On the effects of communication failures in a multi-agent consensus network

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***Abstract*—This paper investigates the effects of either an edge or a node disconnection on a multi-agent consensus network, consisting of N agents that are modeled as simple scalar and discrete-time integrators. The communication among the agents is described by a weighted undirected graph. The first part of the paper addresses the case of an edge disconnection and summarizes the main results obtained in [29]. In particular, we show that if an edge disconnection does not affect the connectedness of the whole network, it does not even affect the final consensus value. Discernibility of the faulty network from the original one is investigated both in case the states of all the agents are available and in case only the states of a subset of the agents are available. Several equivalent conditions are derived and it is proved that the necessary and sufficient conditions for discernibility and for discernibility from the observation of a subset of agents are exactly the same and can be checked on the original state matrix and on its eigenvectors. The second part of the paper provides some original results about the effects of a node disconnection. In general, even when the connectedness of the remaining communication graph (namely the graph describing the interactions of the remaining $N-1$ agents) is preserved, the network converges to a different consensus value. Also in this case, discernibility of the faulty network from the original one is investigated both in case the states of all the agents are available and in case only the states of a subset of the agents are available. Several equivalent conditions are provided to characterize both properties. Finally, a procedure to restore the original consensus value, after a node disconnection, is provided.**

***Index Terms*—Multi-agent system, consensus network, Laplacian matrix, edge disconnection, node disconnection.**

I. Introduction

During the last 10-15 years the interest in distributed control and coordination of networks of agents has increasingly grown [4], [5], [15], [24], [25]. Emerging networking applications that allow peer to peer communication, such as ad hoc wireless communication networks and sensor networks, stimulated research on multi-agent systems design with decentralized architectures [1].

One remarkable result of the recent developments on multi-agent systems is that it is often possible to achieve the same performance as a single, complex and expensive system by suitably designing a network of much cheaper and simpler devices [28]. One side effect of this alternative set-up is that each single node of the network may easily undergo a breakdown for various reasons, such as low power, a damage, an internal failure, and this phenomenon may alter or even disrupt the mission success.

On the other hand, it is also clear that the effectiveness and performance of distributed algorithms are intrinsically related to the communication graph structure, and a temporary or permanent interruption in the communication between two (or more) agents may seriously affect or even compromise the outcome of an algorithm.

A key tool for the design of distributed cooperation between agents is *consensus* on a shared variable, which is updated by performing an elaboration of local information. Consensus-based applications range in a wide number of technological fields, such as electrical power grids [27] and transportation networks [26], cooperative robotics, surveillance, and environmental monitoring [4], and, generally speaking, every time decentralized communication architectures are sought. In this context, an edge disconnection can be compensated through the remaining communication paths, provided that the communication graph after the disconnection remains connected. Similarly, a node disconnection can be comfortably managed by the network through the activity of the remaining nodes, even if this typically leads to converge to a final decision that is different from the one the system would have converged to, had the failure not occurred. Nonetheless it is important to detect such phenomena to prevent an incorrect computation or improper evolution of the network (see e.g. [18] Section II-B and references therein). Variations of the network topology can have a major impact on stability and/or performance, but most of all they can affect the network secure and reliable operation [2].

The effects of a topological variation in a network of linear systems, and in particular, the problem of detecting when a topological variation has occurred, have been the subject of some recent publications. In detail, in [19] some preliminary results about the detectability of a link failure in multi-agent systems described by simple continuous-time integrators and assuming directed and unweighted communication graphs have been provided. In [20] the previous results have been extended to the case of of multiple link failures, by assuming again that the agents are described by scalar integrators and the communication graph is directed and unweighted. Specifically, sufficient graphical conditions for the detectability of a group of edges in the network information flow digraph have been proposed. In [23], the same authors extend the FDI algorithms for efficient sensor location in a (directed or undirected) integrator network where link failures can be either unidirectional or bidirectional. Note that all the proposed results are based on sufficient conditions for detectability and identification. In [7] link failure detection is investigated using probabilistic inference. In [6] the problem of detecting and localizing changes in the dynamics of links in networks of LTI systems is studied without the knowledge of the dynamics of the network. Node disconnection and edge failure have been investigated in [2], where the detection problem has been characterized by means of algebraic conditions on the eigenspace components related to the nominal network topology. Specifically, the authors investigated discernibility of two distinct networks by making use either of the full state information or of the same linear output function. The results are first obtained for generic state space models and subsequently specialized for the case of multi-agent networks adopting a DeGroot's type consensus algorithm. In [21], the possibility of distinguishing digraphs from the output response of some observed agents in a multi-agent network under the agreement protocol is studied.

An experimental study on the effect of a node removal from a consensus-based WSN is reported in [13]. A survey on fault diagnosis approaches for WSN and a list of open research challenges are given in [30]. Detection algorithms in the framework of multi-agent systems, distributed computing and wireless sensor networks are typically useful to rearrange the mission of the team among the remaining nodes and to correct the network evolution according to the detected working condition ([18] Section II-B).

Another framework where topology variations detection algorithms can be effectively applied, to increase reliability, is the emerging field of cyber-physical systems. Indeed, these systems are often characterized by large dimensionality and geographical sparsity, so that a further threat for the network is the tampering by an intruder [17].

Finally, other interesting papers on the failure detection problem for a network of dynamical systems are [20], [22].

For a more complete overview of the problem of detecting topological variations in a network, the interested reader is referred to [2] and references therein.

The paper is organized as follows. In section II the original discrete-time consensus network is introduced. First, the main results obtained in [29] for edge disconnection are presented. Specifically, Section III formalizes the effects of an edge disconnection and shows that if the edge disconnection does not affect the connectedness of the whole network it does not even affect the final consensus value. Discernibility of the faulty network from the original one is investigated in Section IV, both in case the states of all the agents are available and in case only the states of a subset of the agents are available. Several equivalent conditions are derived and it is proved that the necessary and sufficient conditions for discernibility and for discernibility from the observation of a subset of agents are exactly the same and can be checked on the original state matrix and on its eigenvectors. The second part of the paper provides some original results about the effects of a node disconnection. Section V shows that even when the connectedness of the communication graph describing the interactions of the remaining $N-1$ agents is preserved, the network converges to a drifted value of consensus. Discernibility of the faulty network from the original one, after a node failure, is investigated in Section VI, both in case the states of all the agents are available and in case only the states of a subset of the agents are available. Several equivalent conditions are provided to characterize both properties. Finally, a procedure to restore the original consensus value, after a node disconnection, is provided in Section VII.

Despite the good number of necessary and sufficient conditions provided in the paper for the discernibility of the faulty network from the original one, both in case of edge disconnection and in case of node disconnection, both assuming that all state variables are measurable/accessible and assuming that only a subset of them is, the research on this subject is still at an early stage. Specifically, necessary and sufficient conditions for the fault detection and identification need to be provided, as well as algorithms that efficiently identify which edge or node got disconnected. Also, it would be interesting to identify classes of graphs for which fault detection and identification after an edge/node disconnection is always guaranteed. These open problems will be the subject of future research.

