ZERO-TIME-CONTROLLABILITY AND DEAD-BEAT CONTROL OF TWO-DIMENSIONAL BEHAVIORS

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Abstract. In this paper two-dimensional (2D) discrete behaviors, defined on the grid $\mathbb{Z}_+ \times \mathbb{Z}$ and having the time as (first) independent variable, are investigated. For these behaviors, by emphasizing the causality notion that is naturally associated with the time variable, we introduce two new concepts of controllability. Algebraic characterizations of time-controllability and of zero-time-controllability are provided, and it is shown that behaviors endowed with these properties admit special decompositions. Next, the dead-beat control (DBC) problem and the concept of admissible DBC are introduced and related to the zero-time-controllability property. Differently from what happens with one-dimensional behaviors, zero-time-controllability does not ensure the existence of regular DBC's, and stronger algebraic properties need to be imposed on the behavior. Finally, necessary and sufficient conditions for the existence of a DBC that makes the resulting behavior both strongly autonomous and nilpotent are provided.

Key words. 2D behavior, nilpotent autonomous behavior, time-controllability, zero-time- controllability, dead-beat controller, full interconnection, regular interconnection.

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1. Introduction.

Research interests in multidimensional (nD) and, in particular, in two-dimensional (2D) systems started in the seventies [1, 5, 23], stimulated by the large number of application areas (image processing, seismology, learning and repetitive processes, ecological modeling, ...) where these models naturally arise. It was only in the early nineties, with the fundamental work of P. Rocha [16, 20], that the behavioral approach to 2D systems laid down its firm foundations. One of the main advantages of the behavioral approach, namely the fact that it allows for a complete analysis of the systems properties without assuming any causality notion, nor any input/output relationship between the system variables, was the reason of its success in this research area. Indeed, the possibility of defining and characterizing properties, like autonomy and controllability, and to state and solve control and estimation problems, without introducing any a priori concept of causality, allowed for significant advancements in the study and comprehension of multidimensional systems.

In most of the results obtained for 2D and nD behaviors, the independent variables play an equal role. In several engineering applications, however, one of the independent variables represents time, and its role is distinguished from that of all the others, first of all because for that variable the concept of causality always makes sense. The study of multidimensional systems in the behavioral approach, under the assumption that one of the independent variables is time, originated in [26], and was later pursued in [24, 25]. In these papers, Sasane and co-authors investigated the concepts of time-autonomy and time-controllability for systems described by partial differential equations, and provided necessary and/or sufficient conditions for these properties to hold.

In recent times, 2D behaviors for which one of the independent variables is time have been investigated in detail in [9] (in the discrete case), by focusing on the concept of time-relevant autonomous behavior and on the related stability problems. Also, quite recently, stimulated by a preliminary version of this manuscript, Oberst and

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Scheicher [13] have developed a general framework for the discrete nD case, and have provided characterizations of time-autonomy and time-controllability.

The purpose of this paper is to further investigate the class of discrete 2D behaviors having the time as an independent variable. Since our interest is in *dead-beat control*, a concept that makes no sense for behaviors defined all over the discrete grid, we have chosen to consider behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$, by henceforth assuming that the first, namely the time, variable is defined on $\mathbb{Z}_+$, while the other coordinate (that may be regarded as a space variable, without loss of generality) is defined on the whole integer set.

By assuming this perspective, we have given a special role to *vertical strips* in the grid $\mathbb{Z}_+ \times \mathbb{Z}$, namely to the sets $\{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z} : 0 \le h \le N-1\}$, and to the *half-planes* $\{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z} : h \ge N\}$, for $N \in \mathbb{Z}_+$. The former provide the sets where "initial conditions" on the system variables are given, while the latter are the supports of "long term evolutions", where both concepts clearly refer to the time coordinate. Conformably, the definitions of time-controllable behavior and of zero-time-controllable behavior have been introduced in this paper.

While standard controllability [20, 32] corresponds to the possibility of patching two (arbitrarily shifted) behavior trajectories on two distinct sets, $\mathcal{T}_1$ and $\mathcal{T}_2$, provided that they are sufficiently distant, time-controllability only requires that this patching is possible for very special subsets of $\mathbb{Z}_+ \times \mathbb{Z}$, namely when $\mathcal{T}_1$ is a vertical strip and $\mathcal{T}_2$ is a half-plane. On the other hand, zero-time-controllability is a special case of time-controllability, as we impose that every "initial strip" of behavior trajectory can be patched with the zero trajectory in the "future half-plane". The class of autonomous nilpotent behaviors, by this meaning autonomous behaviors whose trajectories are identically zero in some half-plane $\{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z} : h \ge N\}$, represents a special class of zero-time-controllable behaviors.

In this paper, we derive algebraic characterizations of 2D behaviors endowed with these properties, first focusing on the autonomous case and then broadening the results to the general case. Moreover, these properties are related to the existence of suitable behavior sum decompositions.

The concept of dead-beat controller (DBC), that is a controller acting by full interconnection and making the resulting controlled system nilpotent, is introduced. By extending the analysis recently carried on in [3] for 1D behaviors, we introduce the concept of admissible DBC. It turns out that, as in the 1D case, behaviors endowed with an admissible DBC are those and those only that are zero-time-controllable. Differently from what happens with 1D behaviors, however, zero-time-controllability alone is not sufficient to ensure the existence of regular DBC's, and stronger algebraic properties need to be imposed on the behavior.

Finally, by assuming a perspective similar to the one taken in [10, 19], we investigate the conditions that allow to obtain strongly autonomous nilpotent behaviors, by means of either admissible or regular DBC's.

The paper is organized as follows. In section 2, preliminary results about polynomial matrices with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ are introduced. Even if the proofs of these results are not available in the literature, they may easily be derived (*mutatis mutandis*) from the analogous ones obtained for polynomial or Laurent polynomial matrices in two indeterminates, and hence we will state them without an explicit proof. Basic properties of behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$ and the corresponding algebraic characterizations are discussed in section 3. Sections 4 and 5 introduce and characterize time-controllable and zero-time-controllable behaviors, first in the autonomous case and then in the general one. Finally, the DBC problem is stated and solved in section 6. Some comments regarding how the concepts of controllability, time-controllability and zero-time-controllability can be adapted to the special case of 2D behaviors generated by the Fornasini-Marchesini state-space model [5] are discussed in section 7.

2. The ring of polynomial matrices with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$. In the following, we concentrate on the ring of polynomials with real coefficients in the nonnegative powers of z_1 and in the integer powers of z_2, namely on $\mathbb{R}[z_1, z_2, z_2^{-1}]$. Since the former variable is associated with the time variable, while the latter with a space variable, we refer to such polynomials as *time-space polynomials* (for short, *TS-polynomials*). Clearly, the ring of TS-polynomials is properly included in the ring of *Laurent polynomials* (*L-polynomials*) with real coefficients in the variables z_1 and z_2, namely $\mathbb{R}[z_1, z_1^{-1}, z_2, z_2^{-1}]$. So, in some situations it may be convenient to regard TS-polynomials as L-polynomials.

Given any nonzero TS-polynomial $p(z_1, z_2) = \sum_{(i,j)\in\Sigma_p} p_{ij} z_1^i z_2^j$, with Σ_p a finite subset[1] of $\mathbb{Z}_+ \times \mathbb{Z}$, we define the *TS-variety* of p as

$$\mathcal{V}_{TS}(p) := \{(\alpha, \beta) \in \mathbb{C} \times \mathbb{C} : \beta \neq 0, p(\alpha, \beta) = 0\},$$

i.e. as the set of all complex pairs (α, β), with nonzero β, such that (s.t.) $\sum_{(i,j)\in\Sigma_p} p_{ij}$ $\alpha^i \beta^j = 0$. If $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times q}$ is a full column rank matrix, by $\mathcal{V}_{TS}(H)$ we mean the intersection of the TS-varieties of its maximal (namely, qth) order minors[2].

(Factor and zero) primeness properties may be introduced for the class of TS-polynomial matrices by suitably extending the analogous definitions for L-polynomial matrices (see [7, 33, 34]). If we restrict our attention to full column rank matrices, we say that a full column rank TS-polynomial matrix $H(z_1, z_2)$ is *right factor prime* (rFP) if in every factorization $H(z_1, z_2) = \bar{H}(z_1, z_2)\Delta(z_1, z_2)$ over the ring $\mathbb{R}[z_1, z_2, z_2^{-1}]$, with $\Delta(z_1, z_2)$ nonsingular square, the matrix $\Delta(z_1, z_2)$ is *unimodular* in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ (by this meaning that $\det \Delta(z_1, z_2) = c z_2^k$, for some $c \neq 0$ and some $k \in \mathbb{Z}$). Also, $H(z_1, z_2)$ is *right zero prime* (rZP) if it admits a left inverse in $\mathbb{R}[z_1, z_2, z_2^{-1}]$, namely there exists $L(z_1, z_2)$ with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ s.t. $L(z_1, z_2)H(z_1, z_2) = I$.

A full column rank TS-polynomial matrix $H(z_1, z_2)$ is said to be *right monomic* (rM) if it is right zero prime if regarded as an L-polynomial matrix, namely it admits an L-polynomial (but not necessarily TS-polynomial) left inverse. This amounts to saying that there exists $L(z_1, z_2)$ with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ s.t. $L(z_1, z_2)H(z_1, z_2) = z_1^h I$, for some $h \in \mathbb{Z}_+$. In particular, a square TS-polynomial matrix $\Delta(z_1, z_2)$ is called *square monomic* if it is unimodular if regarded as an L-polynomial matrix, and hence $\det \Delta(z_1, z_2) = c z_1^h z_2^k$, for suitable $c \neq 0, h \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. It is worthwhile noticing that a $p \times q$ matrix $H(z_1, z_2)$ is rM if and only if it can be column-bordered up to a $p \times p$ square monomic matrix, by this meaning that there exists $C(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times(p-q)}$ s.t.

$$\Delta(z_1, z_2) := \begin{bmatrix} H(z_1, z_2) & C(z_1, z_2) \end{bmatrix}$$

satisfies $\det \Delta(z_1, z_2) = c z_1^h z_2^k$, for suitable $c \neq 0, h \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$.

All these properties may be easily characterized[3] in terms of the TS-variety of $H(z_1, z_2)$ as, indeed, $H(z_1, z_2)$ is

[1] If $p_{ij} \neq 0$ for every $(i, j) \in \Sigma_p$, we refer to Σ_p as the *support* of the TS-polynomial $p(z_1, z_2)$.

[2] The concept of TS-variety of a full rank matrix is already known in the literature as *characteristic variety*, see [12].

[3] Specific proofs of the characterizations of these and the following properties for TS-polynomial

- rFP if and only if the variety $\mathcal{V}_{TS}(H)$ consists of a finite number of points;
- rM if and only if $\mathcal{V}_{TS}(H) \subseteq \{0\} \times (\mathbb{C} \setminus \{0\})$;
- rZP if and only if $\mathcal{V}_{TS}(H)$ is empty.

It is worthwhile noticing that right zero primeness implies right factor primeness as well as right monomicity, however right factor primeness and right monomicity are not necessarily related. For instance,

$$H(z_1, z_2) = \begin{bmatrix} z_1^2 \\ z_1 z_2 \end{bmatrix}$$

is clearly rM, since $\mathcal{V}_{TS}(H) = \{(0, \beta) : \beta \in \mathbb{C} \setminus \{0\}\}$, but clearly this is not a rFP matrix. On the other hand,

$$H(z_1, z_2) = \begin{bmatrix} z_1 + 1 \\ 1 - z_2 \end{bmatrix}$$

is clearly rFP since $\mathcal{V}_{TS}(H) = \{(-1, 1)\}$ is a finite set, but it is not right monomic.

Of course, the concepts of *left* factor/zero prime or monomic (lFP, lZP and lM, respectively) TS-polynomial matrix can be introduced for full row rank matrices in a similar way, and enjoy analogous properties and characterizations.

Analogously to what happens with L-polynomial matrices, every TS-polynomial matrix $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p \times q}$ of rank r can always be factorized as

$$H(z_1, z_2) = L(z_1, z_2)\Delta(z_1, z_2)R(z_1, z_2), \tag{2.1}$$

for some suitable TS-polynomial matrices, with $L(z_1, z_2)$ $p \times r$ right factor prime, $\Delta(z_1, z_2)$ $r \times r$ nonsingular square, and $R(z_1, z_2)$ $r \times q$ left factor prime. This factorization is essentially unique, by this meaning that these three factors are uniquely determined up to (left and/or right) unimodular matrices.

The concepts of *left annihilator* and, in particular, of *minimal left annihilator* (MLA, for short) of a given TS-polynomial matrix $H(z_1, z_2)$ extend the concepts originally introduced in [16] for polynomial matrices in two indeterminates, and can be summarized as follows: if $H(z_1, z_2)$ is a $p \times q$ TS-polynomial matrix of rank r, a TS-polynomial matrix $L(z_1, z_2)$ is a left annihilator of $H(z_1, z_2)$ if $L(z_1, z_2)H(z_1, z_2) = 0$. A left annihilator $L_m(z_1, z_2)$ of $H(z_1, z_2)$ is an MLA if it is of full row rank and for any other left annihilator $L(z_1, z_2)$ of $H(z_1, z_2)$ we have $L(z_1, z_2) = P(z_1, z_2)L_m(z_1, z_2)$ for some TS-polynomial matrix $P(z_1, z_2)$. It can be easily proved that, when $r < p$, an MLA always exists, it is a $(p - r) \times p$ left factor prime matrix and is uniquely determined modulo a unimodular left factor. If the given $H(z_1, z_2)$ is of full row rank, then for consistency we define its MLA as the "void" matrix with 0 rows and p columns [15].

Right annihilators and *minimal right annihilators* (MRAs) can be similarly defined and enjoy analogous properties.

