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On the input/output decoupling of Boolean Control Networks

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Abstract: In this paper we investigate the input/output decoupling problem for Boolean Control Networks. To keep up with the spirit of the original definition for linear state-space models, that pertains the relationship between inputs and outputs independently of the state variables, we first provide two properties that formalize in different ways the idea that each single component of the output depends on the values of the corresponding input, but not on the values of the other inputs. These properties are introduced by referring to the classical representation of a Boolean Control Network in terms of Boolean input, state and output vectors, whose mutual relationships are expressed through the logical operators AND, OR, etc.. In this set-up we prove that there is some natural ordering among these properties, namely that one of them implies the other. In the second part of the paper we show that by resorting to the algebraic representation of Boolean Control Networks a complete characterization of these properties is possible. The algebraic characterizations obtained through this approach provide easy to check algorithms to evaluate whether a Boolean Control Network is input/output decoupled or not. Finally, graph-theoretic characterizations of the two input/output decoupling properties are provided.

1 Introduction

Originally conceived by mathematicians and engineers to model artificial systems whose describing variables take only two values ("on" or "off", "high" or "low"), logic networks are nowadays largely employed to describe gene regulatory networks, as a bibliographic search immediately reveals. The big success of Boolean Networks (BNs) first, and of Probabilistic Boolean Networks (PBNs) and Boolean Control Networks (BCNs) later, as tools to describe and simulate the behavior of genetic regulatory networks must be credited to Stuart Kauffman [1]. Kauffman was the first to realize that regulatory genes inside the cells act just like switches, that may take either an "on" or an "off" status (1 and 0, respectively). Accordingly, he proposed PBNs as models for genetic networks (see also [2]). Inspired by a few milestones papers, there was a flourishing of literature adopting BNs, PBNs and BCNs to model gene regulatory networks and this represents nowadays a very active research area [3–10].

In the last decade, stimulated also by the successful use of logic networks in biological and medical modeling, Daizhan Cheng and co-authors developed an algebraic approach to BNs and BCNs [11–14] that is based on the possibility of representing each state of a finite state system as a canonical vector, and consequently logic relationships by means of logic matrices. Indeed, a Boolean network with n state variables exhibits 2^n possible configurations, and if any such configuration is represented by means of a canonical vector of size 2^n, all the logic maps that regulate the state-updating can be equivalently described by means of $2^n \times 2^n$ logic matrices. As a result, each Boolean network is converted into a discrete-time linear system. Similarly, a Boolean control network can be represented as a discrete-time bilinear system or, equivalently, as a family of BNs, each of them associated with a specific value of the input variables, and in that sense as a Boolean switched system.

In this set-up, logic-based problems are converted into algebraic problems and hence solved by resorting to mathematical tools similar to those available for linear state-space models. This has made it possible to formalize and solve classical system theoretic problems, like stability and stabilizability, controllability, observability, fault detection and optimal control [15–22].

While the disturbance decoupling problem for BCNs has been successfully investigated in a number of contributions [23–26], to the best of our knowledge the input/output decoupling problem is still unexplored, even if some related problem regarding input decomposition has been addressed in [27]. An explanation may be searched for in the fact that, even if the broad idea of being able to control a single output with a single control input, with no interference from the other inputs, is quite immediate, the details of the problem formalization in the context of Boolean Control Networks are not obvious, and different formulations are possible.

To keep up with the spirit of the original definition, that pertains only the transfer matrix and hence the relation between inputs and outputs, independently of the state variables, in this paper we propose two definitions that formalize in different ways the idea that each single component of the output depends on the value of the corresponding input, but not on the values of the other inputs. The practical meaning of input/output decoupling in terms of biological systems and in particular of gene regulatory networks is quite immediate and rather intriguing: in an input/output decoupled network each of the output variables whose physical status we are measuring (genes that are active or not, proteins that are produced or not, the open/closed state of an ion channel, the basal/high activity of an enzyme) depends only on the status of a specific input variable (a protein that is activated or not, a high/low stress level, a therapy/medicine that is applied or not, a light signal that is on or off...). It is clear that, in all these contexts, the possibility of putting in place strategies that are able to selectively target only one of the output variables, by using a single control input, is highly desirable.

These definitions are introduced by referring to the classical representation of a Boolean Control Network in terms of Boolean input, state and output vectors, and of logical relationship expressed through the logical AND, OR, etc. operators. In this set-up it is possible to prove that there is some ordering between the two properties, namely that one of them implies the other. However, it is very difficult to provide characterizations of BCNs that are input/output decoupled in some sense. In the second part of the paper we show that, by resorting to the algebraic representation of Boolean Control Networks, necessary and sufficient conditions for these properties to hold can be derived. The obtained algebraic characterizations provide easy to check algorithms to evaluate whether a Boolean Control

Network is decoupled or not. Finally, the derived conditions are expressed in terms of certain digraphs associated with the BCN.

Notation. $\mathbb{Z}_+$ denotes the set of nonnegative integers. Given two integers $k, n \in \mathbb{Z}_+$, with $k \le n$, by the symbol $[k, n]$ we denote the set of integers $\{k, k+1, \dots, n\}$. We consider Boolean vectors and matrices, whose entries take values in $\mathcal{B} := \{0, 1\}$, with the usual (entrywise) Boolean operations: AND ($\wedge$), OR ($\vee$), NOT ($\bar{\cdot}$) and their compositions.

δ_k^i denotes the ith *canonical vector* of size k. $\mathcal{L}_k$ is the set of all k-dimensional canonical vectors, and $\mathcal{L}_{k\times n} \subset \mathcal{B}^{k\times n}$ the set of all $k \times n$ *logical matrices*, whose n columns are canonical vectors of size k. Any matrix $L \in \mathcal{L}_{k\times n}$ can be represented as a row whose entries are canonical vectors in $\mathcal{L}_k$, namely as $L = [\delta_k^{i_1} \quad \delta_k^{i_2} \quad \dots \quad \delta_k^{i_n}]$, for suitable indices $i_1, i_2, \dots, i_n \in [1, k]$. The (ℓ, j)th entry of a matrix L is denoted by $[L]_{\ell,j}$, while the ℓth entry of a vector $\mathbf{v}$ is either v_ℓ or $[\mathbf{v}]_\ell$. The latter notation will always be used when the expression of the vector is composite or complex. The pth column of a matrix L is $\mathrm{col}_p(L)$.

Given a matrix $M \in \mathcal{B}^{N\times N}$, we associate with it a *directed graph (digraph)* $\mathcal{D}(M)$, with vertices $1, \dots, N$. There is an arc (j, ℓ) from j to ℓ if and only if the (ℓ, j)th entry of M is unitary.