Notation. $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of nonnegative integer and real numbers, respectively. We let $\mathbf{e}_i$ denote the i-th element of the canonical basis in $\mathbb{R}^k$ (k being clear from the context), with all entries equal to zero except for the i-th one which is unitary. $\mathbf{1}_k$ denotes the k-dimensional real vector whose entries are all 1. Given a real matrix A, the (i,j)-th entry of A is denoted either by a_{ij} or by $[A]_{ij}$, and its *transpose* by $A^\top$. Given a vector $\mathbf{v}$, the i-th entry of $\mathbf{v}$ is denoted by v_i or by $[\mathbf{v}]_i$. The *spectrum* of $A \in \mathbb{R}^{n\times n}$, denoted by $\sigma(A)$, is the set of its eigenvalues and the *spectral radius* of A, denoted by ρ_A, is the maximum modulus of the elements of $\sigma(A)$. For a *nonnegative matrix* $A \in \mathbb{R}_+^{n\times n}$, i.e., a matrix whose entries are nonnegative real numbers, the spectral radius is always an eigenvalue. A nonnegative matrix $A \in \mathbb{R}_+^{n\times n}$ whose entries are all positive is called a *positive matrix.* A nonnegative nonzero matrix $A \in \mathbb{R}_+^{n\times n}, n > 1$, is *irreducible* if no permutation matrix P can be found such that

$$P^\top A P = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. By the Perron-Frobenius theorem [3], [8], [15], for an irreducible nonnegative matrix A the spectral radius ρ_A is a simple real dominant eigenvalue, and the corresponding left and right eigenvectors are positive. Also, nonnegative eigenvectors of a nonnegative irreducible matrix necessarily correspond to the spectral radius. Given $\mathbf{v} = [v_i] \in \mathbb{R}^n$, the symbol $\mathrm{diag}(\mathbf{v})$ denotes the n-dimensional diagonal matrix whose (i,i)-th entry is v_i.

An *undirected and weighted graph* $\mathcal{G}$ is a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}_+^{N\times N}$ is the matrix of the weights of $\mathcal{G}$. The (symmetric) matrix $\mathcal{A}$ is called *adjacency matrix* of the graph. The (i,j)-th entry (and hence the (j,i)-th entry) of $\mathcal{A}$, $[\mathcal{A}]_{ij}$, is nonzero if and only if the arc (j,i) belongs to $\mathcal{E}$. We assume that $[\mathcal{A}]_{ii} = 0$, for all $i \in \{1, 2, \dots, N\}$. A *path connecting* j *and* i is an ordered sequence of arcs $(j, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k), (i_k, i) \in \mathcal{E}$. The graph $\mathcal{G}$ is *connected* if, for every pair of vertices j and i, there is a path connecting them. $\mathcal{G}$ is connected if and only if its adjacency matrix $\mathcal{A}$ is irreducible.
The *Laplacian matrix* [9] associated with the adjacency matrix $\mathcal{A}$ is defined as $\mathcal{L} := \mathcal{C} - \mathcal{A}$, where $\mathcal{C}$ is the (diagonal) connectivity matrix, whose diagonal entries are the sums of the corresponding row entries of $\mathcal{A}$, namely $\mathcal{C} = \mathrm{diag}(\mathcal{A}\mathbf{1}_N)$. Clearly, as the adjacency matrix $\mathcal{A}$ is symmetric, the associated Laplacian $\mathcal{L}$ is symmetric, too.

II. The original consensus network

Consider a multi-agent system consisting of N agents, $N > 1$. The state of each i-th agent, $i \in \{1, .., N\}$, is described by the scalar variable x_i that updates according to the following simple discrete-time linear state-space model [16]:

$$x_i(t+1) = x_i(t) + v_i(t), \qquad t \in \mathbb{Z}_+,$$

where v_i represents the input of the i-th agent. The communication among the N agents is described by an undirected weighted graph $\mathcal{G}$ with adjacency matrix $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}_+^{N\times N}$. The entry $[\mathcal{A}]_{ij}$ of the matrix $\mathcal{A}$ is positive if the agents i and j exchange information, and is zero otherwise. Each i-th agent implements the following linear consensus protocol [16] to generate its input v_i at every time $t \in \mathbb{Z}_+$:

$$v_i(t) = \kappa \sum_{j=1}^{N} [\mathcal{A}]_{ij}(x_j(t) - x_i(t)), \tag{1}$$

where $\kappa > 0$ is a parameter known as *coupling strength*. If the agents' states are piled up to create the N-dimensional state vector $\mathbf{x} \in \mathbb{R}^N$, the overall multi-agent system updates according to the following equation

$$\mathbf{x}(t+1) = (I_N - \kappa\mathcal{L})\mathbf{x}(t) =: A\mathbf{x}(t), \tag{2}$$

where $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{N\times N}$ is the Laplacian associated with the adjacency matrix $\mathcal{A}$. The properties of the Laplacian ensure that $A\mathbf{1}_N = \mathbf{1}_N$, which implies that 1 is an eigenvalue of A. System (2) typically represents the case when agents/nodes exchange information with their neighbours, with the final goal of asymptotically converging to the same constant value. More formally, the multi-agent system (2) is a *consensus network* if for every initial state $\mathbf{x}(0)$ there exists $\alpha = \alpha(\mathbf{x}(0)) \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} \mathbf{x}(t) = \alpha\mathbf{1}_N. \tag{3}$$

α is the *consensus value* [15] for system (2), corresponding to the given initial state. If the communication graph is connected, namely $\mathcal{A}$ and hence $\mathcal{L}$ are irreducible [15], and the coupling strength κ satisfies:

$$0 < \kappa < \frac{1}{\max_{i=1,\dots,N} \ell_{ii}}, \tag{4}$$

ℓ_{ii} being the i-th diagonal entry of $\mathcal{L}$, system (2) is a consensus network [15], and the consensus value is

$$\alpha = \mathbf{w}_A^\top \mathbf{x}(0), \tag{5}$$

where $\mathbf{w}_A = \frac{1}{N}\mathbf{1}_N$ is the left eigenvector of A corresponding to 1 and satisfying $\mathbf{w}_A^\top \mathbf{1}_N = 1$. Therefore the consensus value represents the average value of the agents' initial conditions $x_i(0), i \in \{1, 2, \dots, N\}$.

Assumption 1. In the paper we assume that the communication graph $\mathcal{G}$ is connected (equivalently, $\mathcal{L}$, and hence $\mathcal{A}$, are irreducible matrices) and κ satisfies the inequalities in (4). Consequently, $A = I_N - \kappa\mathcal{L}$ is a nonnegative irreducible matrix. The Perron-Frobenius theorem and identity $A\mathbf{1}_N = \mathbf{1}_N$ ensure that 1 is a simple dominant eigenvalue of A. The eigenspace associated with the unitary eigenvalue is $\langle \mathbf{1}_N \rangle$, and the nonnegative eigenvectors of A necessarily correspond to $\lambda = 1$ and hence are scalar multiple of $\mathbf{1}_N$.

In [29] we have investigated the effects on consensus of an edge disconnection (namely the case when two agents stop exchanging information between each other) at some arbitrary time instant $\tau \geq 0$, and the possibility of detecting such a failure in the communication network. We will briefly recall here the most important results obtained in [29] and subsequently focus on the case of a node disconnection (namely the case when, at some time $t = \tau \geq 0$, one of the agents stops interacting with the others).