3. Basic facts about 2D behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$. In this contribution, by a 2D *behavior* $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$ we mean the set of solutions

$$\mathbf{w} = \{\mathbf{w}(h, k)\}_{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}}$$

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matrices are not available in the literature, but they can be easily obtained through a minor change of the proofs already available for polynomial and L-polynomial matrices in two variables [7, 33, 34]. Also, for more abstract and general proofs one can refer to [12].

of a family of linear 2D difference equations of the following type:

$$\sum_{(i,j)\in\Sigma_H} H_{ij}\mathbf{w}(h+i,k+j)=0, \qquad \forall\ (h,k)\in\mathbb{Z}_+\times\mathbb{Z}, \tag{3.1}$$

where the H_{ij}'s are real matrices with $\mathtt{w}$ columns[4] (and say p rows), and the index set Σ_H is a finite subset of $\mathbb{Z}_+\times\mathbb{Z}$. For a 2D behavior $\mathfrak{B}$ described as in (3.1), we adopt the shorthand notation

$$\mathfrak{B}=\ker H(\sigma_1,\sigma_2), \tag{3.2}$$

where $H(z_1,z_2)=\sum_{(i,j)\in\Sigma_H} H_{ij}z_1^i z_2^j$ is a TS-polynomial matrix, and σ_1 and σ_2 denote the two backward shift operators along the coordinate axes of the discrete grid $\mathbb{Z}\times\mathbb{Z}$, defined as follows[5]:

$$(\sigma_1\mathbf{w})(h,k):=\mathbf{w}(h+1,k),\ (\sigma_2\mathbf{w})(h,k):=\mathbf{w}(h,k+1).$$

Given two TS-polynomial matrices $H_1(z_1,z_2)$ and $H_2(z_1,z_2)$, with the same number of columns $\mathtt{w}$, it can be shown[6] that $\ker H_1(\sigma_1,\sigma_2)\subseteq\ker H_2(\sigma_1,\sigma_2)$ if and only if $H_2(z_1,z_2)=P(z_1,z_2)H_1(z_1,z_2)$, for some TS-polynomial matrix $P(z_1,z_2)$ of suitable size. Also, $\ker H(\sigma_1,\sigma_2)=\{0\}$ if and only if $H(z_1,z_2)$ is rZP, and consequently the TS-polynomial matrices $R(z_1,z_2)$ and $H(z_1,z_2)R(z_1,z_2)$, with $R(z_1,z_2)$ of full row rank, have the same kernel if and only if $H(z_1,z_2)$ is right zero prime. Finally, the map $H(\sigma_1,\sigma_2)$ is surjective if and only if $H(z_1,z_2)$ is of full row rank.

We introduce the following definition of controllability that is equivalent to "controllability (4)" given in [32], and takes into account the fact that we are working with behaviors defined on the half-plane $\mathbb{Z}_+\times\mathbb{Z}$, which are σ_1-,σ_2- and σ_2^{-1}- invariant (see also [21]).

Definition 3.1. *A 2D behavior $\mathfrak{B}=\ker H(\sigma_1,\sigma_2)$, with $H(z_1,z_2)\in\mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times\mathtt{w}}$, is said to be* controllable *if there exists some nonnegative integer $L\in\mathbb{Z}_+$ s.t. for every $\mathcal{T}_1,\mathcal{T}_2\subset\mathbb{Z}_+\times\mathbb{Z}$, with $\operatorname{dist}(\mathcal{T}_1,\mathcal{T}_2):=\min\{|h_1-h_2|+|k_1-k_2| : (h_1,k_1)\in\mathcal{T}_1,(h_2,k_2)\in\mathcal{T}_2\}>L$, and for every pair of trajectories $\mathbf{w},\mathbf{w}^*\in\mathfrak{B}$, one can find $\bar{\mathbf{w}}\in\mathfrak{B}$ s.t.*

$$\begin{array}{rcll}\bar{\mathbf{w}}(h,k) &=& \mathbf{w}(h,k), & \forall\ (h,k)\in\mathcal{T}_1,\\ \bar{\mathbf{w}}(h,k) &=& \mathbf{w}^*(h-\bar{h}_2,k), & \forall\ (h,k)\in\mathcal{T}_2,\end{array}$$

where $\bar{h}_2:=\min\{h\in\mathbb{Z}_+ : \exists k : (h,k)\in\mathcal{T}_2\}$.

We have the following characterization of controllability (see Corollary 7 of [32]).

Proposition 3.2. *Given a 2D behavior $\mathfrak{B}=\ker H(\sigma_1,\sigma_2)$, with $H(z_1,z_2)\in\mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times\mathtt{w}}$, the following facts are equivalent:*

i) $\mathfrak{B}$ is controllable;

ii) $\mathfrak{B}$ can be described as the kernel of a lFP TS-polynomial matrix, or, equivalently, $H(z_1,z_2)=L(z_1,z_2)R(z_1,z_2)$ for some suitable TS-polynomial matrices, with $L(z_1,z_2)$ rZP and $R(z_1,z_2)$ lFP.

[4] In the sequel, the symbol $\mathtt{w}$ will always denote the size of the vector $\mathbf{w}$.

[5] The forward shift operators σ_1^{-1} and σ_2^{-1} are similarly defined. Notice that σ_i, $i=1,2$, and σ_2^{-1} map $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$ into $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$, but this is not true for σ_1^{-1}.

[6] Also in this case one can find a general proof of this result, for modules over commutative rings, in [12], Theorem (61), page 36.

Given a 2D behavior $\mathfrak{B}$, we define its *controllable part* $\mathfrak{B}_c$ as the largest controllable behavior included in $\mathfrak{B}$ [16, 31]. If $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, and we factorize the TS-polynomial matrix $H(z_1, z_2)$ as $H(z_1, z_2) = L(z_1, z_2)\Delta(z_1, z_2)R(z_1, z_2)$ for some suitable TS-polynomial matrices, with $L(z_1, z_2)$ $p \times r$ right factor prime, $\Delta(z_1, z_2)$ $r \times r$ nonsingular square, and $R(z_1, z_2)$ $r \times \mathtt{w}$ left factor prime (see (2.1)), then $\mathfrak{B}_c = \ker R(\sigma_1, \sigma_2)$.

It is worthwhile to remark that, in the literature about two-dimensional behaviors, a stronger concept of controllability has also been defined and characterized, namely the property of *rectifiability* (or *strong controllability*). We refer the interested reader to [10, 17, 18, 19, 22, 34] for the details. In this paper, we are only interested in remarking that a 2D behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$ is rectifiable if and only if it can be described as the kernel of a lZP TS-polynomial matrix.

We now introduce autonomous behaviors.

Definition 3.3. *[6, 21] Given a 2D behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, a set of variables $\{w_i : i \in \mathcal{I}\}$, $\mathcal{I}$ a proper subset of $\{1, 2, \dots, \mathtt{w}\}$, is said to be a* set of free variables *for $\mathfrak{B}$ if the map $\pi_{\mathcal{I}} : \mathfrak{B} \to (\mathbb{R}^{|\mathcal{I}|})^{\mathbb{Z}_+\times\mathbb{Z}}$, that projects every behavior trajectory onto the components indexed on $\mathcal{I}$, is surjective.*
$\mathfrak{B}$ is said to be autonomous *if it has no free variables.*

Within the class of 2D autonomous behaviors we single out the nilpotent ones. Before introducing their definition, it is convenient to introduce some notation that will be used extensively in the rest of the paper.

For any pair of nonnegative integers t_0 and t_1, we define the *vertical strip*

$$\mathcal{S}_{t_0,t_1} := \{(h, k) \in \mathbb{Z}_+ \times \mathbb{Z} : t_0 \le h \le t_1\},$$

which is the empty set if $t_1 < t_0$. When $t_0 = t_1$ we use $\mathcal{S}_{t_0}$ to denote the *vertical line* $\{(t_0, k) : k \in \mathbb{Z}\}$, when $t_0 = 0 \le t_1$ we use $\mathcal{S}_{\to t_1}$ and when $t_1 = +\infty$ we use $\mathcal{S}_{t_0\to}$. Given any trajectory $\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$ and any set $\mathcal{S}_{t_0,t_1}$, we denote the trajectory restriction to the set $\mathcal{S}_{t_0,t_1}$ by $\mathbf{w}|_{\mathcal{S}_{t_0,t_1}}$. The *support* of a trajectory $\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$ is the set of points where the trajectory takes nonzero values, i.e. $\{(h, k) \in \mathbb{Z}_+ \times \mathbb{Z} : \mathbf{w}(h, k) \ne 0\}$.

Definition 3.4. *A 2D autonomous behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, is said to be* nilpotent *(with respect to the half-plane $\mathbb{Z}_+ \times \mathbb{Z}$) if there exists $N \in \mathbb{Z}_+$ s.t. all the trajectories $\mathbf{w} \in \mathfrak{B}$ satisfy $\mathbf{w}|_{\mathcal{S}_{N\to}} = 0$, or, equivalently $\mathbf{w}(h, k) = 0,\ \forall\ (h, k) \in \mathcal{S}_{N\to}$.*

Remark 3.5. *It is worthwhile noticing that nilpotency for a 2D behavior defined on $\mathbb{Z}_+ \times \mathbb{Z}$ does not mean that each trajectory has a finite support, as in the 1D case [4], but only that its support intersects finitely many (possibly zero) straight vertical lines $\mathcal{S}_t$ of $\mathbb{Z}_+ \times \mathbb{Z}$ or, equivalently,it is included in $\mathcal{S}_{\to N-1}$, for some $N \in \mathbb{Z}_+$ [7]. Note that this concept of nilpotency is consistent with the interpretation of the first independent variable as a time coordinate.*

The definition of autonomous behavior has also been introduced by resorting to the concept of characteristic set*. A set $\mathcal{S}$ is characteristic for a behavior $\mathfrak{B}$ if the knowledge of any behavior trajectory $\mathbf{w}$ on $\mathcal{S}$ allows to uniquely determine it on $\mathbb{Z}_+\times\mathbb{Z}$. Indeed, a 2D behavior is autonomous if (and only if) it admits a "nontrivial" characteristic set*

[7] Clearly, the special case $N = 0$ corresponds to $\mathcal{S}_{\to N-1} = \emptyset$ and hence to the zero behavior.

$\mathcal{S}$. The interested reader is referred to [16, 27] for the details[8]*. Under this perspective, it is easily seen that a nilpotent behavior is an autonomous behavior having the strip $\mathcal{S}_{\to N-1}$, for an appropriate nonnegative integer N, as characteristic set.*

PROPOSITION 3.6. *A 2D behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p \times \mathtt{w}}$, is*

- *autonomous if and only if $H(z_1, z_2)$ is a full column rank matrix;*
- *nilpotent if and only if $H(z_1, z_2)$ is a right monomic matrix.*

Proof. The proof of the first part follows the same reasonings adopted for the analogous proof for 2D behaviors defined all over the discrete grid $\mathbb{Z} \times \mathbb{Z}$ [16].

The second part can be proved along the lines of the same proof for 2D behaviors defined on the half-plane $\{(h, k) \in \mathbb{Z} \times \mathbb{Z} : h + k \geq 0\}$ (see [2]). $\square$

REMARK 3.7. *It is worth, at this point, to explicitly comment on the relationship between the autonomous behaviors we consider in this paper and time-autonomous behaviors considered in the works of Sasane et al. [25, 24, 26] (for partial differential equations) and in the recent work by Napp et al. [9] (for partial difference equations). In all these works, time-autonomous (also called* time-relevant*) behaviors are autonomous behaviors for which all the half-planes $\{(h, k) : h \leq t\}$ are characteristic. This amounts to saying that an autonomous behavior is time-relevant if and only if $\mathbf{w}(h, k) = 0$, for every $h < 0$ and every k, implies $\mathbf{w} = 0$. In this paper, we assume the same perspective in giving to one of the variables the interpretation of time variable, however as our interest is in the dead-beat control problem, the concept of time-autonomous behavior does not play any role in this problem solution. Nonetheless, it is worth to remark that once we adapt the perspective taken by the aforementioned Authors to the case of behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$, time-autonomy could be defined in terms of characteristic sets of the form $\mathcal{S}_{\to N-1}$, and it is immediately seen that nilpotent behaviors are a special case of time-autonomous behaviors. Indeed, kernels of right monomic matrices satisfy the algebraic characterization of time-relevant autonomous behaviors derived in [9].*

As in the case of standard 2D behaviors [27, 28] (see, also, [9]), within the class of autonomous behaviors we distinguish two special subclasses: *square autonomous behaviors*, that are the kernels of nonsingular square matrices, and *finite dimensional behaviors*, that are the kernels of rFP matrices. Moreover, every autonomous behavior $\mathfrak{B}_a = \ker(L(\sigma_1, \sigma_2)\Delta(\sigma_1, \sigma_2))$, with $L(z_1, z_2)$ rFP and $\Delta(z_1, z_2)$ nonsingular square, can be expressed as the sum of its (uniquely determined) square-autonomous part $\mathfrak{B}_{sq} = \ker\Delta(\sigma_1, \sigma_2)$, and of some finite-dimensional autonomous behavior $\mathfrak{B}_f$. The concept of *autonomy degree* of a multidimensional behavior was first introduced in [30], and later recalled in other contributions, in particular in [14, 19]. In the 2D case, autonomous behaviors can only have two (nontrivial[9]) degrees of autonomy: autonomous behaviors with autonomy degree 2 are simply finite-dimensional autonomous behaviors and are also known as *strongly autonomous behaviors* (see also [10]). All the other autonomous behaviors have a nontrivial square-autonomous part, and hence have autonomy degree 1.

Finite-dimensional behaviors exhibit the property that every sufficiently large finite rectangle in $\mathbb{Z}_+ \times \mathbb{Z}$ is characteristic for them. As a consequence, for a finite-

[8] Clearly, in these references, the definitions of characteristic set and of autonomous behavior have been given for 2D behaviors defined all over the discrete grid $\mathbb{Z} \times \mathbb{Z}$, but their adaptation to the case $\mathbb{Z}_+ \times \mathbb{Z}$ is immediate.

[9] Indeed, the zero behavior has, by definition, an autonomy degree equal to ∞.

dimensional behavior $\mathfrak{B}$, every trajectory $\mathbf{w} \in \mathfrak{B}$ is uniquely identified by its restriction $\mathbf{w}|_{\mathcal{S}_{\to N-1}}$, provided that N is sufficiently large.

4. (Zero-)time-controllability: the autonomous case. For 2D behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$, we naturally introduce the definitions of time-controllability and of zero-time-controllability. Time-controllability is the property of arbitrarily patching any initial strip of a behavior trajectory with any other behavior trajectory, provided that this latter is properly shifted. This concept is the same one first introduced in [26] and then investigated also in [24, 25] for partial differential equations. On the other hand, zero-time-controllability is the property of arbitrarily patching any initial strip of a behavior trajectory with the zero trajectory, which means that any behavior trajectory can be driven to zero, after a finite number of time instants, namely it becomes identically zero on some suitable half-plane $\mathcal{S}_{N+L\to}$. Both these definitions are consistent with the revised definition of controllability (see Definition 3.1) that we have previously introduced for behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$. In particular, time-controllability is a special case of controllability, obtained by assuming $\mathcal{T}_1 = \mathcal{S}_{\to N-1}$ and $\mathcal{T}_2 = \mathcal{S}_{N+L\to}$, for suitable N and L (see Figure 1).