There is a bijective correspondence between Boolean variables $X \in \mathcal{B}$ and vectors $\mathbf{x} \in \mathcal{L}_2$, defined by the relationship

$$\mathbf{x} = \begin{bmatrix} X \\ \bar{X} \end{bmatrix}.$$

We introduce the *(left) semi-tensor product* $\ltimes$ between matrices (in particular, vectors) as follows [12, 22, 28]: given $L_1 \in \mathbb{R}^{r_1\times c_1}$ and $L_2 \in \mathbb{R}^{r_2\times c_2}$ (in particular, $L_1 \in \mathcal{L}_{r_1\times c_1}$ and $L_2 \in \mathcal{L}_{r_2\times c_2}$), we set

$$L_1 \ltimes L_2 := (L_1 \otimes I_{T/c_1})(L_2 \otimes I_{T/r_2}), \qquad T := \text{l.c.m.}\{c_1, r_2\},$$

where l.c.m. denotes the least common multiple. The semi-tensor product represents an extension of the standard matrix product, by this meaning that if $c_1 = r_2$, then $L_1 \ltimes L_2 = L_1 L_2$. Note that if $\mathbf{x}_1 \in \mathcal{L}_{r_1}$ and $\mathbf{x}_2 \in \mathcal{L}_{r_2}$, then $\mathbf{x}_1 \ltimes \mathbf{x}_2 \in \mathcal{L}_{r_1 r_2}$. For the various properties of the semi-tensor product we refer to [12].

By resorting to the semi-tensor product, we can extend the correspondence between $\mathcal{B}$ and $\mathcal{L}_2$ to a bijective correspondence between $\mathcal{B}^n$ and $\mathcal{L}_{2^n}$. This is possible in the following way: given $X = \begin{bmatrix} X_1 & X_2 & \dots & X_n \end{bmatrix}^\top \in \mathcal{B}^n$, set

$$\mathbf{x} := \begin{bmatrix} X_1 \\ \bar{X}_1 \end{bmatrix} \ltimes \begin{bmatrix} X_2 \\ \bar{X}_2 \end{bmatrix} \ltimes \dots \ltimes \begin{bmatrix} X_n \\ \bar{X}_n \end{bmatrix} = \begin{bmatrix} X_1 X_2 \dots X_{n-1} X_n \\ X_1 X_2 \dots X_{n-1} \bar{X}_n \\ X_1 X_2 \dots \bar{X}_{n-1} X_n \\ \vdots \\ \bar{X}_1 \bar{X}_2 \dots \bar{X}_{n-1} \bar{X}_n \end{bmatrix}.$$

We denote by $\mathbb{C}_n$ the map from $\mathcal{B}^n$ to $\mathcal{L}_{2^n}$ that associates with every vector $X \in \mathcal{B}^n$ its canonical representation $\mathbf{x}$, and by

$$\mathbb{B}_n : \mathcal{L}_{2^n} \to \mathcal{B}^n : \mathbf{x} \to X, \tag{1}$$

Note that in [12], at page 69, an algorithm is given to determine the Boolean equivalent $\mathbb{B}_n(\mathbf{x})$ of a given canonical vector $\mathbf{x}$. On the other hand, if we are interested only in determining the ith entry X_i of $X = \mathbb{B}_n(\mathbf{x})$, we can proceed as follows. Set

$$F_i := \mathbf{1}_{2^{i-1}}^T \otimes I_2 \otimes \mathbf{1}_{2^{n-i}}^T, \qquad 1 \le i \le n,$$

then

$$F_i \mathbf{x} = \begin{bmatrix} X_i \\ \bar{X}_i \end{bmatrix}, \qquad 1 \le i \le n,$$

and hence

$$[\mathbb{B}_n(\mathbf{x})]_i = (\delta_2^1)^\top F_i \mathbf{x}.$$

2 Input/output decoupling properties of a Boolean Control Network

A *Boolean Control Network* (BCN) is described by the following equations

$$\begin{aligned} X(t+1) &= f(X(t), U(t)), \\ Y(t) &= h(X(t)), \qquad t \in \mathbb{Z}_+, \end{aligned} \tag{2}$$

where $X(t)$, $U(t)$ and $Y(t)$ denote the n-dimensional state variable, the m-dimensional input and the p-dimensional output at time t, taking values in $\mathcal{B}^n, \mathcal{B}^m$ and $\mathcal{B}^p$, respectively. f and h are logic functions, i.e. $f : \mathcal{B}^n \times \mathcal{B}^m \to \mathcal{B}^n$ and $h : \mathcal{B}^n \to \mathcal{B}^p$. In the following we will steadily assume that $p = m$, namely that the BCN has a number of outputs equal to the number of inputs, and refer to such a common number by m.

In the classical context of linear state space models, a system with the same number of inputs and outputs is input/output decoupled if its (proper, rational and square) transfer matrix is diagonal. When moving to BCNs, the adaption of this concept is not obvious. Indeed, Boolean Control Networks are not linear and it is not possible to split the output trajectories into forced and unforced components. As a result, no transfer matrix can be defined and various definitions of input/output decoupling may be proposed. In this paper, keeping up with the original spirit of the definition for linear state-space models, we propose, compare and investigate two forms of input/output decoupling for BCNs that are characterized by a common feature: to refer uniquely to the inputs and the outputs of the BCN, without imposing any constraint on the partition of the state variables. Both definitions provide different formalizations of the idea that each single component of the output depends on the value of the corresponding input, but not on the values of the other inputs.

The first definition we introduce is the following one. We believe it is the one that best captures the spirit of the classical property and hence we will adopt it as definition of input/output decoupling for a BCN.

Definition 1. *A BCN (2) with inputs and outputs having the same cardinality, m, is said to be* input/output decoupled *if for every index $i \in [1, m]$ and every initial state $X(0) \in \mathcal{B}^n$, if $U(t)$ and $\hat{U}(t), t \in \mathbb{Z}_+$, are two input sequences characterized by the fact that their ith entries coincide at every time instant, i.e.,*

$$U_i(t) = \hat{U}_i(t), \qquad \forall\, t \in \mathbb{Z}_+, \tag{3}$$

then the output sequences $Y(t)$ and $\hat{Y}(t), t \in \mathbb{Z}_+$, generated by the BCN (2) corresponding to the initial state $X(0)$ and the inputs $U(t)$ and $\hat{U}(t), t \in \mathbb{Z}_+$, respectively, satisfy

$$Y_i(t) = \hat{Y}_i(t), \qquad \forall\, t \in \mathbb{Z}_+. \tag{4}$$

A first characterization of the previous definition of input/output decoupling is given in Proposition 1, below. The result is straightforward, and we give its proof for the sake of completeness. However, this equivalent formulation provides further insights into the nature of input/output decoupling.