III. Consensus after an edge disconnection

If the communication exchange between agents r and h (in the undirected weighted graph $\mathcal{G}$) is interrupted, namely the arc $(r,h), r \neq h$, is disconnected, then the Laplacian $\bar{\mathcal{L}}_{hr}$ of the new digraph is related to the Laplacian $\mathcal{L} = [\ell_{ij}]$ of the original digraph $\mathcal{G}$ by the relationship

$$\bar{\mathcal{L}}_{hr} \quad = \quad \mathcal{L} + \ell_{hr}[\mathbf{e}_h - \mathbf{e}_r][\mathbf{e}_h - \mathbf{e}_r]^\top,$$

where $\ell_{hr} = -[\mathcal{A}]_{hr} < 0$. Consequently, the dynamics of the multi-agent system after the edge disconnection is described by the faulty system

$$\mathbf{x}(t+1) = \bar{A}_{hr}\mathbf{x}(t), \tag{6}$$

with state matrix

$$\bar{A}_{hr} = I_N - \kappa\bar{\mathcal{L}}_{hr} = A - \kappa\ell_{hr}[\mathbf{e}_h - \mathbf{e}_r][\mathbf{e}_h - \mathbf{e}_r]^\top. \tag{7}$$

The first fundamental result regarding edge disconnection is the following one, that establishes that as far as the edge removal does not compromise the connectedness of the communication graph, and hence each agent can (possibly indirectly) exchange information with every other agent, the consensus network will converge to the same agreed value it would have converged before the failure.

Proposition 1. *[29] Given the undirected weighted and connected graph $\mathcal{G}$, with Laplacian $\mathcal{L}$, let $\bar{\mathcal{G}}_{hr}$ be the graph obtained from $\mathcal{G}$ by removing the arc (r,h), and let $\bar{\mathcal{L}}_{hr}$ be the associated Laplacian. Set $\bar{A}_{hr}$ as in (7), where $\kappa > 0$ is a fixed coupling strength, that has been chosen to ensure that system (2) is a consensus network. If $\bar{\mathcal{G}}_{hr}$ is still connected, then the faulty system (6) describing the system dynamics starting at $t = \tau \geq 0$, the time at which the node disconnection occurs, is still a consensus network. Also, for every choice of $\mathbf{x}(0)$ and every $\tau \geq 0$ at which the edge disconnection occurs, the network state converges to the same consensus value to which the original network (2) would have converged before the disconnection.*

Remark 2. *It is clear that the connectedness of the communication graph after the edge disconnection is a necessary condition for the network to reach a consensus. On the other hand, somewhat unexpectedly, such a condition also ensures that a consensus is achieved on the same value even after an edge disconnection. It is worth noticing that this is a peculiarity of undirected graphs, since as shown in [29] this result is no longer true, in general, when the communication between agents is directed, and hence not symmetric.*

Assumption 2E. In the rest of the paper we will assume that the graph $\bar{\mathcal{G}}_{hr}$, obtained from $\mathcal{G}$ upon disconnection of the edge (r,h), is still connected (and hence the Laplacian $\bar{\mathcal{L}}_{hr}$ and the state matrix $\bar{A}_{hr}$ are still irreducible).

IV. Detecting an edge disconnection

In this section we focus on the problem of detecting an edge disconnection in a consensus network and consider two possible scenarios: the case when we can observe the states of all the agents and the case when we can observe the states of a subset of the agents. It is worth noticing that if an edge disconnection takes place once the system has already reached consensus, the dynamics of the faulty system will necessarily be identical to the one of the original system and hence the edge disconnection cannot be detected. To account for this fundamental aspect, in [29] we have extended the concepts of discernibility introduced in [2] for the continuous-time case as follows.

Definition 1. *Consider the multi-agent consensus network (2), and the network (6), obtained from (2) upon disconnection of the edge between agent r and agent h, with $\bar{A}_{hr}$ described as in (7). The two networks are said to be* discernible *if for every fault time $\tau \geq 0$ and every state $\mathbf{x}(\tau) \notin \langle \mathbf{1}_N \rangle$, there exists $t > \tau$ such that the state trajectory of the faulty system (6) at time t, $\mathbf{x}(t) = \bar{A}_{hr}^{t-\tau}\mathbf{x}(\tau)$, is different from the state trajectory of the original system at time t.*
On the other hand, if only the states of $p < N$ agents are available, and we assume without loss of generality that they are the first p agents, we say that the two networks are discernible from the observation of the first p agents *if for every fault time $\tau \geq 0$ and every state $\mathbf{x}(\tau) \notin \langle \mathbf{1}_N \rangle$, the first p entries of* any *state trajectory of the faulty system (6) at time $t \geq \tau$, $\mathbf{x}(t) = \bar{A}_{hr}^{t-\tau}\bar{\mathbf{x}}_\tau$, are different from the first p entries of the state trajectory of the original system at time t for at least one time instant t, namely for every $\bar{\mathbf{x}}_\tau \in \mathbb{R}^N$ there exists $t \geq \tau$ such that*

$$\begin{bmatrix} I_p & 0 \end{bmatrix} \bar{A}_{hr}^{t-\tau}\bar{\mathbf{x}}_\tau \neq \begin{bmatrix} I_p & 0 \end{bmatrix} A^{t-\tau}\mathbf{x}(\tau).$$

It is not difficult to show (see [2], [29]) that the two networks are discernible if and only if the only unobservable states of the matrix pair (Δ_{hr}, Γ_N), with

$$\Delta_{hr} := \begin{bmatrix} A & 0 \\ 0 & \bar{A}_{hr} \end{bmatrix}, \qquad \Gamma_N := \begin{bmatrix} I_N & -I_N \end{bmatrix}, \tag{8}$$

are those in $\langle \mathbf{1}_{2N} \rangle$. As far as the second form of discernibility is concerned, it is easily seen that discernibility of the two systems from the observation of the first p agents imposes the observability of the original system. On the other hand, the lack of observability of the faulty system could lead to some pathological situations. So, in the following we will assume:

Assumption 3E. Both the original system (2) and the faulty one (6) are observable from the first p agents, namely both $(A, \begin{bmatrix} I_p & 0 \end{bmatrix})$ and $(\bar{A}_{hr}, \begin{bmatrix} I_p & 0 \end{bmatrix})$ are observable pairs.

Under this assumption, one can resort to elementary arguments based on the observability matrix of the pair (Δ_{hr}, Γ_p), with

$$\Delta_{hr} := \begin{bmatrix} A & 0 \\ 0 & \bar{A}_{hr} \end{bmatrix}, \qquad \Gamma_p := \begin{bmatrix} I_p & 0 & -I_p & 0 \end{bmatrix}, \tag{9}$$

to prove that the two networks are discernible from the observation of the first p agents if and only if the only unobservable states of the matrix pair (Δ_{hr}, Γ_p) are those in $\langle \mathbf{1}_{2N} \rangle$ (and they necessarily correspond to the unitary eigenvalue).