Definition 4.1. *A 2D behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, is said to be*

- time-controllable *if there exists a nonnegative integer $L \in \mathbb{Z}_+$ s.t. for every $N \in \mathbb{N}$, and every pair of trajectories $\mathbf{w}, \mathbf{w}^* \in \mathfrak{B}$, one can find $\bar{\mathbf{w}} \in \mathfrak{B}$ s.t.*

$$\bar{\mathbf{w}}(h,k) = \mathbf{w}(h,k), \qquad \forall\ (h,k) \in \mathcal{S}_{\to N-1}, \tag{4.1}$$

$$\bar{\mathbf{w}}(h,k) = \mathbf{w}^*(h-N-L,k), \qquad \forall\ (h,k) \in \mathcal{S}_{N+L\to}, \tag{4.2}$$

i.e. $\bar{\mathbf{w}}|_{\mathcal{S}_{\to N-1}} = \mathbf{w}|_{\mathcal{S}_{\to N-1}}$ *and* $\sigma_1^{N+L}\bar{\mathbf{w}} = \mathbf{w}^*$;

- zero-time-controllable *if there exists a nonnegative integer $L \in \mathbb{Z}_+$ s.t. for every $N \in \mathbb{N}$ and every $\mathbf{w} \in \mathfrak{B}$, one can find $\bar{\mathbf{w}} \in \mathfrak{B}$ s.t.*

$$\bar{\mathbf{w}}(h,k) = \mathbf{w}(h,k), \qquad \forall\ (h,k) \in \mathcal{S}_{\to N-1}, \tag{4.3}$$

$$\bar{\mathbf{w}}(h,k) = 0, \qquad \forall\ (h,k) \in \mathcal{S}_{N+L\to}, \tag{4.4}$$

i.e. $\bar{\mathbf{w}}|_{\mathcal{S}_{\to N-1}} = \mathbf{w}|_{\mathcal{S}_{\to N-1}}$ *and* $\sigma_1^{N+L}\bar{\mathbf{w}} = 0$.

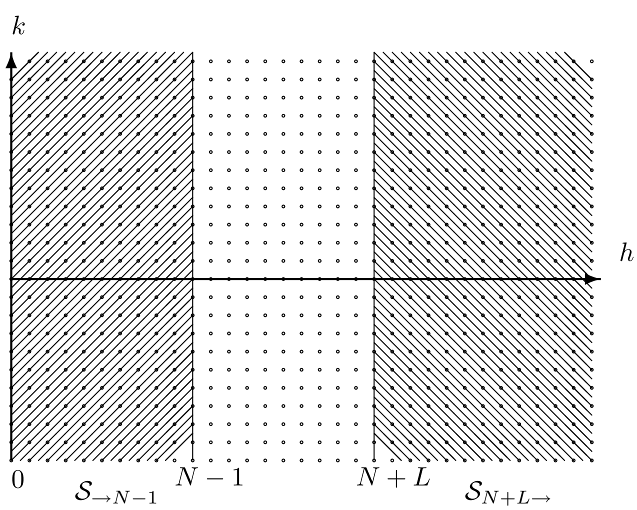

Fig.1: Sets $\mathcal{S}_{\rightarrow N-1}$ and $\mathcal{S}_{N+L\rightarrow}$ involved in (zero-)time-controllability definition.

It is clear that a time-controllable behavior is also zero-time-controllable. However, as we will see, the converse is not true. Zero-time controllability can be seen as a special case of the set-controllability property introduced and explored in [21] for multidimensional behaviors. Indeed, zero-time-controllability can be related, *mutatis mutandis*, to the property of set-controllability to a nilpotent autonomous behavior. Even if set-controllability and the relationship of this property with the behavior decomposition or with the possibility of achieving a certain subbehavior by means of a regular extended interconnection have been fully explored in [21], the obtained results do not admit a straightforward adaptation to our set-up (see Remark 5.2). In addition, they do not provide us with a characterization of zero-time-controllable behaviors in terms of the algebraic properties of the matrices involved in their kernel description.

In this and the following section we provide algebraic characterizations of both time- and zero-time-controllable behaviors, by first addressing the autonomous case. The fact that autonomous behaviors can be found that are either zero-time-controllable or time-controllable should not be regarded as a contradiction. Indeed, even when dealing with 1D state-space models, autonomous nilpotent systems are, in particular, zero-controllable. On the other hand, time-controllability is a weaker property with respect to controllability, since it considers only special sets where the trajectory patching must be performed. This means that time-controllable behaviors are, in general, endowed with a nontrivial autonomous part, as the following example clearly shows. So, also autonomous behaviors can be time-controllable, and this is the case when the restriction to $\mathcal{S}_{\rightarrow N-1}$ of any trajectory belonging to the behavior is independent of its restriction to $\mathcal{S}_{N+L\rightarrow}$.

Example 1. Assume $\mathfrak{B} = \ker\,[\,1-\sigma_2 \quad 1-\sigma_2\,]$. It is easily seen that the following controllable/autonomous decomposition holds true

$$\mathfrak{B} = \mathfrak{B}_c \oplus \mathfrak{B}_a, \qquad \text{with} \qquad \mathfrak{B}_c = \ker\,[\,1 \quad 1\,],\ \mathfrak{B}_a = \ker \begin{bmatrix} 1-\sigma_2 & 0 \\ 0 & 1 \end{bmatrix},$$

so $\mathfrak{B}$ is endowed with a nontrivial autonomous part. However, $\mathfrak{B}$ is time-controllable since the restrictions to the vertical lines $\mathbf{v}_h(k) := \mathbf{w}(h,k)$ of a generic trajectory $\mathbf{w} \in \mathfrak{B}$ are independent one from the other. So, for any $N \in \mathbb{N}$, we can concatenate two arbitrary trajectories $\mathbf{w}$ and $\mathbf{w}^*$ in $\mathfrak{B}$ as described in (4.1)-(4.2) (for $L = 0$).

Aiming at exploring these issues, we need three technical lemmas, whose proofs appear in the Appendix. First, we consider the case of scalar autonomous behaviors.

Lemma 4.2. *Consider a 2D scalar autonomous behavior* $\mathfrak{B} = \ker\,\delta(\sigma_1, \sigma_2)$*, with* $\delta(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]$*, and*

$$\delta(z_1, z_2) = \delta_r(z_2) z_1^r + \delta_{r+1}(z_2) z_1^{r+1} + \ldots + \delta_R(z_2) z_1^R,$$

where $0 \le r \le R$*,* $\delta_i(z_2) \in \mathbb{R}[z_2, z_2^{-1}], i = r, r+1, \ldots, R,$ *and* $\delta_r(z_2)$ *and* $\delta_R(z_2)$ *are both nonzero.*

i) $\mathfrak{B}$ *is time-controllable if and only if* $r = R = 0$*, namely* $\delta(z_1, z_2) \in \mathbb{R}[z_2, z_2^{-1}]$*;*

ii) $\mathfrak{B}$ *is zero-time-controllable if and only if* $r = R$*, namely* $\delta(z_1, z_2)$ *has support included in a straight vertical line.*

By using the characterizations obtained in the scalar case, we can derive the analogous characterizations for square autonomous behaviors.

LEMMA 4.3. *A 2D nonsingular square autonomous behavior* $\mathfrak{B} = \ker\, \Delta(\sigma_1, \sigma_2)$, *with* $\Delta(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{\mathtt{w}\times\mathtt{w}}$ *(and* $\det \Delta(z_1, z_2) \neq 0$*), is*

i) time-controllable if and only if $\ker(\det \Delta(\sigma_1, \sigma_2))$ *is time-controllable, namely* $\det \Delta(z_1, z_2) \in \mathbb{R}[z_2, z_2^{-1}]$*;*

ii) zero-time-controllable if and only if $\ker(\det \Delta(\sigma_1, \sigma_2))$ *is zero-time-controllable, namely* $\det \Delta(z_1, z_2)$ *has support included in a straight vertical line.*

The last lemma provides a necessary condition for (zero-)time-controllability.

LEMMA 4.4. *A 2D behavior* $\mathfrak{B} = \ker\, H(\sigma_1, \sigma_2)$, *with* $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times\mathtt{w}}$*, is*

i) time-controllable only if $H(z_1, z_2) = L(z_1, z_2)\tilde{R}(z_1, z_2)$*, for some right zero prime* $p \times r$ *TS-polynomial matrix* $L(z_1, z_2)$ *and some* $r \times \mathtt{w}$ *TS-polynomial matrix* $\tilde{R}(z_1, z_2)$ *of full row rank;*

ii) zero-time-controllable only if $H(z_1, z_2) = L(z_1, z_2)\tilde{R}(z_1, z_2)$*, for some right factor prime and right monomic* $p \times r$ *TS-polynomial matrix* $L(z_1, z_2)$ *and some* $r \times \mathtt{w}$ *TS-polynomial matrix* $\tilde{R}(z_1, z_2)$ *of full row rank.*

We are now in a position to derive a complete characterization of all autonomous behaviors that are either time-controllable or zero-time-controllable.

PROPOSITION 4.5. *Consider a 2D autonomous behavior* $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$*, with* $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times\mathtt{w}}$*, of full column rank. The following facts are equivalent:*

a1) $\mathfrak{B}$ *is time-controllable;*

a2) $H(z_1, z_2) = L(z_1, z_2)\Delta(z_1, z_2)$*, for some right zero prime* $L(z_1, z_2)$ *and some nonsingular square matrix* $\Delta(z_1, z_2)$*, with* $\det \Delta(z_1, z_2) = \delta(z_2) \in \mathbb{R}[z_2, z_2^{-1}]$.

The following facts are equivalent:

b1) $\mathfrak{B}$ *is zero-time-controllable;*

b2) $H(z_1, z_2) = \tilde{L}(z_1, z_2)\tilde{\Delta}(z_1, z_2)$*, for some right monomic* $\tilde{L}(z_1, z_2)$ *and some nonsingular square matrix* $\tilde{\Delta}(z_1, z_2)$*, with* $\det \tilde{\Delta}(z_1, z_2) = \delta(z_2) \in \mathbb{R}[z_2, z_2^{-1}]$*;*

b3)

$$\mathfrak{B} = \mathfrak{B}_{ztc} = \mathfrak{B}_{tc} + \mathfrak{B}_{nil}, \tag{4.5}$$

where $\mathfrak{B}_{tc}$ *is a time-controllable autonomous behavior, while* $\mathfrak{B}_{nil}$ *is a nilpotent one.*

Proof. a1) ⇒ a2) Assume that $\mathfrak{B}$ is time-controllable. From Lemma 4.4, we have $H(z_1, z_2) = L(z_1, z_2)\Delta(z_1, z_2)$, with $L(z_1, z_2)$ right zero prime and $\Delta(z_1, z_2)$ nonsingular square, so that $\ker(L(\sigma_1, \sigma_2)\Delta(\sigma_1, \sigma_2)) = \ker \Delta(\sigma_1, \sigma_2)$. From Lemma 4.3, it follows that $\det \Delta(z_1, z_2) \in \mathbb{R}[z_2, z_2^{-1}]$.

a2) ⇒ a1) Conversely, if $L(z_1, z_2)$ is right zero prime, $\mathfrak{B} = \ker\, (L(\sigma_1, \sigma_2)\, \Delta(\sigma_1, \sigma_2)) = \ker\, \Delta(\sigma_1, \sigma_2)$. By Lemma 4.3, the assumption on $\det \Delta(z_1, z_2)$ ensures the time-controllability of $\mathfrak{B}$.

b1) ⇒ b2) Assume that $\mathfrak{B}$ is zero-time-controllable. From Lemma 4.4 it follows that $H(z_1, z_2) = L(z_1, z_2)\; \Delta(z_1, z_2)$, for some right monomic and rFP $L(z_1, z_2)$ and some nonsingular square $\Delta(z_1, z_2)$. The square matrix $\Delta(z_1, z_2)$ can always be factorized as

$$\Delta(z_1, z_2) = D(z_1, z_2)\tilde{\Delta}(z_1, z_2),$$

where $D(z_1, z_2)$ is square monomic (possibly unimodular), while $\det \tilde{\Delta}(z_1, z_2)$ is coprime with z_1. Accordingly, $H(z_1, z_2)$ can be rewritten as

$$H(z_1, z_2) = \tilde{L}(z_1, z_2)\tilde{\Delta}(z_1, z_2),$$

where $\tilde{L}(z_1, z_2) := L(z_1, z_2)D(z_1, z_2)$ is right monomic and $\tilde{\Delta}(z_1, z_2)$ is nonsingular square, with determinant devoid of nontrivial monomic factors. Since $\ker\ \tilde{L}(\sigma_1, \sigma_2)$ is a nilpotent behavior, we let h be the minimal positive integer s.t. all the trajectories of $\ker\ \tilde{L}(\sigma_1, \sigma_2)$ are zero in $\mathcal{S}_{h\rightarrow}$. We also set $d_1 := \deg_{z_1} \tilde{\Delta}(z_1, z_2)$.