Proposition 1. *For the BCN (2) with m inputs and m outputs, the following facts are equivalent:*

i) the BCN is input/output decoupled;
ii) for every $i \in [1, m]$ there exists a map ϕ_i such that for every initial condition $X(0) \in \mathcal{B}^n$ and every input sequence $U(t), t \in \mathbb{Z}_+$, one has

$$Y_i(t) = \phi_i(t; X(0), U_i(0), U_i(1), \dots, U_i(t-1)),\ t \ge 1. \tag{5}$$

Proof: i) $\Rightarrow$ ii) Due to the causality of the BCN, it is obvious that there exists a map ϕ such that

$$Y(t) = \phi(t; X(0), U(0), U(1), \ldots, U(t-1)), \qquad t \geq 1.$$

We want to prove that if the BCN is input/output decoupled, then the evaluation of the ith entry of the output at any time $t \geq 1$ requires the knowledge only of the ith entries of the input samples up to time $t-1$. If this were not the case then there would be some initial state $X(0)$, some input sequence $U(t), t \in \mathbb{Z}_+$, a logic map ϕ_i, some (minimum) time instant $\tau \geq 1$ and indices $j_1, \ldots, j_k, k \geq 1, j_\ell \neq i$, such that

$$\begin{aligned} Y_i(\tau) \;&=\; \phi_i(\tau; X(0), U_i(0), U_i(1), \ldots, U_i(\tau-2), U_i(\tau-1), \\ &\quad U_{j_1}(\tau-1), \ldots, U_{j_k}(\tau-1)). \end{aligned}$$

Condition i) ensures that any input sequence $U^\ell(t), t \in \mathbb{Z}_+, \ell \in [1, k]$, satisfying

$$\begin{aligned} U^{(\ell)}(t) &= U(t), && \forall\, t \in \mathbb{Z}_+, t \neq \tau, \\ [U^{(\ell)}(\tau)]_j &= U_j(\tau), && \forall\, j \in [1, m], j \neq j_\ell, \\ [U^\ell(\tau)]_{j_\ell} &= \bar{U}_{j_\ell}(\tau), \end{aligned}$$

generates an output satisfying $[Y^{(\ell)}(\tau)]_i = Y_i(\tau)$. Therefore, $Y_i(\tau)$ is necessarily independent of $U_{j_1}(\tau-1), \ldots, U_{j_k}(\tau-1)$ and (5) holds.

ii) $\Rightarrow$ i) If ii) holds true for every initial condition and every input sequence, then for every $X(0)$ and every pair of input sequences $U(t)$ and $\hat{U}(t), t \in \mathbb{Z}_+$, satisfying (3) one has

$$\begin{aligned} Y_i(t) &= \phi_i(t; X(0), U_i(0), U_i(1), \ldots, U_i(t-1)) \\ &= \phi_i(t; X(0), \hat{U}_i(0), \hat{U}_i(1), \ldots, \hat{U}_i(t-1)) = \hat{Y}_i(t), \end{aligned}$$

for every $t \geq 1$. On the other hand, $Y(0) = h(X(0)) = \hat{Y}(0)$, and hence (4) holds. ∎

Example 1. *Consider the BCN with $m = 2$ inputs and outputs:*

$$\begin{aligned} X_1(t+1) &= X_1(t), \\ X_2(t+1) &= X_2(t) \wedge U_1(t), \\ X_3(t+1) &= \bar{X}_3(t) \vee U_2(t), \\ Y_1(t) &= X_1(t) \wedge X_2(t), \\ Y_2(t) &= X_3(t). \end{aligned}$$

It is very easy to see that condition ii) of Proposition 1 holds and hence the BCN is input/output decoupled (in fact, in this case, also the state vector components are partitioned into two disjoint groups).

Example 2. *Consider the BCN with $m = 2$ inputs and outputs:*

$$\begin{aligned} X_1(t+1) &= X_1(t), \\ X_2(t+1) &= X_1(t) \wedge \bar{X}_2(t) \wedge U_1(t), \\ X_3(t+1) &= U_2(t), \\ Y_1(t) &= X_1(t) \wedge X_2(t), \\ Y_2(t) &= X_2(t) \vee X_3(t). \end{aligned}$$

It is easy to see that the BCN is not input/output decoupled. Assume, for instance, $X(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$ and consider $i = 2$ and the two constant sequences $U(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, t \in \mathbb{Z}_+$, and $\hat{U}(t) = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top, t \in \mathbb{Z}_+$.

Clearly, $U_2(t) = \hat{U}_2(t)$ for every $t \in \mathbb{Z}_+$. However, it is immediate to verify that $X(1) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$ and $Y(1) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, while $\hat{X}(1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$ and $\hat{Y}(1) = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$. Therefore, $Y_2(1) \neq \hat{Y}_2(1)$, thus contradicting the input/output decoupling property.

A second definition of input/output decoupling is the one given in Definition 2, below.

Definition 2. *A BCN (2) with input and output having the same cardinality, m, is said to be* one-step transition input/output decoupled *if, for every index $i \in [1, m]$, every pair of states $X, \hat{X} \in \mathcal{B}^n$ and every pair of input vectors $U, \hat{U} \in \mathcal{B}^m$ satisfying conditions*

$$[h(X)]_i = [h(\hat{X})]_i, \qquad \text{and} \qquad U_i = \hat{U}_i, \tag{6}$$

ensure that

$$[h(f(X, U))]_i = [h(f(\hat{X}, \hat{U}))]_i. \tag{7}$$

The idea behind the previous definition is that if we start with two states X and $\hat{X}$ whose corresponding outputs share the same ith entry, and we apply two input samples, U and $\hat{U}$, that have the same ith component, then the two successor states $f(X, U)$ and $f(\hat{X}, \hat{U})$ will in turn generate two output vectors that share the same ith entry. One-step transition input/output decoupling is a sufficient condition for input/output decoupling, as shown in Proposition 2.

Proposition 2. *Given a BCN (2) with m inputs and m outputs, if the BCN is one-step transition input/output decoupled then it is input/output decoupled.*

Proof: Let i be arbitrary in $[1, m]$ and assume that the BCN is one step transition input/output decoupled. We want to show that for every initial state $X(0) \in \mathcal{B}^n$, if $U(t)$ and $\hat{U}(t), t \in \mathbb{Z}_+$, are two input sequences satisfying (3), then the corresponding outputs, $Y(t)$ and $\hat{Y}(t), t \in \mathbb{Z}_+$, satisfy (4). We prove the result by induction on t. To this end we let $X(t)$ and $\hat{X}(t), t \in \mathbb{Z}_+$, be the state sequences generated by the BCN starting from $X(0)$ and corresponding to $U(t)$ and $\hat{U}(t), t \in \mathbb{Z}_+$, respectively.

If $t = 0$ then obviously $[h(X(0))]_i = [h(\hat{X}(0))]_i$, on the other hand $U_i(0) = \hat{U}_i(0)$, and hence by the one-step transition input/output decoupling assumption

$$Y_i(1) = [h(f(X(0), U(0)))]_i = [h(f(X(0), \hat{U}(0)))]_i = \hat{Y}_i(1).$$

So, assume now that the result is true for every $t \leq \tau$, namely that $Y_i(t) = \hat{Y}_i(t)$ for every $t \leq \tau$. We want to prove that $Y_i(\tau+1) = \hat{Y}_i(\tau+1)$. Since $Y_i(\tau) = \hat{Y}_i(\tau)$, this means that $[h(X(\tau))]_i = [h(\hat{X}(\tau))]_i$. On the other hand $U_i(\tau) = \hat{U}_i(\tau)$, and hence by the (one-step transition input/output decoupling) assumption $Y_i(\tau+1) = [h(f(X(\tau), U(\tau)))]_i = [h(f(\hat{X}(\tau), \hat{U}(\tau)))]_i = \hat{Y}_i(\tau+1)$. This completes the proof. ∎

Example 3. *Consider the BCN of Example 1. It is a matter of simple calculations to verify that the BCN is also one-step transition input/output decoupled.*

Finally, we show that a weaker version of the one-step transition input/output decoupling property represents a necessary condition for input/output decoupling.