A. Discernibility after edge disconnection

The following proposition provides equivalent conditions for discernibility after edge disconnection, in case all the agents' states are accessible.

Proposition 3. *[29] Given the networks (2) and (6), this latter obtained from the former after the disconnection of the edge (r,h), suppose that Assumptions 1 and 2E hold. Then, the following facts are equivalent:*

i) *the networks (2) and (6) are discernible;*
ii) *the unobservable states of the pair (Δ_{hr}, Γ_N) are those in $\langle \mathbf{1}_{2N} \rangle$;*
iii) *the only eigenvalue of the unobservable subspace of the pair $(A, \kappa\ell_{hr}[\mathbf{e}_h - \mathbf{e}_r][\mathbf{e}_h - \mathbf{e}_r]^\top)$ is 1;*
iv) *there is no eigenvalue-eigenvector pair $(\lambda, \mathbf{v})$, with $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^N$, except for $\lambda = 1$ and $\mathbf{v} \in \langle \mathbf{1}_N \rangle$, such that*

$$A\mathbf{v} = \lambda \mathbf{v} \quad \text{and} \quad [\mathbf{v}]_h = [\mathbf{v}]_r. \tag{10}$$

In the special case when A has one eigenvalue $\lambda \neq 1$ of multiplicity greater than 1, then an eigenvector $\mathbf{v}$ of A corresponding to λ can be found such that condition (10) is satisfied, thus making the original network and the faulty network not discernible (see Lemma 4, below). So, a necessary condition for discernibility is that A has all eigenvalues with unitary multiplicity. This implies that $\mathcal{L}$ needs to have all eigenvalues with unitary multiplicity.

Lemma 4. *[29] If the (symmetric) matrix $A = I_N - \kappa\mathcal{L}$, with κ satisfying (4), has an eigenvalue $\lambda \neq 1$ of multiplicity greater than 1, then there exists an eigenvector $\mathbf{v}$ corresponding to λ such that condition (10) holds, and therefore the networks (2) and (6), this latter obtained from the former after the disconnection of the edge (r, h), are not discernible.*

We now explore the more interesting case of discernibility from the observation of the first p agents.

B. Discernibility from the observation of the first p agents after edge disconnection

It is worth noticing that since A is a nonnegative irreducible matrix, having $\mathbf{1}_N$ as dominant eigenvector corresponding to $\lambda = 1$, clearly $\lambda = 1$ is always an observable eigenvalue of the pair $(A, \begin{bmatrix} I_p & 0 \end{bmatrix})$, and hence if the pair $(A, \begin{bmatrix} I_p & 0 \end{bmatrix})$ would not be observable, the eigenvalues of the non-observable subsystem would necessarily have modulus smaller than 1. Note that for the same reason, 1 is always an observable eigenvalue of the matrix $\bar{A}_{hr}$, as far as it remains irreducible. Finally, the irreducibility assumption on both A and $\bar{A}_{hr}$ ensures that the eigenspace of both A and $\bar{A}_{hr}$ corresponding to $\lambda = 1$ is $\langle \mathbf{1}_N \rangle$. So, the only unobservable eigenvectors of (Δ_{hr}, Γ_p) corresponding to the unitary eigenvalue are those belonging to $\langle \mathbf{1}_{2N} \rangle$. As a result, the case $\lambda = 1$ does not require any check. One only needs to evaluate what happens of the PBH observability matrix of the pair (Δ_{hr}, Γ_p) when $\lambda \neq 1$. By putting together these reasonings with Proposition 3 in [2] and some technical lemma [29], we have obtained the following result.

Proposition 5. *[29] Given the networks (2) and (6), this latter obtained from the former after the disconnection of the edge (r, h), suppose that Assumptions 1, 2E and 3E hold. Then, the following facts are equivalent:*

i) *The networks (2) and (6), this latter obtained from the former after the disconnection of the edge (r, h), are discernible from the observation of the first p agents;*
ii) *the unobservable states of the pair (Δ_{hr}, Γ_N) are those in $\langle \mathbf{1}_{2N} \rangle$;*
iii) *for every $\lambda \in \sigma(A) \cap \sigma(\bar{A}_{hr}), \lambda \neq 1$,*

$$\text{rank} \begin{bmatrix} \lambda I_N - A & 0 \\ 0 & \lambda I_N - \bar{A}_{hr} \\ I_p \quad 0 & -I_p \quad 0 \end{bmatrix} = 2N;$$

iv) *there are no $\lambda \in \mathbb{R}$ and nonzero vectors $\mathbf{v}, \bar{\mathbf{v}} \in \mathbb{R}^N$, except for $\lambda = 1$ and $\mathbf{v} = \bar{\mathbf{v}} \in \langle \mathbf{1}_N \rangle$, such that*

$$\begin{cases} A\mathbf{v} = \lambda\mathbf{v}, & \bar{A}_{hr}\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}} \\ \begin{bmatrix} I_p & 0 \end{bmatrix} \mathbf{v} & = \begin{bmatrix} I_p & 0 \end{bmatrix} \bar{\mathbf{v}}; \end{cases} \tag{11}$$

v) *there is no eigenvalue-eigenvector pair $(\lambda, \mathbf{v})$, with $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^N$, except for $\lambda = 1$ and $\mathbf{v} \in \langle \mathbf{1}_N \rangle$, such that (10) holds.*

V. Consensus after a node disconnection

When studying the effects of a node disconnection on a consensus network, we have to face a more complicated situation. Indeed, a node failure corresponds to the disconnection of all the edges having one extreme in that node, and hence the connectivity of the original graph is clearly lost. In order to still achieve some form of consensus, we have to require that all the remaining nodes communicate with each other, but in general we cannot expect that the consensus value will be the same as the one the original network was converging to.

In order to study the effects of the disconnection of some node $h \in \mathcal{V}$ and our possibilities to detect such a fault, we consider as new graph after the node disconnection the graph $\bar{\mathcal{G}}_h^{ext}$, obtained from $\mathcal{G}$ by removing all the arcs having the node h at one of its ends. Note that this is a graph whose vertex set is still $\mathcal{V} = \{1, 2, \ldots, N\}$, and whose set of arcs $\bar{\mathcal{E}}_h^{ext}$ coincides with $\mathcal{E} \setminus \{(h, j), (j, h) : j \in \{1, 2, \ldots, N\}, j \neq h\}$. Accordingly, we define the Laplacian $\bar{\mathcal{L}}_h^{ext}$ of the new graph in terms of the Laplacian $\mathcal{L} = [\ell_{ij}]$ of the original graph $\mathcal{G}$ as follows:

$$\begin{aligned} \bar{\mathcal{L}}_h^{ext} &= \mathcal{L} - \sum_{j \neq h} [\mathcal{A}]_{hj} (\mathbf{e}_h \mathbf{e}_h^\top - \mathbf{e}_h \mathbf{e}_j^\top) \\ &- \sum_{j \neq h} [\mathcal{A}]_{jh} (\mathbf{e}_j \mathbf{e}_j^\top - \mathbf{e}_j \mathbf{e}_h^\top) \in \mathbb{R}^{N \times N}. \end{aligned} \tag{12}$$