Let $\mathbf{w}$ be a trajectory in $\ker \tilde{\Delta}(\sigma_1, \sigma_2) \subseteq \mathfrak{B}$, and let N be a positive integer greater than or equal to h. By the zero-time-controllability of $\mathfrak{B}$, for a suitable $L \geq 0$, a trajectory $\bar{\mathbf{w}} \in \mathfrak{B}$ exists s.t. $\mathbf{w} = \bar{\mathbf{w}}$ in $\mathcal{S}_{\rightarrow N+d_1-1}$ and $\sigma_1^{N+d_1+L}\bar{\mathbf{w}} = 0$. Set $\bar{\mathbf{v}} := \tilde{\Delta}(\sigma_1, \sigma_2)\bar{\mathbf{w}}$ and $\mathbf{v} := \tilde{\Delta}(\sigma_1, \sigma_2)\mathbf{w} = 0$. By the previous assumptions these two trajectories coincide on $\mathcal{S}_{\rightarrow N-1}$. On the other hand, they both belong to $\ker\ \tilde{L}(\sigma_1, \sigma_2)$, and hence are both zero in $\mathcal{S}_{N\rightarrow}$ (as $N \geq h$). Therefore, $\bar{\mathbf{v}} = \mathbf{v} = 0$, which proves that $\bar{\mathbf{w}}$ belongs to $\ker \tilde{\Delta}(\sigma_1, \sigma_2)$, too. But this means that the trajectory $\mathbf{w} \in \ker \tilde{\Delta}(\sigma_1, \sigma_2)$ can be replaced by the trajectory $\bar{\mathbf{w}} \in \ker \tilde{\Delta}(\sigma_1, \sigma_2)$, thus proving zero-time-controllability of $\ker \tilde{\Delta}(\sigma_1, \sigma_2)$. From Lemma 4.3, $\det \tilde{\Delta}(z_1, z_2) = z_1^r p(z_2)$, for suitable $r \geq 0$ and $p(z_2) \in \mathbb{R}[z_2, z_2^{-1}]$. And since $\det \tilde{\Delta}(z_1, z_2)$ is coprime with z_1, r must be 0.

b2) $\Rightarrow$ b3) Suppose that $\mathfrak{B} = \ker(\tilde{L}(\sigma_1, \sigma_2)\tilde{\Delta}(\sigma_1, \sigma_2))$, with $\tilde{L}(z_1, z_2)$ right monomic and $\tilde{\Delta}(z_1, z_2)$ nonsingular square, with $\det \tilde{\Delta}(z_1, z_2) \in \mathbb{R}[z_2, z_2^{-1}]$. If $A(z_1, z_2)$ is s.t. $A(z_1, z_2)\tilde{L}(z_1, z_2) = z_1^h I_{\mathtt{w}}$, for some $h \in \mathbb{Z}_+$, we have

$$\begin{aligned} \mathfrak{B}_1 &:= \ker(\sigma_1^h \tilde{\Delta}(\sigma_1, \sigma_2)) = \ker(A(\sigma_1, \sigma_2)\tilde{L}(\sigma_1, \sigma_2)\tilde{\Delta}(\sigma_1, \sigma_2)) \\ &\supseteq \ker(\tilde{L}(\sigma_1, \sigma_2)\tilde{\Delta}(\sigma_1, \sigma_2)) = \mathfrak{B}, \end{aligned}$$

and from the relationship

$$\begin{bmatrix} -z_1^h I_{\mathtt{w}} & -\tilde{\Delta}(z_1, z_2) & z_1^h \tilde{\Delta}(z_1, z_2) \end{bmatrix} \begin{bmatrix} \tilde{\Delta}(z_1, z_2) & 0 \\ 0 & z_1^h I_{\mathtt{w}} \\ I_{\mathtt{w}} & I_{\mathtt{w}} \end{bmatrix} = 0$$

where the matrix on the left is an MLA of the matrix on the right, it follows (see Lemma A.1, in the Appendix) that

$$\mathfrak{B}_1 = \ker \tilde{\Delta}(\sigma_1, \sigma_2) + \ker(\sigma_1^h I_{\mathtt{w}}).$$

Consequently,

$$\begin{aligned} \mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B} &= (\ker \tilde{\Delta}(\sigma_1, \sigma_2) \cap \mathfrak{B}) + (\ker(\sigma_1^h I_{\mathtt{w}}) \cap \mathfrak{B}) \\ &= \ker \tilde{\Delta}(\sigma_1, \sigma_2) + \ker \begin{bmatrix} \sigma_1^h I_{\mathtt{w}} \\ \tilde{L}(\sigma_1, \sigma_2)\tilde{\Delta}(\sigma_1, \sigma_2) \end{bmatrix}, \end{aligned}$$

and the thesis follows, since $\ker \tilde{\Delta}(\sigma_1, \sigma_2)$ is a time-controllable behavior, while

$$\begin{bmatrix} z_1^h I_{\mathtt{w}} \\ \tilde{L}(z_1, z_2)\tilde{\Delta}(z_1, z_2) \end{bmatrix}$$

is rM.

b3) $\Rightarrow$ b1) If the decomposition (4.5) holds true, $\mathfrak{B}$ is trivially zero-time-controllable. □

5. (Zero-)time-controllability: the general case. Based on the analysis carried on in the previous section, we can now provide necessary and sufficient conditions for a general (not necessarily autonomous) behavior $\mathfrak{B}$ to be (zero-)time-controllable. The following characterization refers both to the algebraic properties of the TS-polynomial matrices involved in the kernel description of $\mathfrak{B}$ and to the possible decompositions of the behavior in terms of its controllable part $\mathfrak{B}_c$ and of a (zero-)time-controllable autonomous one. In fact, this result can be regarded as a special case of the classical theorem stating that every 2D behavior can be expressed as the sum of its controllable part and of an autonomous behavior [28, 31]. Indeed, it turns out that (zero-)time-controllable behaviors are those and those only for which a decomposition of this kind can be found, with the autonomous part enjoying the same (zero-)time-controllability property.

THEOREM 5.1. *Given a 2D behavior* $\mathfrak{B} = \ker H(\sigma_1,\sigma_2)$, *with* $H(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times \mathtt{w}}$, *the following facts are equivalent:*

i) $\mathfrak{B}$ *is time-controllable (zero-time-controllable);*

ii) $H(z_1,z_2)$ *can be expressed as* $H(z_1,z_2) = \tilde{L}(z_1,z_2)\tilde{\Delta}(z_1,z_2)R(z_1,z_2)$, *where* $\tilde{L}(z_1,z_2)$ *is right zero prime (right monomic),* $\tilde{\Delta}(z_1,z_2)$ *is nonsingular square with* $\det\tilde{\Delta}(z_1,z_2) \in \mathbb{R}[z_2,z_2^{-1}]$, *and* $R(z_1,z_2)$ *is left factor prime;*

iii) $\mathfrak{B} = \mathfrak{B}_c + \mathfrak{B}_{tc,a}$ ($\mathfrak{B} = \mathfrak{B}_c + \mathfrak{B}_{ztc,a}$), *where* $\mathfrak{B}_c$ *is the* controllable part *of* $\mathfrak{B}$, *while* $\mathfrak{B}_{tc,a}$ ($\mathfrak{B}_{ztc,a}$) *is a time-controllable square autonomous (zero-time-controllable autonomous) behavior.*

Proof. i) $\Rightarrow$ ii) Assume w.l.o.g. that $H(z_1,z_2)$ factorizes as $H(z_1,z_2) = L(z_1,z_2)\Delta(z_1,z_2)R(z_1,z_2)$ for some suitable TS-polynomial matrices, with $L(z_1,z_2)$ $p\times r$ right factor prime, $\Delta(z_1,z_2)$ $r\times r$ nonsingular square, and $R(z_1,z_2)$ $r\times \mathtt{w}$ left factor prime. By resorting to the same reasoning adopted in the proof of b1) $\Rightarrow$ b2) of Proposition 4.5, we factorize $\Delta(z_1,z_2)$ as $\Delta(z_1,z_2) = D(z_1,z_2)\tilde{\Delta}(z_1,z_2)$, with $D(z_1,z_2)$ square monomic and $\det\tilde{\Delta}(z_1,z_2)$ coprime with z_1, and get

$$H(z_1,z_2) = \tilde{L}(z_1,z_2)\tilde{\Delta}(z_1,z_2)R(z_1,z_2), \tag{5.1}$$

with $\tilde{L}(z_1,z_2) := L(z_1,z_2)D(z_1,z_2)$ of full column rank, $\tilde{\Delta}(z_1,z_2)$ nonsingular square with $\det\tilde{\Delta}(z_1,z_2)$ coprime with z_1, and $R(z_1,z_2)$ left factor prime. By Proposition 4.5, if we prove that the time-controllability (zero-time-controllability) of $\mathfrak{B}$ implies that of $\ker(\tilde{L}(\sigma_1,\sigma_2)\tilde{\Delta}(\sigma_1,\sigma_2))$, we can deduce that $\tilde{L}(z_1,z_2)$ and $\tilde{\Delta}(z_1,z_2)$ (by the essential uniqueness of the decomposition (5.1)) have the desired properties.

Let us prove this fact in the time-controllable case. Let $\mathbf{v}$ and $\mathbf{v}^*$ be two trajectories in $\ker(\tilde{L}(\sigma_1,\sigma_2)\tilde{\Delta}(\sigma_1,\sigma_2))$. By the surjectivity of $R(\sigma_1,\sigma_2)$, we can find two trajectories, $\mathbf{w}$ and $\mathbf{w}^*$ (clearly belonging to $\mathfrak{B}$) s.t. $\mathbf{v} = R(\sigma_1,\sigma_2)\mathbf{w}$ and $\mathbf{v}^* = R(\sigma_1,\sigma_2)\mathbf{w}^*$. By the time-controllability of $\mathfrak{B}$, some $\tilde{L} \geq 0$ exists s.t. for every $\tilde{N} > 0$ (and hence, in particular, for sufficiently large $\tilde{N}$) a trajectory $\bar{\mathbf{w}} \in \mathfrak{B}$ can be found, satisfying $\bar{\mathbf{w}}|_{\mathcal{S}_{\to N-1}} = \mathbf{w}|_{\mathcal{S}_{\to N-1}}$ and $\sigma_1^{\tilde{N}+\tilde{L}}\bar{\mathbf{w}} = \mathbf{w}^*$. It is easy to verify that $\bar{\mathbf{v}} := R(\sigma_1,\sigma_2)\bar{\mathbf{w}}$ is the trajectory of $\ker(\tilde{L}(\sigma_1,\sigma_2)\tilde{\Delta}(\sigma_1,\sigma_2))$ we are searching for, as it patches the initial portion of $\mathbf{v}$ (in $\mathcal{S}_{\to N-1}$, where $N = \tilde{N} - \deg_{z_1} R(z_1,z_2)$) with a suitable shifted version of $\mathbf{v}^*$ (shifted by $N+L$ steps, where $N+L = \tilde{N}+\tilde{L}$). This proves the time-controllability of $\ker(\tilde{L}(\sigma_1,\sigma_2)\tilde{\Delta}(\sigma_1,\sigma_2))$, and therefore that $\det\tilde{\Delta}(z_1,z_2) \in \mathbb{R}[z_2,z_2^{-1}]$ and $\tilde{L}(z_1,z_2)$ is right zero prime.
The proof for zero-time-controllability follows the same lines, upon replacing the trajectories $\mathbf{v}^*$ and $\mathbf{w}^*$ with the zero ones.

ii) $\Rightarrow$ iii) We assume, first, that $\tilde{L}(z_1,z_2)$ is right zero prime, $\tilde{\Delta}(z_1,z_2)$ nonsingular square with $\det\tilde{\Delta}(z_1,z_2) \in \mathbb{R}[z_2,z_2^{-1}]$, and $R(z_1,z_2)$ is left factor prime. Consequently,

$\mathfrak{B}$ can be equivalently described as $\mathfrak{B} = \ker(\tilde{\Delta}(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2))$, where $\tilde{\Delta}(z_1, z_2)$ has size $r \times r$, while $R(z_1, z_2)$ has size $r \times \mathtt{w}$. By the left factor primeness of $R(z_1, z_2)$, there exists a TS-polynomial matrix $C(z_1, z_2)$ such that

$$\begin{bmatrix} R(z_1, z_2) \\ C(z_1, z_2) \end{bmatrix}$$

is a nonsingular square matrix with determinant in $\mathbb{R}[z_2, z_2^{-1}]$. Set

$$\Delta_{sq}(z_1, z_2) := \begin{bmatrix} \tilde{\Delta}(z_1, z_2) & 0 \\ 0 & I_{\mathtt{w}-\mathtt{r}} \end{bmatrix} \begin{bmatrix} R(z_1, z_2) \\ C(z_1, z_2) \end{bmatrix}.$$

Obviously, $\Delta_{sq}(z_1, z_2)$ is nonsingular square and its determinant belongs to $\mathbb{R}[z_2, z_2^{-1}]$. Consequently, $\ker\Delta_{sq}(\sigma_1, \sigma_2)$ is a time-controllable (square autonomous) behavior. We aim to show that $\mathfrak{B}_c + \ker\Delta_{sq}(\sigma_1, \sigma_2) = \mathfrak{B}$. To this end (see Lemma A.1, in the Appendix), it is sufficient to show that there exists an MLA

$$T(z_1, z_2) = [\, X(z_1, z_2) \quad Y(z_1, z_2) \quad Z(z_1, z_2) \,]$$

of

$$\left[\begin{array}{c|c} R(z_1, z_2) & 0 \\ \hline 0 & \Delta_{sq}(z_1, z_2) \\ \hline I_{\mathtt{w}} & I_{\mathtt{w}} \end{array}\right] \in \mathbb{R}[z_1, z_2]^{(r+2\mathtt{w}) \times 2\mathtt{w}},$$

such that $\ker Z(\sigma_1, \sigma_2) = \mathfrak{B}$. Indeed, by choosing

$$T(z_1, z_2) = [\, \tilde{\Delta}(z_1, z_2) \quad | \quad [\, I_r \quad 0 \,] \quad | \quad -\tilde{\Delta}(z_1, z_2)R(z_1, z_2) \,]$$

we get the matrix we were looking for.