Proposition 3. *Given a BCN (2) with m inputs and m outputs, if the BCN is input/output decoupled then, for every index $i \in [1, m]$,* *every state $X_0 \in \mathcal{B}^n$ and every pair of input vectors $U, \hat{U} \in \mathcal{B}^m$ satisfying condition*

$$U_i = \hat{U}_i, \tag{8}$$

ensure that

$$[h(f(X_0, U))]_i = [h(f(X_0, \hat{U}))]_i. \tag{9}$$

Proof: If for some index $i = i^* \in [1, m]$ a state X_0 and two input vectors U and $\hat{U}$ could be found for which (8) holds, but (9) does not, then, assuming $X(0) = X_0$, every pair of input sequences $U(t)$ and $\hat{U}(t), t \in \mathbb{Z}_+$, satisfying the following conditions:

$$U(0) = U, \qquad \hat{U}(0) = \hat{U}, \qquad U(t) = \hat{U}(t), \ \ \forall\, t \geq 1,$$

would make condition (4) violated for $i = i^*$ at $t = 1$. ∎

While proving the mutual relationship between these properties by referring to the Boolean description (2) is a difficult task to achieve, in the next section we will show how the algebraic representation of BCNs allows to provide a characterization of all the properties introduced in this section, and hence to understand how they are mutually related. Specifically, it will be shown that Proposition 2 cannot be reversed, and hence BCNs can be found that are input/output decoupled but not one-step transition input/output decoupled. Meanwhile the necessary condition for input/output decoupling given in Proposition 3 will turn out to be also sufficient.

3 The algebraic representation of BCNs and the characterization of the decoupling properties

The algebraic representation of a BCN introduced in [12–14] is based on two fundamental ideas: the possibility of representing Boolean vectors by means of canonical vectors and the use of the semi-tensor product $\ltimes$. As a result, logical relations among Boolean vectors are expressed as algebraic equations and BCNs are converted into discrete-time bilinear systems. Indeed, every BCN (2) can be described [12] as

$$\begin{aligned} \mathbf{x}(t+1) &= L \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t), \\ \mathbf{y}(t) &= H \ltimes \mathbf{x}(t) = H\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \end{aligned} \tag{10}$$

where $\mathbf{x}(t) \in \mathcal{L}_N, \mathbf{u}(t)$ and $\mathbf{y}(t) \in \mathcal{L}_M$, with $N := 2^n$ and $M := 2^m$. $L \in \mathcal{L}_{N \times NM}$ and $H \in \mathcal{L}_{M \times N}$ are matrices whose columns are canonical vectors of size N and M, respectively. For every choice of the input variable at time t, namely for every $\mathbf{u}(t) = \delta_M^j, j \in [1, M]$, $L \ltimes \mathbf{u}(t) =: L_j$ is a matrix in $\mathcal{L}_{N \times N}$. So, we can think of the state equation of the BCN (10) as a Boolean switched system [29],

$$\mathbf{x}(t+1) = L_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{11}$$

where $\sigma(t), t \in \mathbb{Z}_+$, is a switching sequence taking values in $[1, M]$. For every $j \in [1, M]$, the Boolean Network

$$\mathbf{x}(t+1) = L_j\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{12}$$

represents the jth subsystem of (11).

In order to provide a characterization of input/output decoupling we introduce, for every $i \in [1, m]$ and every $b \in \mathcal{B} = \{0, 1\}$, the following sets (see (1) for the definition of $\mathbb{B}_m$)

$$\begin{aligned} I_{U_i=b} &:= \{j \in [1, M] : [\mathbb{B}_m(\delta_M^j)]_i = b\}, & (13)\\ I^X_{Y_i=b} &:= \{q \in [1, N] : [\mathbb{B}_m(H\delta_N^q)]_i = b\}. & (14) \end{aligned}$$

Note that there is a major difference between these two sets. Indeed, $I_{U_i=b}$ represents the set of canonical input vectors whose Boolean equivalent has the ith entry equal to b, while $I^X_{Y_i=b}$ represents the set of canonical state vectors whose associated output has Boolean equivalent with ith entry equal to b. So, while in the former case we simply refer to the Boolean equivalent of a canonical vector, in the latter case we refer to the Boolean equivalent of its H-image. We observe that

$$\begin{aligned} I_{U_i=0} \cap I_{U_i=1} &= \emptyset, & I_{U_i=0} \cup I_{U_i=1} &= [1, M], \\ I^X_{Y_i=0} \cap I^X_{Y_i=1} &= \emptyset, & I^X_{Y_i=0} \cup I^X_{Y_i=1} &= [1, N]. \end{aligned}$$

So, in both cases we have a partition of the set of possible indices. However, while $|I_{U_i=0}| = |I_{U_i=1}| = M/2 = 2^{m-1}$, in general the sets $I^X_{Y_i=b}, b \in \mathcal{B}$, have arbitrary cardinality.

Given $i \in [1, m]$, we say that indices $\ell, j \in [1, M]$ *belong to the same ith input class* (and denote it by $\ell \underset{I_{U_i}}{\sim} j$), if there exists $b \in \mathcal{B}$ such that $\ell, j \in I_{U_i=b}$. This amounts to saying that the Boolean equivalents of the two canonical (input) vectors have the same ith entry, i.e., $[\mathbb{B}_m(\delta_M^\ell)]_i = [\mathbb{B}_m(\delta_M^j)]_i$.

Similarly, two indices $p, q \in [1, N]$ *belong to the same ith output indistinguishability class* (and denote it by $p \underset{I^X_{Y_i}}{\sim} q$), if there exists $b \in \mathcal{B}$ such that $p, q \in I^X_{Y_i=b}$. This amounts to saying that the Boolean equivalents of the outputs associated with the two canonical (state) vectors have the same ith entry, i.e., $[\mathbb{B}_m(H\delta_N^p)]_i = [\mathbb{B}_m(H\delta_N^q)]_i$.

By referring to the above sets and notation, it is immediate to restate the notion of input/output decoupled BCN (Definition 1) we introduced in the previous section in terms of the algebraic description (10).