Note that $\bar{\mathcal{L}}_h^{ext}$ has the h-th column and row that are both zero. Also, we let $\bar{\mathcal{G}}_h$ be the subgraph obtained from $\mathcal{G}$ (or from $\bar{\mathcal{G}}_h^{ext}$) by removing the node h, and $\bar{\mathcal{L}}_h$ the associated Laplacian (therefore $\bar{\mathcal{L}}_h \mathbf{1}_{N-1} = 0$). Clearly, $\bar{\mathcal{L}}_h$ is the $(N-1) \times (N-1)$ principal submatrix of $\bar{\mathcal{L}}_h^{ext}$ obtained by removing the h-th column and row.
Accordingly, the matrices that describe the dynamics of all the N agents and of the remaining $N-1$ agents after the node disconnection are, respectively,

$$\begin{aligned} \bar{A}_h^{ext} &:= I_N - \kappa \bar{\mathcal{L}}_h^{ext} \\ &= A + \kappa \Big(\sum_{j \neq h} [\mathcal{A}]_{hj} (\mathbf{e}_h \mathbf{e}_h^\top - \mathbf{e}_h \mathbf{e}_j^\top) \\ &+ \sum_{j \neq h} [\mathcal{A}]_{jh} (\mathbf{e}_j \mathbf{e}_j^\top - \mathbf{e}_j \mathbf{e}_h^\top) \Big) \end{aligned} \tag{13}$$

$$\begin{aligned} \bar{A}_h &:= I_{N-1} - \kappa \bar{\mathcal{L}}_h \\ &= S_h^\top A S_h + \kappa \Big[\sum_{j<h} [\mathcal{A}]_{jh} \mathbf{e}_j \mathbf{e}_j^\top + \sum_{j>h} [\mathcal{A}]_{jh} \mathbf{e}_{j-1} \mathbf{e}_{j-1}^\top \Big] \end{aligned} \tag{14}$$

where S_h is the $N \times (N-1)$ selection matrix that selects all the columns of A except for the h-th one (the $N \times (N-1)$ submatrix of I_N obtained by removing the h-th column). As we will see, depending on the specific problem we will address, it will be more convenient to refer to the dynamics of all the N agents, after the node disconnection, or to the dynamics of the $N-1$ active agents.

The first result we provide for node disconnection is Proposition 6 below. Compared to Proposition 1, addressing the effects of an edge disconnection, the following result states that a node disconnection does not prevent consensus of the resulting faulty network, provided that it remains connected, but differently from the case of an edge disconnection the agents converge to a consensus value that is different from the original one.

Proposition 6. *Given the undirected weighted and connected graph $\mathcal{G}$, with Laplacian $\mathcal{L}$, let $\bar{\mathcal{G}}_h$ be the subgraph obtained from $\mathcal{G}$ by removing the node h, and let $\bar{\mathcal{L}}_h$ be the associated Laplacian. Set $\bar{A}_h := I_{N-1} - \kappa\bar{\mathcal{L}}_h$, where $\kappa > 0$ is a fixed coupling strength, that has been chosen to ensure that system (2) is a consensus network. If $\bar{\mathcal{G}}_h$ is still connected, then*

i) *the $N-1$-dimensional system $\bar{\mathbf{x}}(t+1) = (I_{N-1} - \kappa\bar{\mathcal{L}}_h)\bar{\mathbf{x}}(t) = \bar{A}_h\bar{\mathbf{x}}(t)$ is still a consensus network;*

ii) *unless the index h is such that all the eigenvectors of A, corresponding to nonunitary and nonzero eigenvalues, have the h-th entry that is zero, for every $\tau \geq 0$ at which the node disconnection occurs there exist initial states $\mathbf{x}(0)$ corresponding to which the consensus value obtained by the faulty network is different from the original one (5).*

Proof. i) Since the original network is a consensus network, $\mathcal{L}$ is irreducible and κ satisfies the constraint (4) (see Assumption 1). On the other hand, by assumption, the Laplacian $\bar{\mathcal{L}}_h$ is still irreducible and clearly if we denote by $\bar{\ell}_{ij}$ the (i,j)-th entry of $\bar{\mathcal{L}}_h$, then $\max_{i=1,\dots,N-1} \bar{\ell}_{ii} \leq \max_{i=1,\dots,N} \ell_{ii}$, thus ensuring that

$$0 < \kappa < \frac{1}{\max_{i=1,\dots,N} \bar{\ell}_{ii}}.$$

Consequently, also the new network is a consensus network.
ii) As A is nonnegative, irreducible and symmetric, $\mathbb{R}^N$ admits a basis consisting of eigenvectors of A (equivalently, of eigenvectors of $\mathcal{L}$). We let $\mathbf{1}_N$ be the eigenvector of A corresponding to 1, and $\mathbf{v}_2, \dots, \mathbf{v}_N$ be the eigenvectors of A corresponding to the remaining eigenvalues of A, say $\lambda_2, \dots, \lambda_N$, that are real eigenvalues of modulus smaller than 1. Every initial condition $\mathbf{x}(0)$ can be expressed as

$$\mathbf{x}(0) = \alpha\mathbf{1}_N + \sum_{i=2}^{N} \alpha_i \mathbf{v}_i,$$

where $\alpha = \mathbf{w}_A^\top\mathbf{x}(0) = \frac{1}{N}\mathbf{1}_N^\top\mathbf{x}(0)$. We note that if the disconnection takes place at $\tau \geq 0$ then

- $\mathbf{x}(\tau) = \alpha\mathbf{1}_N + \sum_{i=2}^{N} \alpha_i \lambda_i^\tau \mathbf{v}_i$;
- the consensus value reached by the new network coincides with
$$\lim_{t\to+\infty} \bar{\mathbf{x}}(t) = \lim_{t\to+\infty} \bar{A}_h^{t-\tau}\bar{\mathbf{x}}(\tau) = \lim_{t\to+\infty} \bar{A}_h^{t}\bar{\mathbf{x}}(\tau),$$
where $\bar{\mathbf{x}}(\tau) = S_h^\top\mathbf{x}(\tau) = \alpha\mathbf{1}_{N-1} + \sum_{i=2}^{N} \alpha_i\lambda_i^\tau S_h^\top\mathbf{v}_i$, and hence it coincides with $\bar{\mathbf{w}}_{\bar{A}}^\top\bar{\mathbf{x}}(\tau) = \frac{1}{N-1}\mathbf{1}_{N-1}^\top\bar{\mathbf{x}}(\tau)$.