Suppose, now, that $\mathfrak{B} = \ker(\tilde{L}(\sigma_1, \sigma_2)\tilde{\Delta}(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2))$, with $\tilde{L}(z_1, z_2)$ right monomic, $R(z_1, z_2)$ left factor prime, $\tilde{\Delta}(z_1, z_2)$ nonsingular square and $\det \tilde{\Delta}(z_1, z_2) \in \mathbb{R}[z_2, z_2^{-1}]$. Let $A(z_1, z_2)$ be a matrix s.t. $A(z_1, z_2)\ \tilde{L}(z_1, z_2) = z_1^h I_r$, for a suitable $h \geq 0$. By using a reasoning similar to the one adopted in the proof of Proposition 4.5, we first define $\mathfrak{B}_1 := \ker(\sigma_1^h \tilde{\Delta}(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2))$, then decompose it as

$$\mathfrak{B}_1 = \ker(\tilde{\Delta}(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2)) + \ker(\sigma_1^h I_{\mathtt{w}})$$

and then deduce that

$$\mathfrak{B} = \ker(\tilde{\Delta}(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2)) + \ker \begin{bmatrix} \sigma_1^h I_{\mathtt{w}} \\ \tilde{L}(\sigma_1, \sigma_2)\tilde{\Delta}(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2) \end{bmatrix}.$$

By the previous part of the proof, the behavior $\ker(\tilde{\Delta}(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2))$ can be expressed as

$$\ker(\tilde{\Delta}(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2)) = \mathfrak{B}_c + \mathfrak{B}_{tc,a}.$$

On the other hand the full column rank matrix

$$\begin{bmatrix} z_1^h I_{\mathtt{w}} \\ \tilde{L}(z_1, z_2)\tilde{\Delta}(z_1, z_2)R(z_1, z_2) \end{bmatrix},$$

is clearly right monomic and hence its kernel is a nilpotent behavior. So, we have proved that

$$\mathfrak{B} = (\mathfrak{B}_c + \mathfrak{B}_{tc,a}) + \mathfrak{B}_{nil}.$$

By Proposition 4.5, $\mathfrak{B}_{tc,a} + \mathfrak{B}_{nil}$ is zero-time-controllable, and this completes the proof.

iii) $\Rightarrow$ i) Obvious: it suffices to perform separate reasonings on the trajectories $\mathbf{w}_c$ and $\mathbf{w}_{tc,a}$ (or $\mathbf{w}_{ztc,a}$) of the two summand behaviors. □

Remark 5.2. *In [21] it was shown that a behavior $\mathfrak{B}$ is set-controllable to a sub-behavior $\mathfrak{B}' \subset \mathfrak{B}$ if and only if $\mathfrak{B} = \mathfrak{B}_c + \mathfrak{B}'$. It is worthwhile to underline that the equivalence of zero-time-controllability with the behavior decomposition*

$$\mathfrak{B} = \mathfrak{B}_c + (\mathfrak{B}_{tc,a} + \mathfrak{B}_{nil}) = \mathfrak{B}_{tc} + \mathfrak{B}_{nil},$$

proved in the previous theorem, represents the perfect analogue of that result in our setup. Indeed, the target of controlling the behavior trajectories to zero in a suitable half-plane $\mathcal{S}_{N+L\rightarrow}$ can be interpreted as set-controllability to a nilpotent behavior $\mathfrak{B}'$. On the other hand, due to the fact that we work on $\mathbb{Z}_+ \times \mathbb{Z}$ and because of the privileged role of the time coordinate, controllability property in the present context is naturally replaced by time-controllability.

6. Dead-beat controllers. By a *controller* $\mathcal{C}$ of a given 2D behavior $\mathfrak{B} = \ker\, H(\sigma_1, \sigma_2)$, with $H(z_1,\ z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, we mean a system (a set of 2D difference equations) that constrains the system trajectories, and hence is described by a difference equation of the following type

$$C(\sigma_1, \sigma_2)\mathbf{w}(h, k) = 0, \qquad \forall\ (h, k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{6.1}$$

for a suitable TS-polynomial matrix $C(z_1, z_2)$. The overall *controlled behavior*, i.e. the behavior of the system obtained by *full interconnection* of the behavior $\mathfrak{B}$ and the controller (6.1), is described by

$$\begin{bmatrix} H(\sigma_1, \sigma_2) \\ C(\sigma_1, \sigma_2) \end{bmatrix} \mathbf{w}(h, k) = 0, \qquad \forall\ (h, k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{6.2}$$

it is denoted by $\mathcal{K}$, and it is clearly the intersection of $\mathfrak{B}$ and $\mathcal{C} := \ker C(\sigma_1, \sigma_2)$.

The target of the dead-beat control problem is to design, if possible, a controller $\mathcal{C}$ s.t. the controlled behavior $\mathcal{K}$ is an autonomous nilpotent (with respect to $\mathbb{Z}_+ \times \mathbb{Z}$) behavior.

Definition 6.1. *Given a 2D behavior $\mathfrak{B} = \ker\, H(\sigma_1, \sigma_2)$, with $H(z_1,\ z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, a controller $\mathcal{C}$ is said to be a* dead-beat controller *(DBC) for the system if there exists $N \in \mathbb{Z}_+$ s.t. all the trajectories of the resulting controlled behavior have support included in the vertical strip $\mathcal{S}_{\rightarrow N-1}$, which amounts to saying that $\mathcal{K}$ is nilpotent.*

By referring to the description (6.2) of the controlled behavior $\mathcal{K}$, a characterization of the DBC's follows immediately as a corollary of Proposition 3.6.

Corollary 6.2. *Given a 2D behavior $\mathfrak{B} = \ker\, H(\sigma_1, \sigma_2)$, with $H(z_1,\ z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, a TS-polynomial matrix $C(z_1, z_2)$, defines a DBC (6.1) for the system if and only if*

$$\Gamma(z_1, z_2) := \begin{bmatrix} H(z_1, z_2) \\ C(z_1, z_2) \end{bmatrix} \tag{6.3}$$

is right monomic.

As it is well known and as it happens in the 1D case (see [3] and [29]), every 2D system admits a dead-beat controller. In fact, the goal of forcing to zero all the trajectories of $\mathcal{K}$ in a finite number of steps, is always achievable, independently of the behavior properties: it is sufficient to choose, for instance, the controller $C(z_1, z_2) = I_{\mathtt{w}}$ to ensure that $\mathcal{K}$ is the zero behavior, and hence, in particular, it is nilpotent.

For this reason in [3] the concept of *admissible* DBC has been introduced, to identify what properties a system controlled by a DBC should reasonably exhibit. When dealing with 1D state-space models, if we apply the DBC starting at some time $N > 0$, the trajectory of the controlled systems coincides with the original state trajectory in the initial window $[0, N-1]$ and then goes to zero in a finite number of steps. So, when moving to the 1D behavioral setting, what we require is that the DBC performs its task without constraining the initial portion of the trajectories in $\mathfrak{B}$. So, if it starts working at time $t = M$, it does not affect the samples of the trajectory in some initial window $[0, N-1]$ provided that $M - N$ is large enough. Of course, a similar perspective applies to 2D behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$, provided that we replace the time t with the vertical lines $\mathcal{S}_t$, $t \in \mathbb{Z}_+$.

By assuming this perspective, we want to introduce the concept of *admissible 2D DBC*, which establishes a bridge between control regarded as behavior interconnection and control regarded as steering one trajectory to another desired one. To this end, we need some mathematical preliminaries. Given a controller $\mathcal{C}$, described by the difference equation (6.1), we introduce the *delayed controllers*[10] $\mathcal{C}_i, i \in \mathbb{Z}_+$, described by the difference equation

$$\sigma_1^i C(\sigma_1, \sigma_2)\mathbf{w}(h, k) = 0, \qquad (h, k) \in \mathbb{Z}_+ \times \mathbb{Z}.$$

If we denote by $\mathcal{K}_i$ the controlled behavior obtained corresponding to $\mathcal{C}_i$, then

$$\mathcal{K}_i = \ker \begin{bmatrix} H(\sigma_1, \sigma_2) \\ \sigma_1^i C(\sigma_1, \sigma_2) \end{bmatrix}. \tag{6.4}$$

Clearly, $\mathcal{C} = \mathcal{C}_0$ and $\mathcal{K} = \mathcal{K}_0$.

The controller $\mathcal{C}_i$ acts on the trajectories $\mathbf{w}$ of $\mathfrak{B}$ as the original controller $\mathcal{C}$ does, but instead of performing the control action on the whole $\mathbb{Z}_+ \times \mathbb{Z}$, it acts on $\mathcal{S}_{i\rightarrow}$. Note, however, that this does not mean that the controlled trajectories are unconstrained on the separation sets preceding $\mathcal{S}_i$. It is easily seen that if $\mathcal{C}$ is a DBC for the given system, then every $\mathcal{C}_i$ is a DBC (see Lemma A.2 in the Appendix).

Now that we have defined the notion of delayed controller, we can formalize the concept of admissible DBC that we have previously described in rough terms. A DBC $\mathcal{C}$ for $\mathfrak{B}$ is admissible if, when we apply its delayed versions $\mathcal{C}_i$, for sufficiently high values of i, we can drive to zero any behavior trajectory, meanwhile preserving the values it takes on a sufficiently large initial strip.

DEFINITION 6.3. *Given a 2D behavior $\mathfrak{B}$, a dead-beat controller $\mathcal{C}$ described as in (6.1) is said to be* admissible *if there exists $L \in \mathbb{Z}_+$ s.t. for every $\mathbf{w} \in \mathfrak{B}$ and every $N \in \mathbb{N}$, there exists $\bar{\mathbf{w}} \in \mathcal{K}_{L+N}$, the nilpotent behavior obtained corresponding to the controller $\mathcal{C}_{L+N}$, s.t. $\bar{\mathbf{w}}(h, k)|_{\mathcal{S}_{\rightarrow N-1}} = \mathbf{w}(h, k)|_{\mathcal{S}_{\rightarrow N-1}}$.*

[10] The expression "delayed" makes sense in this context, as we give to the first independent variable the interpretation of time variable.

We are in a position to relate zero-time-controllability of $\mathbf{w}$ to the existence of an admissible DBC.

THEOREM 6.4. *A 2D behavior $\mathfrak{B} = \ker H(\sigma_1,\sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times \mathtt{w}}$, has an admissible DBC if and only if $\mathfrak{B}$ is zero-time-controllable. If this is the case, then every DBC is admissible.*

Proof. Assume, first, that the system has an admissible DBC $\mathcal{C}$, described by the TS-polynomial matrix $C(z_1,z_2)$, and, hence

$$\mathcal{K} = \ker \begin{bmatrix} H(\sigma_1,\sigma_2) \\ C(\sigma_1,\sigma_2) \end{bmatrix}$$

is a nilpotent behavior. This implies that there exists $M \in \mathbb{Z}_+$ s.t. all trajectories in $\mathcal{K}$ are zero on $\mathcal{S}_{M\rightarrow}$. On the other hand, for every $i \in \mathbb{Z}_+$

$$\mathcal{K}_i \subseteq \ker \left(\sigma_1^i \begin{bmatrix} H(\sigma_1,\sigma_2) \\ C(\sigma_1,\sigma_2) \end{bmatrix} \right),$$

and the behavior on the right hand-side is nilpotent, with trajectories which are identically zero at least on $\mathcal{S}_{M+i\rightarrow}$. So, also the trajectories of $\mathcal{K}_i$ have finite support included in $\mathcal{S}_{\rightarrow M+i-1}$, and this is true for every $i \in \mathbb{Z}_+$.

Since $\mathcal{C}$ is an admissible DBC, there exists $L \in \mathbb{Z}_+$ s.t. for every $N \in \mathbb{N}$ and every $\mathbf{w} \in \mathfrak{B}$ a trajectory $\bar{\mathbf{w}} \in \mathcal{K}_{L+N} \subseteq \mathfrak{B}$ can be found, coinciding with $\mathbf{w}$ in $\mathcal{S}_{\rightarrow N-1}$. Such a trajectory $\bar{\mathbf{w}}$ is surely zero on $\mathcal{S}_{L+N+M\rightarrow}$. So, we have proved that there exists $L^* \in \mathbb{Z}_+$, specifically $L^* := M+L$, s.t. for every $\mathbf{w} \in \mathfrak{B}$ there exists a trajectory $\bar{\mathbf{w}} \in \mathcal{K}_{L^*+N} \subseteq \mathfrak{B}$ coinciding with $\mathbf{w}$ in $\mathcal{S}_{\rightarrow N-1}$ and zero in $\mathcal{S}_{L^*+N\rightarrow}$. This proves that $\mathfrak{B}$ is zero-time-controllable.

We have already pointed out that a DBC always exists, independently of the behavior properties. We want to show that when $\mathfrak{B}$ is zero-time-controllable, every DBC is admissible (this, obviously, implies that there exists an admissible one). Let $C(z_1,z_2)$ be a TS-polynomial matrix that describes a DBC. We have only to verify that it is admissible. By the zero-time-controllability property, there exists a nonnegative integer L s.t. for every $N \in \mathbb{N}$ and every $\mathbf{w} \in \mathfrak{B}$, one can find $\bar{\mathbf{w}} \in \mathfrak{B}$ satisfying

$$\bar{\mathbf{w}}|_{\mathcal{S}_{\rightarrow N-1}} = \mathbf{w}|_{\mathcal{S}_{\rightarrow N-1}}, \qquad \sigma_1^{N+L}\bar{\mathbf{w}} = 0. \tag{6.5}$$

We want to show that this same nonnegative integer L makes the definition of admissible DBC satisfied. To this end we have to show that for every $N \in \mathbb{N}$ and every $\mathbf{w} \in \mathfrak{B}$, there exists $\bar{\mathbf{w}} \in \mathcal{K}_{L+N}$ coinciding with $\mathbf{w}$ in $\mathcal{S}_{\rightarrow N-1}$. As

$$\mathcal{K}_{L+N} = \ker \begin{bmatrix} H(\sigma_1,\sigma_2) \\ \sigma_1^{N+L} C(\sigma_1,\sigma_2) \end{bmatrix},$$

it is immediately seen that the two conditions in (6.5), ensure that $\bar{\mathbf{w}} \in \mathfrak{B}$ belongs both to $\mathfrak{B} = \ker H(\sigma_1,\sigma_2)$ and to $\ker(\sigma_1^{L+N} C(\sigma_1,\sigma_2))$. Therefore $\bar{\mathbf{w}} \in \mathcal{K}_{L+N}$ and this makes the definition of admissible DBC satisfied. □

EXAMPLE 1. *(revisited)* We have already seen that the trajectories of $\mathfrak{B} = \ker \begin{bmatrix} 1-\sigma_2 & 1-\sigma_2 \end{bmatrix}$ have the property that their restrictions to vertical lines are independent one from the others. In particular, for any $N \in \mathbb{N}$ and any $\mathbf{w} \in \mathfrak{B}$ the "truncated" trajectory

$$\begin{aligned} \bar{\mathbf{w}}_N(h,k) &= \mathbf{w}(h,k), && \forall\ (h,k) \in \mathcal{S}_{\rightarrow N-1}, \\ \bar{\mathbf{w}}_N(h,k) &= 0, && \forall\ (h,k) \in \mathcal{S}_{N\rightarrow}, \end{aligned}$$

belongs to $\mathfrak{B}$, too. Indeed, all such truncated trajectories belong to the controlled behavior $\mathcal{K}_N$, obtained corresponding to the (admissible) controller $\mathcal{C}_N$, where $\mathcal{C} = \ker I_{\mathtt{w}}$. So, the definition of admissible DBC holds for $L = 0$.

On the other hand, when $\mathfrak{B}$ is not zero-time-controllable, every delayed DBC necessarily introduces additional constraints on the initial portions of those trajectories of $\mathfrak{B}$ it can drive to zero, and hence it violates the concept of admissibility.