Proposition 4. *A BCN (10) is input/output decoupled if and only if for every $i \in [1, m]$ and every $\mathbf{x}(0) \in \mathcal{L}_N$, we have that every pair of input sequences $\mathbf{u}(t) = \delta_M^{\ell_t}, t \in \mathbb{Z}_+$, and $\hat{\mathbf{u}}(t) = \delta_M^{j_t}, t \in \mathbb{Z}_+$, satisfying*

$$\ell_t \underset{I_{U_i}}{\sim} j_t, \qquad \forall\, t \in \mathbb{Z}_+, \tag{15}$$

ensure that the two corresponding state sequences $\mathbf{x}(t) = \delta_N^{p_t}, t \in \mathbb{Z}_+$, and $\hat{\mathbf{x}}(t) = \delta_N^{q_t}, t \in \mathbb{Z}_+$, satisfy

$$p_t \underset{I^X_{Y_i}}{\sim} q_t, \qquad \forall\, t \in \mathbb{Z}_+. \tag{16}$$

In order to determine a practical way to verify whether the previous equivalent characterization holds for every index $i \in [1, m]$, we proceed as follows. Assume that i is a fixed index in $[1, m]$. We define three Boolean matrices:

$H_i \in \mathcal{L}_{2 \times N}$ is the logical matrix whose pth column is $\delta_2^1 = \mathbb{C}_1(1)$ if $p \in I^X_{Y_i=1}$, and is $\delta_2^2 = \mathbb{C}_1(0)$ if $p \in I^X_{Y_i=0}$;

M_{i1} is the Boolean sum of all the blocks $L_j, j \in [1, M]$, that correspond to indices $j \in I_{U_i=1}$, while

M_{i0} is the Boolean sum of all the blocks $L_j, j \in [1, M]$, that correspond to indices $j \in I_{U_i=0}$. In formulas

$$\mathrm{col}_p(H_i) = H_i\delta_N^p \ := \begin{cases} \delta_2^1, & p \in I^X_{Y_i=1}, \\ \delta_2^2, & p \in I^X_{Y_i=0}; \end{cases} \tag{17}$$

$$M_{i1} := \bigvee_{j \in I_{U_i=1}} L_j; \tag{18}$$

$$M_{i0} := \bigvee_{j \in I_{U_i=0}} L_j. \tag{19}$$

Note that both M_{i1} and M_{i0} are Boolean matrices devoid of zero columns. Before proceeding, we would like to comment on the meaning of the previous matrices. The logical matrix H_i defines a map from $\mathcal{L}_N$ to $\mathcal{L}_2$ whose purpose is to indicate whether a certain state vector δ_N^p generates an output vector whose Boolean equivalent has ith entry which is unitary ($H_i\delta_N^p = \delta_2^1$) or zero ($H_i\delta_N^p = \delta_2^2$). The matrix M_{i1} "assembles" all the matrices L_j that describe the action on the state vectors of inputs whose Boolean equivalent has unitary ith entry, while M_{i0} does the same for all the matrices L_j associated with input vectors δ_M^j whose Boolean equivalent has zero ith entry. Clearly, M_{i1} and M_{i0} are Boolean, not necessarily logical, matrices and $M_{ib}\delta_N^p$, $b \in \mathcal{B}, p \in [1, N]$, is a Boolean vector whose qth entry is 1 if and only if there exists an input vector, whose Boolean equivalent has ith entry equal to b, that makes it possible the transition from δ_N^p to δ_N^q. By making use of the previous matrices we can provide a necessary condition for input/output decoupling.

Proposition 5. *If a BCN* (10) *is input/output decoupled then for every $i \in [1, m]$ the two matrices H_iM_{i1} and H_iM_{i0} are logical matrices of size $2 \times N$.*

Proof: By Proposition 3 we know that when a BCN is input/output decoupled then, for every index $i \in [1, m]$ and every state $X_0 \in \mathcal{B}^n$, all input vectors $U \in \mathcal{B}^m$ having the same value of ith entry lead to states $f(X_0, U)$ whose associated outputs share the same ith entry. If we refer to the algebraic representation of the BCN (10), this means that for every $i \in [1, m]$ and every $p \in [1, N]$, the pth columns of all matrices $H_iL_j, j \in I_{U_i=1}$, must be either all equal to δ_2^1 or all equal to δ_2^2. But this amounts to saying that the pth column of H_iM_{i1} must be a canonical vector, and since this must be true for every $p \in [1, N]$, this corresponds to saying that H_iM_{i1} is a logical matrix, namely $H_iM_{i1} \in \mathcal{L}_{2\times N}$. Obviously the same argument applies to the case when we consider the columns of the matrices $H_iL_j, j \in I_{U_i=0}$, and this leads to the condition $H_iM_{i0} \in \mathcal{L}_{2\times N}$. ■

By proceeding along the same lines, we can provide an equivalent characterization of the one step transition input/output decoupling property, in terms of matrices H_i, M_{i1} and $M_{i0}, i \in [1, m]$.

Proposition 6. *A BCN* (10) *is one step transition input/output decoupled if and only if, for every $i \in [1, m]$,*

i) H_iM_{i1} and H_iM_{i0} are $2 \times N$ logical matrices;
ii) for every $p, q \in [1, N]$, condition $H_i\delta_N^\ell = H_i\delta_N^j$ implies

$$H_iM_{i1}\delta_N^p = H_iM_{i1}\delta_N^q \qquad \text{and} \qquad H_iM_{i0}\delta_N^p = H_iM_{i0}\delta_N^q.$$

Proof: By definition, a BCN is one-step transition input/output decoupled if, for every index $i \in [1, m]$, every pair of states $X, \hat{X} \in \mathcal{B}^n$ and every pair of input vectors $U, \hat{U} \in \mathcal{B}^m$ satisfying conditions $[h(X)]_i = [h(\hat{X})]_i$, and $U_i = \hat{U}_i$, ensure that $[h(f(X, U))]_i = [h(f(\hat{X}, \hat{U}))]_i$.

We observe that if we refer to the algebraic representation (10), and we denote the states by $\mathbf{x} = \delta_N^p$ and $\hat{\mathbf{x}} = \delta_N^q$, while the inputs by $\mathbf{u} = \delta_M^\ell$ and $\hat{\mathbf{u}} = \delta_M^j$, then the previous definition becomes: if $H_i\delta_N^p = H_i\delta_N^q$ and $\ell, j \in I_{U_i=b}$ for some $b \in \mathcal{B}$, then $H_iL_\ell\delta_N^p = H_iL_j\delta_N^q$.

[Necessity] If the BCN is one-step transition input/output decoupled, then it is input/output decoupled and hence condition i) holds by Proposition 5. On the other hand, it is easy to see that if $H_i\delta_N^p = H_i\delta_N^q$ and $\ell, j \in I_{U_i=b}$ for some $b \in \mathcal{B}$ ensure that $H_iL_\ell\delta_N^p = H_iL_j\delta_N^q$, then, in particular, $H_iL_\ell\delta^p$ takes the same value for every $\ell \in I_{U_i=b}$, which leads to saying that $H_iM_{ib}\delta^p = \vee_{\ell\in I_{U_i=b}} H_iL_\ell\delta_N^p$ is a canonical vector. On the other hand, by the same reasoning, also $H_iM_{ib}\delta^q$ is a canonical vector and it must coincide with $H_iM_{ib}\delta^p$. Therefore condition $H_i\delta_N^p = H_i\delta_N^q$ implies

$$H_iM_{ib}\delta_N^p = H_iM_{ib}\delta_N^q.$$

And since b can be arbitrarily chosen in $\mathcal{B}$, condition ii) follows.