On the other hand, (see also [29]) $\mathbf{w}_A^\top\mathbf{v}_i = 0$ for every $i \in \{2, \dots, N\}$. Therefore $\bar{\mathbf{w}}_{\bar{A}}^\top S_h^\top\mathbf{v}_i = -\frac{[\mathbf{v}_i]_h}{N-1}$ for every $i \in \{2, \dots, N\}$. This implies that

$$\bar{\mathbf{w}}_{\bar{A}}^\top\bar{\mathbf{x}}(\tau) = \alpha + \sum_{i=2}^{N} \alpha_i\lambda_i^\tau \bar{\mathbf{w}}_{\bar{A}}^\top S_h^\top\mathbf{v}_i = $$
$$= \mathbf{w}_A^\top\mathbf{x}(0) - \sum_{i=2}^{N} \alpha_i\lambda_i^\tau \frac{[\mathbf{v}_i]_h}{N-1}. \quad (15)$$

So, unless each $[\mathbf{v}_i]_h, i = 2, 3, \dots, N$, corresponding to $\lambda_i \neq 0$ is zero, there exist choices of the coefficients $\alpha_i, i = 2, 3, \dots, N$, such that $\sum_{i=2}^{N} \alpha_i\lambda_i^\tau \frac{[\mathbf{v}_i]_h}{N-1} \neq 0$, and hence $\bar{\mathbf{w}}_{\bar{A}}^\top\bar{\mathbf{x}}(\tau) \neq \mathbf{w}_A^\top\mathbf{x}(0)$. □

Assumption 2N. In the rest of the paper we will assume that the Laplacian $\bar{\mathcal{L}}_h$ (and hence the state matrix $\bar{A}_h$) of the (reduced) network, obtained from (2) upon disconnection of the node h, is irreducible.

VI. Detecting a node disconnection

A. Detecting a node disconnection: the concept of discernibility

In the following, to simplify the notation, we will assume that the node that disconnects is the N-th one, namely the last one. Clearly, we can always reduce ourselves to this situation by means of a suitable relabelling of the nodes. Also, we will omit the suffix $h = N$ in the Laplacian of the faulty network and related matrices. Accordingly, we can partition the original matrix A as follows

$$A = \begin{bmatrix} B & -\kappa\boldsymbol{\mu} \\ -\kappa\boldsymbol{\mu}^\top & 1-\kappa\ell_{NN} \end{bmatrix}, \quad (16)$$

where $B := I_{N-1} - \kappa S_N^\top\mathcal{L}S_N$, $\boldsymbol{\mu} \in \mathbb{R}^{N-1}$ and ℓ_{NN}, the (N,N)-th entry of $\mathcal{L}$, belongs to $\mathbb{R}$. The disconnection of node N at some time $t = \tau \geq 0$ therefore leads to the new state space description for the set of all N agents:

$$\mathbf{x}(t+1) = \bar{A}^{ext}\mathbf{x}(t), \quad (17)$$

with $\bar{A}^{ext}$ described as

$$\bar{A}^{ext} = \begin{bmatrix} B - \kappa\,\mathrm{diag}(\boldsymbol{\mu}) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{A} & 0 \\ 0 & 1 \end{bmatrix}. \quad (18)$$

Remark 7. *By Assumption 2N, $\bar{A}$ is a nonnegative irreducible matrix, and by adopting the same reasonings previously adopted for A we can claim that $\lambda = 1$ is its dominant eigenvalue, while $\mathbf{1}_{N-1}$ is its dominant eigenvector and there are no other eigenvectors of $\bar{A}$ corresponding to $\lambda = 1$ except for multiples of $\mathbf{1}_{N-1}$. On the other hand, $\bar{A}^{ext}\mathbf{1}_N = \mathbf{1}_N$, but 1 is not a simple eigenvalue of $\bar{A}^{ext}$. Indeed, also $\mathbf{e}_N$ is an eigenvector of $\bar{A}^{ext}$ corresponding to 1. Indeed, $\lambda = 1$ is an eigenvalue of $\bar{A}^{ext}$ with algebraic and geometric multiplicities both equal to 2 and the eigenspace associated with $\lambda = 1$ is $\langle \mathbf{1}_N, \mathbf{e}_N \rangle = \langle \begin{bmatrix} \mathbf{1}_{N-1} \\ 0 \end{bmatrix}, \mathbf{e}_N \rangle$.*

We first consider the problem of detecting a node disconnection when the states of all the agents are accessible. Also, in this case, the definition of discernibility has been adapted from the one given in [2], by ruling out the case when the node disconnection takes place after the multi-agent system has already reached consensus, and hence the fault is clearly undetectable.

Definition 2. *Consider the multi-agent consensus network (2), and the N-dimensional network (17) obtained from (2) upon disconnection of the node N, with $\bar{A}^{ext}$ described as in (18). The two networks are said to be* discernible *if for every fault time $\tau \geq 0$ and every state $\mathbf{x}(\tau) \notin \langle \mathbf{1}_N \rangle$, there exists $t \geq \tau$ such that the state trajectory of the faulty system (17) at time t, $\mathbf{x}(t) = (\bar{A}^{ext})^{t-\tau}\mathbf{x}(\tau)$, is different from the state trajectory of the original system at time t.*

By making use of arguments similar to those adopted in [2] and [29], that simply rely on the kernel of the observability matrix associated with the pair (Δ, Γ_N), with

$$\begin{aligned}\Delta &:= \begin{bmatrix} A & 0 \\ 0 & \bar{A}^{ext} \end{bmatrix} \\ &= \begin{bmatrix} B & -\kappa\boldsymbol{\mu} & 0 & 0 \\ -\kappa\boldsymbol{\mu}^\top & 1-\kappa\ell_{NN} & 0 & 0 \\ 0 & 0 & B-\kappa\,\mathrm{diag}(\boldsymbol{\mu}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \Gamma_N &:= \begin{bmatrix} I_N & -I_N \end{bmatrix} = \begin{bmatrix} I_{N-1} & 0 & -I_{N-1} & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix},\end{aligned}$$

we can easily claim that the two networks are discernible if and only if the only unobservable states of the matrix pair (Δ, Γ_N) are those in $\langle \mathbf{1}_{2N} \rangle$. This characterization is the starting point to derive the following result.

Proposition 8. *Given the N-dimensional networks (2) and (17), this latter obtained from the former after the disconnection of the node N, the following facts are equivalent:*

i) *the networks (2) and (17) are discernible;*
ii) *the unobservable states of the pair (Δ, Γ_N) are those in $\langle \mathbf{1}_{2N} \rangle$;*
iii) *there is no eigenvalue-eigenvector pair $(\lambda, \mathbf{v})$ of the matrix B, with $\lambda \in \mathbb{R}, \lambda \neq 1$, and $\mathbf{v} \in \mathbb{R}^{N-1}, \mathbf{v} \neq 0$, satisfying*
$$[\mathbf{v}]_i = 0, \forall\, i \in \{1,\dots,N-1\} \text{ such that } [\mathcal{A}]_{iN} \neq 0; \tag{19}$$
iv) *there is no eigenvalue-eigenvector pair $(\lambda, \mathbf{z})$ of the matrix A, with $\lambda \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^N, \mathbf{z} \neq 0$, satisfying*
$$\begin{cases} [\mathbf{z}]_N = 0, \\ [\mathbf{z}]_i = 0 \;\forall\, i \in \{1,\dots,N-1\} \text{ s.t. } [\mathcal{A}]_{iN} \neq 0; \end{cases} \tag{20}$$
v) *set $I := \{i \in \{1,2,\dots,N-1\} : [\mathcal{A}]_{iN} = 0\}$, $J := \{1,\dots,N\} \setminus I$, and let P be an $N \times N$ permutation matrix that brings the nodes in I as the first $|I|$ nodes, and leaves the N-th node as the last one, and set*
$$P^\top A P = \begin{bmatrix} A_{II} & A_{IJ} \\ A_{JI} & A_{JJ} \end{bmatrix}.$$
Then there is no eigenvalue-eigenvector pair $(\lambda, \mathbf{v})$ of the matrix A_{II}, with $\lambda \in \mathbb{R}$, satisfying $A_{JI}\mathbf{v} = 0$, namely the pair (A_{II}, A_{JI}) is observable.