Example 2. Consider the behavior $\mathcal{B} = \ker [\, 1-\sigma_1 \quad 1-\sigma_1 \,]$, obtained by replacing σ_2 with σ_1 in the behavioral description of Example 1. $\mathcal{B}$ is not zero-time-controllable, since $[\, 1-z_1 \quad 1-z_1 \,] = (1-z_1)\,[\, 1 \quad 1 \,]$, and $1-z_1$ cannot be expressed as $z_1^r \delta(z_2)$, for some $r \in \mathbb{Z}_+$ and some $\delta(z_2) \in \mathbb{R}[z_2, z_2^{-1}]$. We aim at showing that any DBC $\mathcal{C}$ for $\mathcal{B}$ is not admissible. Let $\mathcal{C} = \ker C(\sigma_1, \sigma_2)$ be a given DBC for $\mathcal{B}$, $\mathcal{K}$ the corresponding (nilpotent) controlled behavior, and denote by $M \in \mathbb{Z}_+$ the smallest nonnegative integer such that $\mathbf{w} \in \mathcal{K}$ implies $\mathbf{w}|_{\mathcal{S}_{M\rightarrow}} = 0$. For any $N \in \mathbb{N}$ and $L \in \mathbb{Z}_+$, the controlled behavior $\mathcal{K}_{N+L}$ corresponding to the delayed DBC $\mathcal{C}_{N+L}$ satisfies

$$\begin{aligned} \mathbf{w} \in \mathcal{K}_{N+L} \quad &\Rightarrow \quad \mathbf{w}|_{\mathcal{S}_{M+N+L\rightarrow}} = 0 \\ &\Rightarrow \quad w_1(M+N+L,k) + w_2(M+N+L,k) = 0, \forall\, k \in \mathbb{Z}. \end{aligned} \tag{6.6}$$

The trajectories of $\mathcal{B}$ are those (and those only) satisfying

$$w_1(h,k) + w_2(h,k) = w_1(0,k) + w_2(0,k),\ \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

so, in particular, they satisfy

$$w_1(M+N+L,k) + w_2(M+N+L,k) = w_1(0,k) + w_2(0,k), \forall (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{6.7}$$

and since $\mathcal{K}_{M+N+L} \subseteq \mathcal{B}$, from (6.6) and (6.7) it follows

$$\mathbf{w} \in \mathcal{K}_{N+L} \ \Rightarrow\ w_1(0,k) + w_2(0,k) = 0,\ \forall\ k \in \mathbb{Z}.$$

Consequently, if $\mathbf{w} \in \mathcal{B}$ is any trajectory satisfying $w_1(0,k) + w_2(0,k) \neq 0$ for some $k \in \mathbb{Z}$, it cannot be driven to zero by any delayed DBC $\mathcal{C}_{N+L}$. Equivalently, any delayed DBC constrains the controlled trajectories to satisfy $w_1(0,k) + w_2(0,k) = 0$ for all $k \in \mathbb{Z}$, and therefore constrains the initial portion of $\mathbf{w}$ to satisfy additional constraints w.r.t. the constraints imposed by $\mathcal{B}$.

In the literature about behaviors, particular attention has been devoted to the so-called *regular interconnections* [29]. The idea underlying the concept of regular interconnection is rather simple: the connection of a plant and a controller is a regular one if the controller laws are not redundant with respect to the system laws. In the specific context of multidimensional behaviors, the possibility of obtaining certain controlled behaviors by means of a regular (possibly extended) interconnection has been explored, for instance, in [10, 11, 17, 19, 21]. We want now to investigate under what conditions a (zero-time-controllable) 2D behavior admits a "regular" dead-beat controller.

Definition 6.5. *Given a 2D behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p \times \mathtt{w}}$, a DBC $\mathcal{C}$ described as in (6.1) is* regular *if*

$$\text{rank} \begin{bmatrix} H(z_1,z_2) \\ C(z_1,z_2) \end{bmatrix} = \text{rank} H(z_1,z_2) + \text{rank} C(z_1,z_2).$$

It turns out (see next theorem) that a necessary condition for a regular DBC to exist is that $\mathfrak{B}$ is zero-time-controllable. However, the converse is not true, and further conditions must be imposed on the polynomial matrices that describe the behavior $\mathfrak{B}$ in order to ensure that such a regular DBC exists.

THEOREM 6.6. *Consider a 2D behavior* $\mathfrak{B} = \ker H(\sigma_1,\sigma_2)$, *with* $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, *and assume w.l.o.g. that* $H(z_1,z_2)$ *factorizes as* $H(z_1,z_2) = L(z_1,z_2)$ $\Delta(z_1,z_2)R(z_1,z_2)$ *for some suitable TS-polynomial matrices, with* $L(z_1,z_2)$ $p\times r$ *right factor prime,* $\Delta(z_1,z_2)$ $r\times r$ *nonsingular square, and* $R(z_1,z_2)$ $r\times \mathtt{w}$ *left factor prime,* r *being the rank of* $H(z_1,z_2)$. *The following facts are equivalent:*

i) the behavior $\mathfrak{B}$ *admits a regular DBC;*

ii) $L(z_1,z_2)$ *is rM,* $\Delta(z_1,z_2)$ *is square monomic, and* $R(z_1,z_2)$ *is lM.*

Proof. i) $\Rightarrow$ ii) Let $\mathcal{C} = \ker C(\sigma_1,\sigma_2)$ be a regular DBC, and assume w.l.o.g. that $C(z_1,z_2) = L_c(z_1,z_2)\Delta_c(z_1,z_2)R_c(z_1,z_2)$, for some suitable TS-polynomial matrices, with $L_c(z_1,z_2)$ $p_c\times r_c$ right factor prime, $\Delta_c(z_1,z_2)$ $r_c\times r_c$ nonsingular square, and $R_c(z_1,z_2)$ $r_c\times \mathtt{w}$ left factor prime, r_c being the rank of $C(z_1,z_2)$. By the definition of regular interconnection, and the fact that the controlled behavior is nilpotent, and hence in particular autonomous, it must be $r+r_c = \mathtt{w}$. Also, as the controlled behavior is nilpotent, the TS-polynomial matrix

$$\begin{aligned} R_M(z_1,z_2) &:= \begin{bmatrix} L(z_1,z_2)\Delta(z_1,z_2)R(z_1,z_2) \\ L_c(z_1,z_2)\Delta_c(z_1,z_2)R_c(z_1,z_2)\end{bmatrix} \\ &= \begin{bmatrix} L(z_1,z_2) & 0 \\ 0 & L_c(z_1,z_2)\end{bmatrix}\begin{bmatrix} \Delta(z_1,z_2) & 0 \\ 0 & \Delta_c(z_1,z_2)\end{bmatrix}\begin{bmatrix} R(z_1,z_2) \\ R_c(z_1,z_2)\end{bmatrix}\end{aligned}$$

must be rM. This implies that the first of the three matrices appearing in the above factorization must be rM, while the two square factors must be square monomic. This, in turn, implies that $L(z_1,z_2)$ must be rM, $\Delta(z_1,z_2)$ square monomic, while $R(z_1,z_2)$ must be lM, since it can be row-bordered up to a square monomic matrix.

ii) $\Rightarrow$ i) Conversely, assume that the TS-polynomial matrices involved in the factorization of the TS-polynomial matrix $H(z_1,z_2)$ satisfy the algebraic properties given in ii). Then $R(z_1,z_2)$ can be row-bordered up to a square monomic matrix, namely there exists a TS-polynomial matrix $C(z_1,z_2)$, of size $(\mathtt{w}-\mathtt{r})\times \mathtt{w}$, s.t.

$$\begin{bmatrix} R(z_1,z_2) \\ C(z_1,z_2)\end{bmatrix}$$

is a square monomic $\mathtt{w}\times\mathtt{w}$ matrix. Then it is easily seen that

$$\begin{aligned} R_M(z_1,z_2) &:= \begin{bmatrix} L(z_1,z_2)\Delta(z_1,z_2)R(z_1,z_2) \\ C(z_1,z_2)\end{bmatrix} \\ &= \begin{bmatrix} L(z_1,z_2) & 0 \\ 0 & I_{\mathtt{w}-r}\end{bmatrix}\begin{bmatrix} \Delta(z_1,z_2) & 0 \\ 0 & I_{\mathtt{w}-r}\end{bmatrix}\begin{bmatrix} R(z_1,z_2) \\ C(z_1,z_2)\end{bmatrix}\end{aligned}$$

is right monomic, and hence $\mathcal{C} = \ker C(\sigma_1,\sigma_2)$ is a DBC. On the other hand, as $\mathrm{rank} C(z_1,z_2) = \mathtt{w} - \mathrm{rank} H(z_1,z_2)$, it is also a regular one. □

REMARK 6.7. *It is worthwhile to compare the previous result with an important result derived in [21]. In Theorem 5.7 of [21] it has been proved that a behavior* $\mathfrak{B}$ *is set-controllable to some subbehavior* $\mathfrak{B}'$ *if and only if* $\mathfrak{B}'$ *is achievable from* $\mathfrak{B}$ *by regular extended (or latent) interconnection. Extended and latent interconnections are* *connections that resort, both for the original behavior and for the controller behavior, to latent variable representations and hence interconnect behaviors having $\mathfrak{B}$ and $\mathcal{C}$ as projections on the external variables. By adapting this result to our set-up, we can say that a 2D behavior is zero-time-controllable if and only if there exists a regular extended (or latent) interconnection that makes the resulting (external) controlled behavior nilpotent. A similar result has been derived in [3] for 1D behaviors, as indeed the existence of a regular DBC, obtained by connecting a behavior and a controller both described by means of a latent variable representation, is indeed equivalent to zero-time-controllability. However, as the previous result clearly shows, this is no longer true if we consider regular full interconnections, and zero-time-controllability is only a necessary condition for the existence of a regular DBC.*

To conclude, we want to focus our attention on a special class of DBC's, namely DBC's that make the resulting controlled behavior not only nilpotent but also strongly autonomous. As previously recalled, there has been a number of significant contributions focusing on the design of controllers in such a way that the resulting controlled behavior is not only autonomous but strongly autonomous (and with additional desired properties like stability) (see [10, 19, 18]). As we will see, as far as we are interested in obtaining a strongly autonomous and nilpotent behavior by means of an admissible DBC, zero-time-controllability proves to be a sufficiently strong property, while if we aim at obtaining such a controlled behavior by means of a regular interconnection, then we have to further constrain the algebraic properties of the behavior $\mathfrak{B}$.

COROLLARY 6.8. *Consider a 2D behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, and assume w.l.o.g. that $H(z_1, z_2)$ factorizes as $H(z_1, z_2) = L(z_1, z_2)$ $\Delta(z_1, z_2)R(z_1, z_2)$ for some suitable TS-polynomial matrices, with $L(z_1, z_2)$ $p \times r$ right factor prime, $\Delta(z_1, z_2)$ $r \times r$ nonsingular square, and $R(z_1, z_2)$ $r \times \mathtt{w}$ left factor prime, r being the rank of $H(z_1, z_2)$. Then:*

- *i) there exists an admissible DBC that makes the resulting controlled behavior strongly autonomous and nilpotent if and only if $\mathfrak{B}$ is zero-time-controllable;*
- *ii) there exists a regular DBC that makes the resulting controlled behavior strongly autonomous and nilpotent if and only if $L(z_1, z_2)$ is rM, $\Delta(z_1, z_2)$ is unimodular, and $R(z_1, z_2)$ is lZP.*

Proof. i) Assume that $\mathfrak{B}$ is zero-time-controllable. Since we know that every DBC for $\mathfrak{B}$ is admissible, it is sufficient to prove that there exists a DBC that makes

$$R_M(z_1, z_2) := \begin{bmatrix} H(z_1, z_2) \\ C(z_1, z_2) \end{bmatrix}$$

rFP. By Theorem 5.1, we know that $H(z_1, z_2)$ can be expressed as $H(z_1, z_2) = \tilde{L}(z_1, z_2)\tilde{\Delta}(z_1, z_2)R(z_1, z_2)$, where $\tilde{L}(z_1, z_2)$ is right monomic, $\tilde{\Delta}(z_1, z_2)$ is nonsingular square with $\det \tilde{\Delta}(z_1, z_2) \in \mathbb{R}[z_2, z_2^{-1}]$, and $R(z_1, z_2)$ is left factor prime. Factorize $\tilde{L}(z_1, z_2)$ as $\tilde{L}(z_1, z_2) = L(z_1, z_2)\ \Delta_1(z_1, z_2)$, with $L(z_1, z_2)$ rFP and rM, while $\Delta_1(z_1, z_2)$ is square monomic. So, if we choose, $C(z_1, z_2) = L(z_1, z_2)$, we get a DBC that makes the resulting matrix

$$R_M(z_1, z_2) = \begin{bmatrix} \tilde{L}(z_1, z_2)\tilde{\Delta}(z_1, z_2)R(z_1, z_2) \\ L(z_1, z_2) \end{bmatrix}$$

both rFP and rM. The converse is obvious.

ii) The proof follows the same lines of the proof of Theorem 6.6, and we omit it for the sake of brevity. □

Example 3. Consider the controllable behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2) := \ker \left[\, 1 - \sigma_1 \quad 1 - \sigma_2 \,\right]$. Clearly any DBC is admissible for it. However, regular DBC's are not available, as $\mathcal{V}_{TS}(H)$ is not empty, even if $\mathfrak{B}$ is not endowed with any autonomous part.

Example 4. Consider the zero-time-controllable behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2) := \ker \left[\, 1 + \sigma_2 \quad \sigma_1 \,\right]$. The controller defined by $\mathcal{C} := \ker \left[\, 1 \quad 0 \,\right]$ is a regular DBC for $\mathfrak{B}$. However, a regular DBC that makes the resulting system strongly autonomous does not exist, since $H(z_1, z_2) = R(z_1, z_2)$ is not lZP.

Remark 6.9. *It is worthwhile to enlighten some interesting connections between the previous corollary and some results reported in [10, 19]. First of all, point ii) states, in particular, that if a DBC can be found that makes the resulting controlled behavior both nilpotent and strongly (i.e. finite-dimensional) autonomous, then $\mathfrak{B}_c$, the controllable part of $\mathfrak{B}$, must be rectifiable, since $\mathfrak{B}_c = \ker R(\sigma_1, \sigma_2)$ and $R(z_1, z_2)$ is a lZP TS-polynomial matrix. This necessary condition has already been derived in [10] for two-dimensional behaviors (see Theorem 16) and in [19] for nD behaviors (see Theorem 14). Indeed, it was proved that if an autonomous behavior with autonomy degree greater than 1 can be obtained by means of a regular interconnection, then the controllable part of the behavior must be rectifiable.*

On the other hand, an additional necessary and sufficient condition for the existence of a DBC that makes the resulting controlled behavior nilpotent and strongly autonomous could be easily found by exploiting the factorization provided in ii) of the previous corollary. Indeed, it is easy to show, by resorting to the same techniques adopted within the proof of Theorem 5.1, that such a DBC exists for $\mathfrak{B}$ if and only if $\mathfrak{B}_c$ is rectifiable and there exists a nilpotent and strongly autonomous behavior $\mathfrak{B}_{nil,sa}$ such that

$$\mathfrak{B} = \mathfrak{B}_c \oplus \mathfrak{B}_{nil,sa}.$$

Quite interestingly, these necessary and sufficient conditions for the existence of a DBC that makes the resulting nilpotent behavior strongly autonomous are completely analogous to the ones obtained in [10] (see Theorem 22) and in [19] (see Theorem 18) for the stabilization problem of 2D and nD behaviors, respectively.