[Sufficiency] The fact that i) and ii) imply that the BCN (10) is one-step transition input/output decoupled is easily proved by reversing the previous reasoning. ■

Example 4. *Consider the BCN of Example* 1. *It is a matter of simple calculations to verify that the BCN can be described as in* (10) *for $L = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \end{bmatrix}$, with*

$$\begin{aligned} L_1 &:= \begin{bmatrix} \delta_8^1 & \delta_8^1 & \delta_8^3 & \delta_8^3 & \delta_8^5 & \delta_8^5 & \delta_8^5 & \delta_8^5 \end{bmatrix}, \\ L_2 &:= \begin{bmatrix} \delta_8^2 & \delta_8^1 & \delta_8^4 & \delta_8^3 & \delta_8^6 & \delta_8^5 & \delta_8^8 & \delta_8^7 \end{bmatrix}, \\ L_3 &:= \begin{bmatrix} \delta_8^3 & \delta_8^3 & \delta_8^3 & \delta_8^3 & \delta_8^7 & \delta_8^7 & \delta_8^7 & \delta_8^7 \end{bmatrix}, \\ L_4 &:= \begin{bmatrix} \delta_8^4 & \delta_8^3 & \delta_8^4 & \delta_8^3 & \delta_8^8 & \delta_8^7 & \delta_8^8 & \delta_8^7 \end{bmatrix}, \end{aligned}$$

and $H = \begin{bmatrix} \delta_4^1 & \delta_4^2 & \delta_4^3 & \delta_4^4 & \delta_4^3 & \delta_4^4 & \delta_4^3 & \delta_4^4 \end{bmatrix}$. The index sets are $I_{U_1=1} = \{1, 2\}, I_{U_1=0} = \{3, 4\}, I_{U_2=1} = \{1, 3\}$, $I_{U_2=0} = \{2, 4\}, I^X_{Y_1=1} = \{1, 2\}, I^X_{Y_1=0} = \{3, 4, 5, 6, 7, 8\}, I^X_{Y_2=1} = \{1, 3, 5, 7\}$, and $I^X_{Y_2=0} = \{2, 4, 6, 8\}$. Therefore we have

$$\begin{aligned} H_1 &:= \begin{bmatrix} \delta_2^1 & \delta_2^1 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 \end{bmatrix}, \\ H_2 &:= \begin{bmatrix} \delta_2^1 & \delta_2^2 & \delta_2^1 & \delta_2^2 & \delta_2^1 & \delta_2^2 & \delta_2^1 & \delta_2^2 \end{bmatrix}, \end{aligned}$$

$M_{11} = L_1 \vee L_2, M_{10} = L_3 \vee L_4, M_{21} = L_1 \vee L_3$, and $M_{20} = L_2 \vee L_4$. Again, simple calculations show that the characterization given in Proposition 6 *holds, and hence the BCN is one-step transition input/output decoupled (and hence input/output decoupled).*

Example 5. *Consider the BCN with $m = 2$ inputs and outputs and $n = 3$ state variables, described as in* (10) *for $L = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \end{bmatrix}$, with*

$$\begin{aligned} L_1 &:= \begin{bmatrix} \delta_8^1 & \delta_8^3 & \delta_8^1 & \delta_8^2 & \delta_8^3 & \delta_8^5 & \delta_8^6 & \delta_8^7 \end{bmatrix}, \\ L_2 &:= \begin{bmatrix} \delta_8^2 & \delta_8^3 & \delta_8^2 & \delta_8^1 & \delta_8^4 & \delta_8^5 & \delta_8^6 & \delta_8^8 \end{bmatrix}, \\ L_3 &:= L_2, \quad L_4 := L_1, \end{aligned}$$

and $H = \begin{bmatrix} \delta_4^1 & \delta_4^1 & \delta_4^3 & \delta_4^3 & \delta_4^2 & \delta_4^4 & \delta_4^4 & \delta_4^4 \end{bmatrix}$. The index sets are the same as in Example 4 *and we have*

$$\begin{aligned} H_1 &:= \begin{bmatrix} \delta_2^1 & \delta_2^1 & \delta_2^2 & \delta_2^2 & \delta_2^1 & \delta_2^2 & \delta_2^2 & \delta_2^2 \end{bmatrix}, \\ H_2 &:= \begin{bmatrix} \delta_2^1 & \delta_2^1 & \delta_2^1 & \delta_2^1 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 \end{bmatrix}, \end{aligned}$$

$M_{11} = L_1 \vee L_2, M_{10}L_3 \vee L_4, M_{21} = L_1 \vee L_3$, and $M_{20} = L_2 \vee L_4$. Simple calculations show that

$$\begin{aligned} H_1M_{11} &= \begin{bmatrix} \delta_2^1 & \delta_2^2 & \delta_2^1 & \delta_2^1 & \delta_2^2 & \delta_2^1 & \delta_2^2 & \delta_2^2 \end{bmatrix} = H_1M_{10}, \\ H_2M_{21} &= \begin{bmatrix} \delta_2^1 & \delta_2^1 & \delta_2^1 & \delta_2^1 & \delta_2^1 & \delta_2^2 & \delta_2^2 & \delta_2^2 \end{bmatrix} = H_2M_{20}. \end{aligned}$$

So, clearly, condition i) in Proposition 6 *holds, but condition ii) does not (indeed, columns 1 and 2 of matrix H_1 coincide, but columns 1 and 2 of matrix H_1M_{11} do not), and hence the BCN verifies the necessary condition for input/output decoupling given in Propositon 5, but it is not one-step transition input/output decoupled.*

Proposition 6 and the previous example show that one-step transition input/output decoupling and the necessary condition for input/output decoupling proposed in Propositon 5 are not equivalent. So, the question naturally arises: is input/output decoupling an intermediate property, different from the other two, or is it equivalent to either one of them? Proposition 7 provides an algebraic characterization of input/output decoupling that will allow to answer this question.

Proposition 7. *A BCN* (10) *is input/output decoupled if and only if for every $i \in [1, m]$, every $k \in \mathbb{Z}_+, k \geq 1$, and every choice of k indices $b_1, b_2, \ldots, b_k \in \mathcal{B}$, the matrix $H_iM_{i,b_k} \ldots M_{i,b_2}M_{i,b_1}$ is a logical matrix.*

Proof: As shown in Proposition 4, a BCN (10) is input/output decoupled if and only if for every $i \in [1, m]$ and every $\mathbf{x}(0) = \delta_N^r \in \mathcal{L}_N$, we have that every pair of input sequences $\mathbf{u}(t) = \delta_M^{\ell_t}, t \in \mathbb{Z}_+$, and $\hat{\mathbf{u}}(t) = \delta_M^{j_t}, t \in \mathbb{Z}_+$, satisfying (15) ensure that the two corresponding state sequences $\mathbf{x}(t) = \delta_N^{p_t}, t \in \mathbb{Z}_+$, and $\hat{\mathbf{x}}(t) = \delta_N^{q_t}, t \in \mathbb{Z}_+$, satisfy (16). For every $t \in \mathbb{Z}_+$, the state variables $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ can be expressed as

$$\mathbf{x}(t) = L_{\ell_{t-1}} \dots L_{\ell_1} L_{\ell_0} \delta_N^r, \quad \hat{\mathbf{x}}(t) = L_{j_{t-1}} \dots L_{j_1} L_{j_0} \delta_N^r,$$

respectively. Therefore condition (16) corresponds to

$$H_i L_{\ell_{t-1}} \dots L_{\ell_1} L_{\ell_0} \delta_N^r = H_i L_{j_{t-1}} \dots L_{j_1} L_{j_0} \delta_N^r,$$

namely to ask that the two vectors are identical canonical vectors. If we define b_{t+1} as the uniquely determined Boolean value such that $\ell_t, j_t \in I_{U_i=b_{t+1}}$, then the previous identity holds if and only if $H_i M_{i,b_t} \dots M_{i,b_2} M_{i,b_1} \delta_N^r$ is a canonical vector for every $t \in \mathbb{Z}_+, t \geq 1$, and since r is arbitrary in $[1, N]$, this means that the matrix $H_i M_{i,b_t} \dots M_{i,b_2} M_{i,b_1}$ must be logical. ■

Theorem 1. *A BCN* (10) *is input/output decoupled if and only for every $i \in [1, m]$ the two matrices $H_i M_{i1}$ and $H_i M_{i0}$ are logical matrices of size* $2 \times N$.