Proof. i) $\Leftrightarrow$ ii) This identity has been commented upon earlier.
ii) $\Leftrightarrow$ iii) By making use of the PBH observability matrix, we can claim that ii) holds if and only if the only pairs $(\lambda, \mathbf{w})$, with $\mathbf{w} \neq 0$, such that
$$\begin{bmatrix} \lambda I_{2N} - \Delta \\ \Gamma_N \end{bmatrix} \mathbf{w} = 0 \tag{21}$$
satisfy $\lambda = 1$ and $\mathbf{w} \in \langle \mathbf{1}_{2N} \rangle$. It is easily seen that (21) holds for some $(\lambda, \mathbf{w})$, with $\mathbf{w} \neq 0$, if and only if either $(\lambda, \mathbf{w}) = (1, \alpha\mathbf{1}_{2N}), \exists\, \alpha$, or $\lambda \neq 1$ and there exists $\mathbf{v} \in \mathbb{R}^{N-1}, \mathbf{v} \neq 0$, such that
$$\begin{cases} (\lambda I_{N-1} - B)\mathbf{v} &= 0 \\ \boldsymbol{\mu}^\top \mathbf{v} &= 0 \\ \mathrm{diag}(\boldsymbol{\mu})\mathbf{v} &= 0. \end{cases} \tag{22}$$
The previous conditions (22) simultaneously hold for some $\lambda \neq 1$ if and only if there exist $\lambda \neq 1$ and $\mathbf{v} \neq 0$ such that
$$\begin{cases} (\lambda I_{N-1} - B)\mathbf{v} \;=\; 0 \\ [\boldsymbol{\mu}]_i \neq 0 \;\Rightarrow\; [\mathbf{v}]_i = 0. \end{cases} \tag{23}$$
Since $\boldsymbol{\mu} = -\begin{bmatrix} [\mathcal{A}]_{1N} \\ \vdots \\ [\mathcal{A}]_{N-1,N} \end{bmatrix}$, this is equivalent to saying that B has an eigenvector $\mathbf{v}$, corresponding to some nonunitary eigenvalue λ, whose entries satisfy (19).
iii) $\Leftrightarrow$ iv) To deny iii) is equivalent to claim the existence of λ (which is necessarily different from 1) and $\mathbf{v} \neq 0$ such that (23) holds. Then it is easy to see that if (23) holds, then iv) is contradicted by $\mathbf{z} := \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}$. Conversely, if there exists some $\mathbf{z}$ and λ contradicting iv), then iii) does not hold for $\mathbf{v} := S_N^\top \mathbf{z}$ and the same λ.
iv) $\Leftrightarrow$ v) Condition iv) is violated if and only if there exist λ (which is necessarily different from 1) and $\mathbf{z} = \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}$, with $\mathbf{v} \neq 0$, such that
$$\begin{bmatrix} A_{II} & A_{IJ} \\ A_{JI} & A_{JJ} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}.$$
This amounts to saying that there exist λ (which is necessarily different from 1) and $\mathbf{v} \neq 0$ such that
$$A_{II}\mathbf{v} = \lambda \mathbf{v} \text{ and } A_{JI}\mathbf{v} = 0.$$
So, the violation of condition iv) is equivalent to assuming that (A_{II}, A_{JI}) is not observable. $\square$

B. Discernibility from the observation of a subset of the agents

We now address discernibility from the observation of a subset of cardinality $p < N$ of the states of the agents. Clearly, two possible situations arise: (1) the available measurements include the state of the disconnected agent; (2) the available measurements do not include the state of the disconnected agent. To make the definition generic, and consider both cases simultaneously, we denote by $\mathcal{M} := \{m_1, m_2 \dots, m_p\}$, with $1 \leq m_1 < m_2 < \dots < m_p \leq N$, the indices of the agents whose states are measurable and let M be the corresponding $p \times N$ state-to-output matrix, whose i-th row is $\mathbf{e}_{m_i}^\top, i \in \{1, 2, \dots, p\}$.

If we assume, as before, that the disconnected node is the N-th one, and the N-dimensional network obtained from (2) upon disconnection of the node N is described as in (17), with $\bar{A}^{ext}$ as in (18), we have the following definition.

Definition 3. *Consider the multi-agent consensus network* (2) *and the N-dimensional network* (17)*, obtained from* (2) *upon disconnection of the node N. Assume that the p measured states are those indexed in $\mathcal{M}$ and hence they represent an output vector obtained from the state vector as follows*

$$\mathbf{y}(t) = M\mathbf{x}(t).$$

We say that the two networks (2) *and* (17) *are* discernible from the observation of the p agents indexed in $\mathcal{M}$ *if for every fault time $\tau \geq 0$ and every state $\mathbf{x}(\tau) \notin \langle \mathbf{1}_N \rangle$, the output of the original system* (2) *and the output of the faulty system* (17) *are different for at least one time instant $t \geq \tau$, namely for every $\bar{\mathbf{x}}_\tau \in \mathbb{R}^N$ there exists $t \geq \tau$ such that*

$$M(\bar{A}^{ext})^{t-\tau}\bar{\mathbf{x}}_\tau \neq MA^{t-\tau}\mathbf{x}(\tau). \tag{24}$$

It is easily seen that since the node that gets disconnected is the N-th one, we can investigate the two previously described situations, namely the one when the available measurements include the state of the disconnected agent, and the case when they do not, by simply assuming in the first case

$$M = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{25}$$

and in the second case

$$M = \begin{bmatrix} I_p & 0 \end{bmatrix}. \tag{26}$$

Indeed, we can always reduce ourselves to one of these two situations by means of a simple relabelling of the agents. Consequently, we will investigate discernibility from the observation of the last p agents (Case 1) assuming M as in (25), and discernibility from the observation of the first p agents (Case 2) assuming M as in (26).
Also in this case, as we did for the case of edge disconnection, we need to introduce the observability assumption on the original system, since this is a necessary condition for discernibility from the observation of the selected p agents, both in Case 1 and in Case 2.

Assumption 3N. The original system (2) is observable from the selected p agents, namely (A, M) is an observable pair.

C. Case 1: Discernibility from the observation of the last p agents

In this subsection, we will assume that the state-to-output matrix M is described as in (25). We also assume, as we did in the case of an edge disconnection, that also the faulty network after the node disconnection $(\bar{A}^{ext}, M)$ is observable (an assumption that is reasonable, since the disconnected node is one of the measured ones, and avoids pathological situations).

Assumption 4N. The faulty system (17) is observable from the last p agents, namely $(A, \begin{bmatrix} 0 & I_p \end{bmatrix})$ is an observable pair.