7. 2D Fornasini-Marchesini state-space models. To conclude, we want to provide a quick overlook of how the controllability properties investigated in this paper adapt to the case of 2D state-space systems, described by the Fornasini-Marchesini model [5]. A quarter-plane causal 2D state-space model is:

$$\mathbf{x}(h+1,k+1)=A_1\mathbf{x}(h,k+1)+A_2\mathbf{x}(h+1,k)+B_1\mathbf{u}(h,k+1)+B_2\mathbf{u}(h+1,k) \tag{7.1}$$

where $\mathbf{x}$ is the n-dimensional state variable, $\mathbf{u}$ is the m-dimensional input variable and the pair of independent coordinates (h,k) takes values in the half-plane $\mathcal{H}_0 := \{(h,k) \in \mathbb{Z} \times \mathbb{Z} : h + k \geq 0\}$. Once we define the t-th separation set

$$\mathcal{C}_t := \{(h,k) \in \mathbb{Z} \times \mathbb{Z} : \; h + k = t\}, \qquad t \in \mathbb{Z}_+,$$

the half plane $\mathcal{H}_0$ can be seen as $\bigcup_{t=0}^{+\infty} \mathcal{C}_t$. By the quarter-plane causal law underlying (7.1), the knowledge of the *initial global state* $\{\mathbf{x}(h,k), \; (h,k) \in \mathcal{C}_0\}$, together with the knowledge of the input evolution in the half plane $\mathcal{H}_0$, allows to uniquely determine the state evolution on $\mathcal{H}_0$. In this set-up, the variable t appearing in the separation sets $\mathcal{C}_t, t \geq 0$, can be thought of as the *time variable*. So, in order to make it possible a comparison with behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$, we first need to *rotate* the discrete grid, in order to overlap the half-plane $\mathcal{H}_0$ with the half plane $\mathbb{Z}_+ \times \mathbb{Z}$, considered in this paper. With this goal in mind, the 2D state space model must be modified in the following way:

$$\begin{aligned}(7.2)\quad \mathbf{x}(h+1,k) = A_1\mathbf{x}(h,k) + A_2\mathbf{x}(h,k+1) + B_1\mathbf{u}(h,k) + B_2\mathbf{u}(h,k+1),\\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},\end{aligned}$$

so that now its separation sets coincide with the vertical lines $\mathcal{S}_t, t \in \mathbb{Z}_+$. This latter model can be rewritten, by making use of the usual shift operators σ_1 and σ_2, as

$$\sigma_1\mathbf{x}(h,k) = A_1\mathbf{x}(h,k) + A_2\sigma_2\mathbf{x}(h,k) + B_1\mathbf{u}(h,k) + B_2\sigma_2\mathbf{u}(h,k), \quad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

and once the signal $\mathbf{w}(h,k) = [\, \mathbf{x}^T(h,k) \quad \mathbf{u}^T(h,k)\,]^T$ has been introduced, it becomes

$$H(\sigma_1,\sigma_2)\mathbf{w}(h,k) = 0, \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

where

$$H(z_1,z_2) := [\, z_1 I_n - (A_1 + A_2 z_2) \quad -(B_1 + B_2 z_2)\,] \in \mathbb{R}[z_1,z_2]^{n\times(n+m)}.$$

Being $H(z_1,z_2)$ of full row rank, in the standard (polynomial) decomposition $H(z_1, z_2) = L(z_1,z_2)\Delta(z_1,z_2)R(z_1,z_2)$ we may assume w.l.o.g that $L(z_1,z_2) = I_n$, $\Delta(z_1, z_2)$ is a greatest left divisor [8] of $H(z_1,z_2)$, and $R(z_1,z_2)$ is a lFP polynomial matrix.

By Proposition 3.2 and Theorem 5.1, the 2D state space model, described in behavioral form, is

- controllable if and only if $\Delta(z_1,z_2)$ is unimodular;
- time-controllable if and only if $\det\Delta(z_1,z_2) = \delta(z_2)$, for some nonzero $\delta(z_2) \in \mathbb{R}[z_2]$;
- zero-time-controllable if and only if $\det\Delta(z_1,z_2) = z_1^h\delta(z_2)$, for some nonzero $\delta(z_2) \in \mathbb{R}[z_2]$ and for some $h \in \mathbb{Z}_+$.

By the structure of $z_1 I_n - A_1 - A_2 z_2$, it is easily seen that $\det\Delta(z_1,z_2)$ cannot have nontrivial divisors of the form $\delta(z_2) \in \mathbb{R}[z_2]$. So, we can finally refine the previous characterizations as follows:

- controllability and time-controllability for 2D state-space models are always equivalent properties, and they both correspond to the left factor primeness of $H(z_1,z_2)$;
- zero-time-controllability is equivalent to square monomicity of $\Delta(z_1,z_2)$.

Appendix A. Technical lemmas and proofs of some results.

Lemma A.1. *For $i = 1,2$, let $H_i(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p_i\times \mathtt{w}}$ be a $p_i \times q$ TS-polynomial matrix, and let $\mathfrak{B} \subset (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$ be a 2D behavior. The following facts are equivalent:*

i) $\mathfrak{B} = \ker H_1(\sigma_1,\sigma_2) + \ker H_2(\sigma_1,\sigma_2)$;

ii) there exist TS-polynomial matrices $X(z_1, z_2)$, $Y(z_1, z_2)$ and $Z(z_1, z_2)$, of suitable dimensions, s.t. the identity

$$[\, X(z_1, z_2) \quad Y(z_1, z_2) \quad Z(z_1, z_2)\,] \begin{bmatrix} H_1(z_1, z_2) & 0 \\ 0 & H_2(z_1, z_2) \\ I_{\mathtt{w}} & I_{\mathtt{w}} \end{bmatrix} = 0$$

holds, the left block matrix is an MLA of the right one, and $\mathfrak{B} = \ker Z(\sigma_1, \sigma_2)$.

Proof. The proof follows the same lines of the analogous proof for standard 2D behaviors defined on $\mathbb{Z} \times \mathbb{Z}$ (see [28]). □

Proof of Lemma 4.2 ii) Suppose first that $\delta(z_1, z_2)$ has support included in a straight vertical line, namely $\delta(z_1, z_2) = \delta_r(z_2) z_1^r$. Then clearly $w \in \mathfrak{B}$ if and only if for every $h \in \mathbb{Z}_+, h \geq r$, the 1D sequence $v_h(k) := w(h, k)$ belongs to ker $\delta_r(\sigma_2)$. So, for every $w \in \mathfrak{B}$ and every $N \in \mathbb{N}$, we can always choose a trajectory in $\mathfrak{B}$ that both coincides with w on $\mathcal{S}_{\to N-1}$ and is zero on $\mathcal{S}_{N\to}$. So, zero-time-controllability holds (with $L = 0$).

Conversely, suppose that $\mathfrak{B}$ is zero-time-controllable, and assume, by contradiction, that $r < R$. Let w be any trajectory in $\mathfrak{B}$ and let N be any positive integer with $N > r$ (if $N \leq r$ every trajectory $w \in \mathfrak{B}$ can be driven to zero in just one step, meanwhile preserving it in $\mathcal{S}_{\to N-1}$). Let $\bar{w}$ be a trajectory of $\mathfrak{B}$ that fits w in $\mathcal{S}_{\to N-1}$ and is zero in $\mathcal{S}_{M\to}$, with $M \geq N$. Such a trajectory must satisfy

$$\delta(\sigma_1, \sigma_2)\bar{w}(h, k) = 0, \qquad \forall\ (h, k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

so, in particular, it must be

$$\delta_r(\sigma_2)\bar{w}(M-1, k) + \delta_{r+1}(\sigma_2)\bar{w}(M, k) + \ldots + \delta_R(\sigma_2)\bar{w}(M-1+R-r, k) = 0, \forall\ k \in \mathbb{Z}.$$

Since $\bar{w}(t, k) = 0$ for every $t \geq M$, this implies that

$$\delta_r(\sigma_2)\bar{w}(M-1, k) = 0, \qquad \forall\ k \in \mathbb{Z}.$$

Consequently, the restriction $\bar{v}_{M-1}(k) := \bar{w}(M-1, k), k \in \mathbb{Z}$, regarded as a 1D trajectory, belongs to ker $\delta_r(\sigma_2)$. Similarly, we have

$$\delta_r(\sigma_2)\bar{w}(M-2, k) + \delta_{r+1}(\sigma_2)\bar{w}(M-1, k) + \ldots + \delta_R(\sigma_2)\bar{w}(M-2+R-r, k) = 0,$$

$\forall\ k \in \mathbb{Z}$, which is equivalent to

$$\delta_r(\sigma_2)\bar{w}(M-2, k) + \delta_{r+1}(\sigma_2)\bar{w}(M-1, k) = 0, \qquad \forall\ k \in \mathbb{Z}.$$

If we apply $\delta_r(\sigma_2)$ to both sides, we get

$$[\delta_r(\sigma_2)]^2\bar{w}(M-2, k) = [\delta_r(\sigma_2)]^2\bar{w}(M-2, k) + \delta_{r+1}(\sigma_2)\delta_r(\sigma_2)\bar{w}(M-1, k) = 0.$$

So, the restriction $\bar{v}_{M-2}(k) := \bar{w}(M-2, k), k \in \mathbb{Z}$, belongs to ker $[\delta_r(\sigma_2)]^2$. By proceeding in this way, we prove that $\bar{v}_r(k) := \bar{w}(r, k), k \in \mathbb{Z}$, belongs to ker $[\delta_r(\sigma_2)]^{M-r}$. But then all the restrictions $\bar{v}_t(k) := \bar{w}(t, k), t = r, r+1, \ldots, M-1$, belong to ker $[\delta_r(\sigma_2)]^{M-r}$. On the other hand, all the restrictions of $\bar{w}$ to the subsequent vertical lines are zero, which amounts to saying that all the trajectories $\bar{v}_t(k), t \geq M$, are zero. As a result, all the trajectories $\bar{v}_t(k), t \geq r$, belong to ker $[\delta_r(\sigma_2)]^{M-r}$.

However, in $\mathfrak{B}$ there exist (infinitely many) trajectories w whose restrictions to the vertical lines $v_t(k) = w(t,k),\ k \in \mathbb{Z},\ t \in \mathbb{Z}_+$, do not belong to $\ker[\delta_r(\sigma_2)]^{M-r}$, at least for $t = r, r+1, \ldots, R-1$. This is easily proved as follows: consider the relationship

$$\delta_r(\sigma_2)v_{r+t}(k) + \ldots + \delta_{R-1}(\sigma_2)v_{R+t-1}(k) = -\delta_R(\sigma_2)v_{R+t}(k), \qquad k \in \mathbb{Z},$$

which holds true for every $t \geq 0$. As the L-polynomial $\delta_R(z)$ induces a surjective map, we can freely choose the sequences $v_i(k), i = r, r+1, \ldots, R-1$, in such a way that they do not belong to $\ker[\delta_r(\sigma_2)]^{M-r}$, and evaluate a corresponding (in general, not unique) $v_R(k)$ from the previous relation evaluated in $t = 0$. By repeating this reasoning from $t = 1$ onward, we can evaluate (possible) evolutions $v_i(k), i \geq R+1$, which are compatible with the behavior equations. Finally, we choose $v_0(k), \ldots, v_{r-1}(k)$ in a completely free manner. So, with this procedure, we have obtained a trajectory $w \in \mathfrak{B}$ whose restrictions to the vertical lines $\mathcal{S}_t, t = r, \ldots, R-1$, do not belong to ker $[\delta_r(\sigma_2)]^{M-r}$. Since any such trajectory $w \in \mathfrak{B}$ could not be replaced by a trajectory $\bar{w} \in \mathfrak{B}$ that is identically zero in $\mathcal{S}_{M\rightarrow}$, this contradicts the zero-time-controllability assumption.

i) If $\mathfrak{B}$ is the kernel of an L-polynomial in the variable z_2 alone, namely, $\delta(z_1, z_2) = \delta_0(z_2)$, this means that $w \in \mathfrak{B}$ if and only if $w|_{\mathcal{S}_t} \in \mathfrak{B}|_{\mathcal{S}_t} = \ker\ \delta_0(\sigma_2)$ for every $t \in \mathbb{Z}_+$. But then surely any initial portion of a trajectory w, say $w|_{\mathcal{S}_{\rightarrow N-1}}$, can be "patched" with the shifted version (by N time instants) of any other trajectory w^*. This amounts to saying that the trajectory $\bar{w}$, defined as

$$\begin{array}{rcll} \bar{w}(h,k) & = & w(h,k), & \forall\ (h,k) \in \mathcal{S}_{\rightarrow N-1}, \\ \bar{w}(h,k) & = & w^*(h-N,k), & \forall\ (h,k) \in \mathcal{S}_{N\rightarrow}, \end{array}$$

belongs to $\mathfrak{B}$. So, time-controllability holds (with $L = 0$).