Proof: By Proposition 5, we already know that the necessity part is true. So, we need only to prove the sufficiency. To this end, we set $i \in [1, m]$ to a fixed value and proceed as follows. First of all, by the way the matrix H_i has been defined, we know that a permutation matrix $\Pi_i \in \mathcal{L}_{N \times N}$ exists such that

$$\hat{H}_i := H_i \Pi_i^\top = [\underbrace{\delta_N^1 \ \dots \ \delta_N^1}_{n_i} \mid \underbrace{\delta_N^2 \ \dots \ \delta_N^2}_{N-n_i}],$$

where $n_i := |I_{Y_i=1}^X|$. This amounts to sort all the canonical vectors representing the states so that the first n_i are those associated with output vectors whose Boolean equivalent has unitary ith entry. Correspondingly, we get

$$\hat{M}_{ib} \ := \ \Pi_i M_{ib} \Pi_i^\top = \begin{bmatrix} M_{ib}^{(11)} & M_{ib}^{(12)} \\ M_{ib}^{(21)} & M_{ib}^{(22)} \end{bmatrix}, \qquad b \in \mathcal{B}.$$

Clearly, for every $k \in \mathbb{Z}_+, k \geq 1$, and every choice of k indices $b_1, b_2, \dots, b_k \in \mathcal{B}$,

$$\hat{H}_i \hat{M}_{i,b_k} \dots \hat{M}_{i,b_2} \hat{M}_{i,b_1} = H_i M_{i,b_k} \dots M_{i,b_2} M_{i,b_1} \Pi_i^\top,$$

and $H_i M_{i,b_k} \dots M_{i,b_2} M_{i,b_1}$ is a logical matrix if and only if $\hat{H}_i \hat{M}_{i,b_k} \dots \hat{M}_{i,b_2} \hat{M}_{i,b_1}$ is a logical matrix. So, in the following we will assume $H_i = \hat{H}_i$ and $M_{ib} = \hat{M}_{ib}, b \in \mathcal{B}$. If the hypothesis holds, namely $H_i M_{i1}$ and $H_i M_{i0}$ are logical matrices, this implies that each matrix $M_{ib}, b \in \mathcal{B}$, satisfies the following conditions:

- one of the two blocks $M_{ib}^{(11)}$ and $M_{ib}^{(21)}$ is zero;
- one of the two blocks $M_{ib}^{(12)}$ and $M_{ib}^{(22)}$ is zero.

We want to prove that $H_i M_{i,b_k} \dots M_{i,b_2} M_{i,b_1}$ is a logical matrix for every $k \in \mathbb{Z}_+, k \geq 1$. To this goal, we proceed by induction. By assumption we know that this is true for $k = 1$. We assume that the result is true for $k = \bar{k} - 1$ and show that this is true for $k = \bar{k}$. Consider the matrix product $H_i M_{i,b_{\bar{k}}} M_{i,b_{\bar{k}-1}} \dots M_{i,b_2} M_{i,b_1}$ and set $W_i := M_{i,b_{\bar{k}}} M_{i,b_{\bar{k}-1}} \dots M_{i,b_2}$.

By inductive assumption, $H_i W_i$ is a logical matrix, and this ensures that

$$W_i = \begin{bmatrix} W_i^{(11)} & W_i^{(12)} \\ W_i^{(21)} & W_i^{(22)} \end{bmatrix}$$

is such that one of the two blocks $W_i^{(11)}$ and $W_i^{(21)}$ is zero, and one of the two blocks $W_i^{(12)}$ and $W_i^{(22)}$ is zero. But then it is immediate to see that the blocks of

$$\begin{aligned} V_i \quad &= \quad \begin{bmatrix} V_i^{(11)} & V_i^{(12)} \\ V_i^{(21)} & V_i^{(22)} \end{bmatrix} := M_{i,b_{\bar{k}}} M_{i,b_{\bar{k}-1}} \dots M_{i,b_2} M_{i,b_1} \\ &= \quad \begin{bmatrix} W_i^{(11)} & W_i^{(12)} \\ W_i^{(21)} & W_i^{(22)} \end{bmatrix} \begin{bmatrix} M_{ib_1}^{(11)} & M_{ib_1}^{(12)} \\ M_{ib_1}^{(21)} & M_{ib_1}^{(22)} \end{bmatrix} \end{aligned}$$

satisfy the usual condition (one of the two blocks $V_i^{(11)}$ and $V_i^{(21)}$ is zero, and one of the two blocks $V_i^{(12)}$ and $V_i^{(22)}$ is zero) thus ensuring that $H_i V_i = H_i M_{i,b_{\bar{k}}} M_{i,b_{\bar{k}-1}} \dots M_{i,b_2} M_{i,b_1}$ is logical. ■

Remark 1. *It is worth noticing that the input/output decoupling property can be tested on the algebraic representation of a BCN by simply evaluating the entries of $2m$ matrices of size $N \times N$. Compared to other criteria based on the algebraic representation* (10)*, that require to compute and inspect a number of matrices that grows with $M = 2^m$ or even a power of M and N (see, e.g., [18, 30]), this criterion is particularly simple and efficient from a computational viewpoint.*

4 Graph-theoretic characterizations of input/output decoupling properties

Given a BCN (10) with $N = 2^n$ states and $M = 2^m$ inputs and outputs, it is possible to associate with it m directed graphs $\mathcal{D}_i, i \in [1, m]$. For each index i, the digraph $\mathcal{D}_i$ has N vertices, denoted by $\{1, 2, \dots, N\}$, and M arcs of two distinct types, obtained in this way: there is an *arc of type* 1 from p to q, with $p, q \in [1, N]$, if and only if $[M_{i1} \delta_N^p]_q = 1$, and there is an *arc of type* 0 from p to q, with $p, q \in [1, N]$, if and only if $[M_{i0} \delta_N^p]_q = 1$. In other words, arcs of type 1 (0, resp.) in the digraph $\mathcal{D}_i$ are those associated with state transitions corresponding to inputs $\mathbf{u} = \delta_M^j$ with $j \in I_{U_i=1}$ ($j \in I_{U_i=0}$, resp.). We note that $\mathcal{D}_i$ is simply the union of the two digraphs $\mathcal{D}(M_{i1})$ and $\mathcal{D}(M_{i0})$, in which, however, we keep track of the specific matrix each arc is associated with.

