007.

[19] N.K. Son and D. Hinrichsen. Robust stability of positive continuous time systems. *Numer. Funct. Anal. Optimiz.*, 17 (5 & 6):649–659, 1996.

[20] S. Su and Z. Lin. Distributed consensus control of multi-agent systems with higher order agent dynamics and dynamically changing directed interaction topologies. *IEEE Trans. Aut. Contr.*, 61(2):515–519, 2016.

[21] Y. Su and J. Huang. Stability of a class of linear switching systems with applications to two consensus problems. *IEEE Trans. Aut. Contr.*, 57 (6):1420–1430, June 2012.

[22] Z. Sun and S.S. Ge. *Switched Linear Systems: Control and Design.* Springer Verlag, London, 2005.

[23] Z. Sun and S.S. Ge. *Stability Theory of Switched Dynamical Systems.* Springer Verlag, London, 2011.

[24] J.N. Tsitsiklis. *Problems in Decentralized Decision Making and Computation.* PhD thesis, Dept. EECS, MIT, 1984.

[25] J. Wang, D. Cheng, and X. Hu. Consensus of multi-agent linear dynamic systems. *Asian J. Contr.*, 10, 2008.

[26] G. Wen, Z. Duan, G. Chen, and W. Yu. Consensus tracking of multi-agent systems with Lipschitz-type node dynamics and switching topologies. *IEEE Trans.Circ. Sys. I: Regular Papers*, 61 (2):499–511, 2014.

[27] G. Wen, G. Hu, W. Yu, and G. Chen. Distributed $\mathcal{H}_\infty$ consensus of higher order multiagent systems with switching topologies. *IEEE Trans. Circ. Sys.*, 61(5):359–363, 2014.

[28] G. Wen and V. Ugrinovskii. Distributed consensus of linear multi-agent systems with switching directed topologies. In *Proc. Australian Contr. Conf.*, pages 146–151, Canberra, Australia, 2014.

[29] P. Wieland, J.-S. Kim, and F. Allgower. On topology and dynamics of consensus among linear high-order agents. *Int. J. Sys. Sci.*, 42 (10):1831–1842, 2011.

[30] P. Wieland, J.-S. Kim, H. Scheu, and F. Allgower. On consensus in multi-agent systems with linear high- order agents. In *Proc. 17th IFAC World Congress*, pages 1541–1546, Seoul, Korea, 2008.

[31] J. Wu, J. Qin, C. Yu, and F. Allgower. Leaderless synchronization of linear multi-agent systems under directed switching topologies: an invariance approach. In *Proc. 52nd IEEE CDC*, pages 6043–6048, Firenze, Italy, 2013.

[32] W. Xia, M. Cao, and K.H. Johansson. Structural balance and opinion separation in trust-mistrust social networks. *IEEE Trans. Contr. Netw. Sys.*, 3 (1):46–56, 2016.