Conversely, suppose that $\mathfrak{B}$ is time-controllable. Then it is zero-time-controllable and this ensures that $r = R$. On the other hand, if $r > 0$ there exists in $\mathfrak{B}$ a trajectory, say w^*, whose restriction to the first straight line $\mathcal{S}_0$ does not belong to ker $\delta_r(\sigma_2)$. Clearly, for every $N \geq r$, no parameter $L \in \mathbb{Z}_+$ can be found s.t. a trajectory $\bar{w} \in \mathfrak{B}$ exists, satisfying

$$\begin{array}{rcll} \bar{w}(h,k) & = & 0, & \forall\ (h,k) \in \mathcal{S}_{\rightarrow N-1}, \\ \bar{w}(h,k) & = & w^*(h-N-L,k), & \forall\ (h,k) \in \mathcal{S}_{N+L\rightarrow}. \end{array}$$

$\square$

Proof of Lemma 4.3 i) Suppose, first, that $\ker(\det \Delta(\sigma_1, \sigma_2))$ is time-controllable, and hence also $\ker(\det \Delta(\sigma_1, \sigma_2) I_{\mathtt{w}})$ is time-controllable. Let $\mathbf{w}$ and $\mathbf{w}^*$ be two sequences in $\mathfrak{B}$ and let N be a positive integer. Since $\mathrm{adj}\Delta(z_1, z_2)$ is a nonsingular square matrix, it defines a surjective map. This ensures that there exist $\mathbf{v}$ and $\mathbf{v}^* \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$ s.t.

$$\mathbf{w} = \mathrm{adj}\Delta(\sigma_1, \sigma_2)\mathbf{v}, \qquad\qquad \mathbf{w}^* = \mathrm{adj}\Delta(\sigma_1, \sigma_2)\mathbf{v}^*.$$

As $\det \Delta(z_1, z_2) I_{\mathtt{w}} = \Delta(z_1, z_2)\ \mathrm{adj}\Delta(z_1, z_2)$, clearly, $\mathbf{v}, \mathbf{v}^* \in \ker(\det \Delta(\sigma_1, \sigma_2) I_{\mathtt{w}})$. Set $\tilde{N} := N + \deg_{z_1}(\mathrm{adj}\Delta(z_1, z_2))$. As $\ker(\det\ \Delta(\sigma_1,\ \sigma_2) I_{\mathtt{w}})$ is time-controllable, we can find $\bar{\mathbf{v}} \in \ker(\det\Delta(\sigma_1,\sigma_2)I_{\mathtt{w}})$ s.t. $\bar{\mathbf{v}} = \mathbf{v}$ in $\mathcal{S}_{\to \tilde{N}-1}$, and $\sigma_1^{\tilde{N}+\tilde{L}}\bar{\mathbf{v}} = \mathbf{v}^*$ for some $\tilde{L} \geq 0$. Accordingly, $\bar{\mathbf{w}} = \mathbf{w}$ on $\mathcal{S}_{\to N-1}$ and $\sigma_1^{\tilde{N}+\tilde{L}}\bar{\mathbf{w}} = \sigma_1^{N+L}\bar{\mathbf{w}} = \mathbf{w}^*$, where $L := \tilde{L} + \deg_{z_1}(\mathrm{adj}\Delta(z_1,z_2))$ (see Figure 2 to visualize the various sets involved in the proof). This proves that $\mathfrak{B}$ is time-controllable.

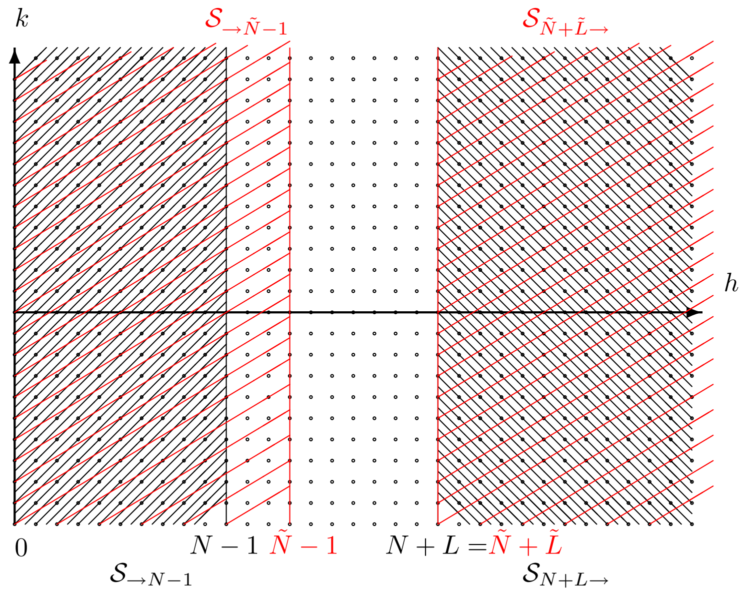

Fig.2: Sets $\mathcal{S}_{\to N-1}, \mathcal{S}_{\to\tilde{N}-1}, \mathcal{S}_{\tilde{N}+\tilde{L}\to}$ and $\mathcal{S}_{N+L\to}$ in the proof of Lemma 4.3.

Conversely, assume that $\mathfrak{B}$ is time-controllable, and let $P_{w_i}(\mathfrak{B})$ denote the projection of $\mathfrak{B}$ on its ith component w_i. It can be easily proved (see also Theorem 3 in [18]) that a nonzero polynomial $q_i(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]$ can be found s.t. $P_{w_i}(\mathfrak{B}) = \ker\ q_i(\sigma_1,\sigma_2)$. Clearly, the direct product of these projections contains $\mathfrak{B}$, which amounts to saying that $\ker D(\sigma_1,\sigma_2) \supseteq \ker\Delta(\sigma_1,\sigma_2)$, where

$$D(z_1,z_2) := \mathrm{diag}\{q_1(z_1,z_2), q_2(z_1,z_2), \dots, q_{\mathtt{w}}(z_1,z_2)\}.$$

Consequently, $D(z_1,z_2) = T(z_1,z_2)\Delta(z_1,z_2)$, for some $T(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{\mathtt{w}\times\mathtt{w}}$. This implies that $\det\Delta(z_1,z_2)$ divides $\det D(z_1,z_2) = \prod_{i=1}^{\mathtt{w}} q_i(z_1,z_2)$.

On the other hand, all the projections $P_{w_i}(\mathfrak{B})$ necessarily inherit the time-controllability property of $\mathfrak{B}$, and since they are scalar autonomous behavior, this implies that, for every $i \in \{1,2,\dots,\mathtt{w}\}$, $q_i(z_1,z_2) = \bar{q}_i(z_2)$, for some $\bar{q}_i(z_2) \in \mathbb{R}[z_2,z_2^{-1}]$. So, as $\det\Delta(z_1,z_2)$ divides $\prod_{i=1}^{\mathtt{w}} \bar{q}_i(z_2)$, it belongs, in turn, to $\mathbb{R}[z_2,z_2^{-1}]$.

ii) The corresponding proof can be derived along the same lines. $\square$

Proof of Lemma 4.4 ii) Assume without loss of generality (w.l.o.g.) that $H(z_1,z_2) = L(z_1,z_2)\ \tilde{R}(z_1,z_2)$, with $L(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times r}$ right factor prime and $\tilde{R}(z_1,z_2) \in \mathbb{R}[z_1,\ z_2,z_2^{-1}]^{r\times\mathtt{w}}$ of full row rank. We aim to show that zero-time-controllability of $\mathfrak{B}$ implies that the TS-variety $\mathcal{V}_{TS}(L)$ does not include pairs (α,β) with both nonzero entries. Assume, by contradiction, that there exist some (possibly complex) pair $(\alpha,\beta) \in \mathcal{V}_{TS}(L)$ (with $\alpha\cdot\beta \neq 0$) and therefore some vector $\mathbf{v} \in \mathbb{C}^r, \mathbf{v} \neq$

0, s.t. $L(\alpha,\beta)\mathbf{v} = 0$. It is easy to verify that the trajectory $\mathbf{z}$, defined by

$$\mathbf{z}(h,k) := \alpha^h \beta^k \mathbf{v}, \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

satisfies $L(\sigma_1,\sigma_2)\mathbf{z}(h,k) = 0$ for every $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}$. In general, α and/or β (and hence $\mathbf{v}$) are complex valued, so $\mathbf{z}$ is complex valued too, and it can be expressed as $\mathbf{z} = \mathbf{z}_R + j\mathbf{z}_I$, where $\mathbf{z}_R$ is the real part of $\mathbf{z}$ and $\mathbf{z}_I$ is its imaginary part. Of course, both these sequences are real valued, belong to ker $L(\sigma_1,\sigma_2)$, and at least one of them (in the following we refer to this trajectory as $\bar{\mathbf{z}} \in (\mathbb{R}^r)^{\mathbb{Z}_+\times\mathbb{Z}}$) has support that intersects infinitely many vertical lines $\mathcal{S}_t$ (since the trajectory $\mathbf{z}$ enjoys this property). Since $\tilde{R}(z_1,z_2)$ is of full row rank, it defines a surjective map in $\mathbb{Z}_+ \times \mathbb{Z}$. This implies that there exists $\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$ s.t.

$$\bar{\mathbf{z}} = \tilde{R}(\sigma_1,\sigma_2)\mathbf{w},$$

and clearly $\mathbf{w} \in \mathfrak{B}$. Suppose now that we want to drive to zero the sequence $\mathbf{w}$ in $\mathcal{S}_{N+L\rightarrow}$, meanwhile preserving it on $\mathcal{S}_{\rightarrow N-1}$, namely we want to find $\bar{\mathbf{w}} \in \mathfrak{B}$ s.t. $\bar{\mathbf{w}} = \mathbf{w}$ in $\mathcal{S}_{\rightarrow N-1}$ and $\bar{\mathbf{w}} = 0$ in $\mathcal{S}_{N+L\rightarrow}$. If N is sufficiently large, we constrain a large enough portion of $\bar{\mathbf{w}}$ to coincide with $\mathbf{w}$ and therefore we constrain, in turn, a large initial portion of the corresponding image, $\tilde{R}(\sigma_1,\sigma_2)\bar{\mathbf{w}}$, to coincide with $\bar{\mathbf{z}}$. But since $R(\sigma_1,\sigma_2)\bar{\mathbf{w}}$ must belong to the finite-dimensional autonomous behavior ker$L(\sigma_1,\sigma_2)$, by constraining its initial portion we essentially impose that the whole trajectory $\tilde{R}(\sigma_1,\sigma_2)\bar{\mathbf{w}}$ coincides with $\bar{\mathbf{z}}$. Since $\bar{\mathbf{z}}$ intersects an infinite number of vertical lines, so does any $\bar{\mathbf{w}}$ s.t. $\tilde{R}(\sigma_1,\sigma_2)\bar{\mathbf{w}} = \bar{\mathbf{z}}$. This implies that $\mathbf{w}$ cannot be replaced by a sequence $\bar{\mathbf{w}}$ whose support is included in a vertical strip, and hence zero-time-controllability does not hold. So, we have proved that no (α,β), with $\alpha\cdot\beta \neq 0$, belongs to $\mathcal{V}_{TS}(L)$, and hence $L(z_1,z_2)$ is right monomic, as well as right factor prime, by the initial assumption.

i) Assume w.l.o.g. that $H(z_1,z_2) = L(z_1,z_2)\tilde{R}(z_1,z_2)$, with $L(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times r}$ right factor prime and $\tilde{R}(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{r\times\mathtt{w}}$ of full row rank. If $\mathfrak{B}$ is time-controllable then it is zero-time-controllable. So, by the first part of the proof, we can surely ensure that $L(z_1,z_2)$ is rM. If it were not rZP, ker $L(\sigma_1,\sigma_2)$ would be a nontrivial nilpotent behavior and there would be some index $\tilde{N}$ s.t. all trajectories in ker $L(\sigma_1,\sigma_2)$ have support included in $\mathcal{S}_{\rightarrow\tilde{N}-1}$. Let $\mathbf{z}^*$ be a nontrivial trajectory in ker $L(\sigma_1,\sigma_2)$. By exploiting again the surjectivity of $\tilde{R}(\sigma_1,\sigma_2)$, we can find $\mathbf{w}^*$ such that $\mathbf{z}^* = \tilde{R}(\sigma_1,\sigma_2)\mathbf{w}^*$. Clearly, $\mathbf{w}^* \in \mathfrak{B}$. Set $N := \tilde{N} + \deg_{z_1}\Delta(z_1,z_2)$. We want to prove that there is no way to find $L \in \mathbb{Z}_+$ and a trajectory $\bar{\mathbf{w}} \in \mathfrak{B}$ s.t.

$$\begin{array}{rcll} \bar{\mathbf{w}}(h,k) & = & 0, & \forall\ (h,k) \in \mathcal{S}_{\rightarrow N-1}, \\ \bar{\mathbf{w}}(h,k) & = & \mathbf{w}^*(h-N-L,k), & \forall\ (h,k) \in \mathcal{S}_{N+L\rightarrow}. \end{array}$$

We first of all note that $\bar{\mathbf{w}} \in \mathfrak{B}$ if and only if $\bar{\mathbf{z}} := \tilde{R}(\sigma_1,\sigma_2)\bar{\mathbf{w}} \in$ ker $L(\sigma_1,\sigma_2)$. But since $\bar{\mathbf{z}}|_{\mathcal{S}_{\rightarrow\tilde{N}-1}} = 0$, by the nilpotency property of ker $L(\sigma_1,\sigma_2)$, the trajectory $\bar{\mathbf{z}}$ must be zero all over $\mathbb{Z}_+\times\mathbb{Z}$. Moreover, from $\sigma_1^{N+L}\bar{\mathbf{w}} = \mathbf{w}^*$, it follows $\mathbf{z}^* = \tilde{R}(\sigma_1,\sigma_2)\mathbf{w}^* = \sigma_1^{N+L}\tilde{R}(\sigma_1,\sigma_2)\bar{\mathbf{w}} = \sigma_1^{N+L}\bar{\mathbf{z}} = 0$, thus leading to a contradiction. This implies that ker $L(\sigma_1,\sigma_2)$ does not include nonzero trajectories, which is equivalent to the fact that $L(z_1,z_2)$ is rZP. □

Lemma A.2. *Given a 2D behavior $\mathfrak{B} =$ ker $H(\sigma_1,\sigma_2)$, with $H(z_1,\ z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times\mathtt{w}}$, if the controller $\mathcal{C}$ described by the difference equation (6.1), with $C(z_1,z_2) \in$*

$\mathbb{R}[z_1, z_2, z_2^{-1}]^{c\times \mathtt{w}}$, *is a DBC for the system, then any controller* $\mathcal{C}_i, i \in \mathbb{Z}_+$, *described by*

$$\sigma_1^i C(\sigma_1, \sigma_2)\mathbf{w}(h, k) = 0, \qquad (h, k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

is a DBC.

Proof. This immediately follows from the fact that if

$$\Gamma(z_1, z_2) := \begin{bmatrix} H(z_1, z_2) \\ C(z_1, z_2) \end{bmatrix}$$

is right monomic, then also

$$\begin{bmatrix} I_p & 0 \\ 0 & z_1^i I_c \end{bmatrix} \Gamma(z_1, z_2)$$

is right monomic. □

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