Finally, we partition the vertices in $\mathcal{D}_i$ into the two ith output indistiguishability classes: the class $I_{Y_i=1}^X$ and the class $I_{Y_i=0}^X$.

By referring to the digraphs $\mathcal{D}_i, i \in [1, m]$, we can characterize both forms of input/ouput decoupling.

Proposition 8. *Consider a BCN* (10) *with $N = 2^n$ states and $M = 2^m$ inputs and outputs, and let $\mathcal{D}_i, i \in [1, m]$, be the associated directed graphs. The BCN is input/output decoupled if and only if for every $i \in [1, m]$, every $b \in \mathcal{B}$ and every vertex $p \in [1, N]$, in the digraph $\mathcal{D}_i$ all arcs of type b starting from the vertex p end in the same ith output indistiguishability class.*

Proof: This trivially follows from the fact that $H_i M_{ib}$ is a logical matrix if and only if each vertex $p \in [1, N]$ is mapped by all arcs of type b either into vertices belonging to $I_{Y_i=1}^X$ (if $\mathrm{col}_p(H_i M_{ib}) = \delta_2^1$) or into vertices belonging to $I_{Y_i=0}^X$ (if $\mathrm{col}_p(H_i M_{ib}) = \delta_2^2$), but not into both classes of vertices. ■

Proposition 9. *Consider a BCN* (10) *with $N = 2^n$ states and $M = 2^m$ inputs and outputs, and let $\mathcal{D}_i, i \in [1, m]$, be the associated directed graphs. The BCN is one-step transition input/output decoupled if and only if for every $i \in [1, m]$ and every $b_1, b_2 \in \mathcal{B}$,* *in the digraph $\mathcal{D}_i$ all arcs of type b_1 starting from the vertices of the class $I^X_{Y_i=b_2}$ end in the same ith output indistiguishability class (either $I^X_{Y_i=b_2}$ or $I^X_{Y_i=\bar{b}_2}$).*

Proof: A BCN (10) is one-step transition input/output decoupled if and only if conditions i) and ii) of Proposition 6 hold. Condition i) is equivalent to the input/output decoupling property of the BCN, and this ensures (see Proposition 8) that for every $i \in [1, m]$, every $b_1 \in \mathcal{B}$ and every vertex $p \in [1, N]$, in the digraph $\mathcal{D}_i$ all arcs of type b_1 starting from the vertex p end in the same ith output indistiguishability class. Condition ii) amounts to saying that for each vertex $q \in [1, N]$, with $q \overset{\sim}{_{I^X_{Y_i}}} p$, all the outgoing arcs of type b_1 end in the same ith output indistiguishability class as the arcs of type b_1 leaving from p. ■

Example 6. *Consider the BCN of Example 5. It is easy to see that in this specific case: (a) $\mathcal{D}_1 = \mathcal{D}_2$; (b) in each digraph $\mathcal{D}_i$ the set of arcs of type 1 coincides with the set of arcs of type 0 (since $M_{11} = M_{10}$ and $M_{21} = M_{20}$). So, the only thing that changes is the partition into indistinguishability classes. The digraph $\mathcal{D}_1$ is in Figure 1: the class $I_{Y_1=1}$ consists of vertices $\{1, 2, 5\}$, while the class $I_{Y_1=0}$ consists of vertices $\{3, 4, 6, 7, 8\}$. On the other hand, the digraph $\mathcal{D}_2$ is in Figure 2: the class $I_{Y_2=1}$ consists of vertices $\{1, 2, 3, 4\}$, while the class $I_{Y_2=0}$ consists of vertices $\{5, 6, 7, 8\}$.*

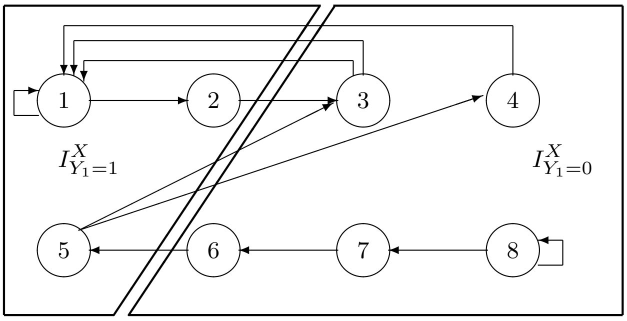

Fig. 1: The graph $\mathcal{D}_1$ for Example 6.

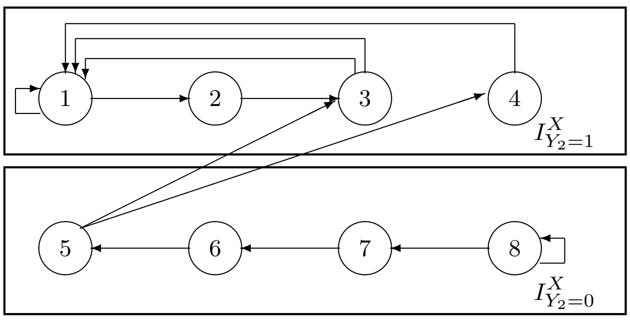

Fig. 2: The graph $\mathcal{D}_2$ for Example 6.

It is immediately apparent that, in both graphs, all arcs leaving a vertex end in a single indistinguishability class and not in two indistinguishability classes. Therefore the BCN is input/output decoupled. However, it it not true that $p, q \in I^X_{Y_i=b}, \exists\, b \in \mathcal{B}$, ensures that all arcs of the same type leaving p and q end in the same indistinguishability class. For instance, in $\mathcal{D}_1$ all arcs leaving 1 end in $I^X_{Y_1=1}$, while all arcs leaving 5 end in $I^X_{Y_1=0}$, even if $1 \overset{\sim}{_{I^X_{Y_1}}} 5$. Consequently, the BCN is not one-step transition input/output decoupled.

5 Conclusions

In this paper we have introduced two types of input/output decoupling properties by referring to the classical representation of a Boolean Control Network in terms of Boolean input, state and output vectors, whose mutual relationships are expressed through logical operators. By resorting to the algebraic representation of a BCN, a complete characterization of these properties has been obtained, thus showing that the one-step transition input/output decoupling property is stronger than the input/output decoupling property. The algebraic characterizations derived through the algebraic approach have led to easy to check testing algorithms. At the end of the paper, equivalent conditions based on certain associated digraphs for these properties to hold have been presented.

It is worth noticing that the input/output decoupling problem has been investigated in the paper by imposing that, for every index i, the ith output only depends on the ith input. However, the result can be easily adjusted to the case of an input-reordering. Indeed if σ represents a permutation of the set $[1, m]$, then the case when the ith output only depends on the $\sigma(i)$th input can be characterised by replacing the matrices $H_i M_{ib}, b \in \mathcal{B}$, with the matrices $H_i M_{\sigma(i)b}$, $b \in \mathcal{B}$.

Future research efforts will aim at determining necessary and sufficient conditions for the existence of state-feedback control laws that make a given BCN input/output decoupled, and to clarify under what additional conditions input/output decoupling necessarily imposes also a partition of the state variables into disjoint groups.

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