To investigate discernibility in Case 1 we introduce the pair

$$\Delta := \begin{bmatrix} A & 0 \\ 0 & \bar{A}^{ext} \end{bmatrix} \quad \Gamma_{p,obs} := \begin{bmatrix} 0 & I_p & 0 & -I_p \end{bmatrix}. \tag{27}$$

Also, it is convenient to adopt a finer block-partition of the matrix A with respect to the one given in (16). This corresponds to distinguishing in the set of the first $N-1$ agents those that are directly measured from those that are not. Specifically, we will assume in Case 1 that

$$A = \begin{bmatrix} A_{11} & A_{12} & -\kappa\boldsymbol{\mu}_{13} \\ A_{12}^\top & A_{22} & -\kappa\boldsymbol{\mu}_{23} \\ -\kappa\boldsymbol{\mu}_{13}^\top & -\kappa\boldsymbol{\mu}_{23}^\top & 1-\kappa\ell_{NN} \end{bmatrix}, \tag{28}$$

where $A_{11} \in \mathbb{R}_+^{(N-p)\times(N-p)}, A_{22} \in \mathbb{R}_+^{(p-1)\times(p-1)}, \boldsymbol{\mu}_{12} \in \mathbb{R}^{N-p}$ and $\boldsymbol{\mu}_{23} \in \mathbb{R}^{p-1}$. Accordingly,

$$\begin{aligned} \bar{A}^{ext} &= \begin{bmatrix} A_{11} - \kappa \operatorname{diag}(\boldsymbol{\mu}_{13}) & A_{12} & 0 \\ A_{12}^\top & A_{22} - \kappa \operatorname{diag}(\boldsymbol{\mu}_{23}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \bar{A} & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{29}$$

We can now introduce the following result.

Proposition 9. *Consider the networks* (2) *and* (17)*, this latter obtained from the former after the disconnection of the node N. Assume that the matrices A and $\bar{A}^{ext}$ are described as in* (28) *and* (29)*, respectively, with $A_{11} \in \mathbb{R}_+^{(N-p)\times(N-p)}$, $A_{22} \in \mathbb{R}_+^{(p-1)\times(p-1)}$, $\boldsymbol{\mu}_{12} \in \mathbb{R}^{N-p}$ and $\boldsymbol{\mu}_{23} \in \mathbb{R}^{p-1}$. Finally, set $\bar{A}_{11} = A_{11} - \kappa \operatorname{diag}(\boldsymbol{\mu}_{13})$ and $\bar{A}_{22} = A_{22} - \kappa \operatorname{diag}(\boldsymbol{\mu}_{23})$. Under Assumptions 1, 2N, 3N and 4N, the following facts are equivalent:*

i) *the networks* (2) *and* (17) *are discernible from the observation of the last p agents;*
ii) *the unobservable states of the pair $(\Delta, \Gamma_{p,obs})$ are those in $\langle \mathbf{1}_{2N} \rangle$;*
iii) *the PBH matrix*

$$\left[\begin{array}{ccc} \lambda I - A_{11} & -A_{12} & 0 \\ -A_{12}^\top & \lambda I - A_{22} & 0 \\ 0 & -A_{12} & \lambda I - \bar{A}_{11} \\ \hline \boldsymbol{\mu}_{13}^\top & \boldsymbol{\mu}_{23}^\top & 0 \\ -A_{12}^\top & \kappa \operatorname{diag}(\boldsymbol{\mu}_{23}) & A_{12}^\top \end{array}\right]$$

is of full column rank for every $\lambda \neq 1$;

iv) *the pair (Φ, H), where*

$$\Phi := \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^\top & A_{22} & 0 \\ 0 & A_{12} & \bar{A}_{11} \end{bmatrix}$$

and

$$H := \begin{bmatrix} \boldsymbol{\mu}_{13}^\top & \boldsymbol{\mu}_{23}^\top & 0 \\ -A_{12}^\top & \kappa \text{ diag}(\boldsymbol{\mu}_{23}) & A_{12}^\top \end{bmatrix},$$

is observable.

Proof. i) $\Leftrightarrow$ ii). Suppose that there exists an unobservable state of the pair $(\Delta, \Gamma_{p,obs})$, say $\begin{bmatrix} \mathbf{x}_\tau \\ \bar{\mathbf{x}}_\tau \end{bmatrix}$, that does not belong to $\langle \mathbf{1}_{2N} \rangle$. Then $\begin{bmatrix} 0 & I_p \end{bmatrix} A^{t-\tau}\mathbf{x}_\tau = \begin{bmatrix} 0 & I_p \end{bmatrix} (\bar{A}^{ext})^{t-\tau}\bar{\mathbf{x}}_\tau$ holds for every $t \geq \tau$. If $\mathbf{x}_\tau$ would belong to $\langle \mathbf{1}_N \rangle$, then the observability of (17) would imply that $\bar{\mathbf{x}}_\tau = \mathbf{x}_\tau$, but this would contradict the assumption that $\begin{bmatrix} \mathbf{x}_\tau \\ \bar{\mathbf{x}}_\tau \end{bmatrix} \not\in \langle \mathbf{1}_{2N} \rangle$. Therefore $\mathbf{x}_\tau \not\in \langle \mathbf{1}_N \rangle$ and this implies that the existence of an unobservable state that does not belong to $\langle \mathbf{1}_{2N} \rangle$ contradicts the discernibility from the observation of the last p agents.
Conversely, assume that there is no discernibility from the observation of the last p agents, and hence there exists a vector $\begin{bmatrix} \mathbf{x}_\tau \\ \bar{\mathbf{x}}_\tau \end{bmatrix}$, with $\mathbf{x}_\tau \not\in \langle \mathbf{1}_N \rangle$, such that $\begin{bmatrix} 0 & I_p \end{bmatrix} A^{t-\tau}\mathbf{x}_\tau = \begin{bmatrix} 0 & I_p \end{bmatrix} (\bar{A}^{ext})^{t-\tau}\bar{\mathbf{x}}_\tau$ holds for every $t \geq \tau$. Clearly, $\begin{bmatrix} \mathbf{x}_\tau \\ \bar{\mathbf{x}}_\tau \end{bmatrix}$ is an unobservable state of the pair $(\Delta, \Gamma_{p,obs})$, and does not belong to $\langle \mathbf{1}_{2N} \rangle$.
ii) $\Leftrightarrow$ iii). Consider the PBH observability matrix associated with the pair (27). We observe that if $\lambda = 1$, then the only right eigenvector of A corresponding to $\lambda = 1$ is $\mathbf{1}_N$, therefore the only vector $\mathbf{v} \in \mathbb{R}^N$ such that

$$\bar{A}^{ext}\mathbf{v} = \mathbf{v} \quad \begin{bmatrix} 0 & I_p \end{bmatrix} \mathbf{v} = \mathbf{1}_p$$

is necessarily (see Remark 7) the vector $\mathbf{1}_N$. Therefore the only unobservable eigenvectors of the pair $(\Delta, \Gamma_{p,obs})$ corresponding to $\lambda = 1$ are those in $\langle \mathbf{1}_{2N} \rangle$. Now, suppose that $\lambda \neq 1$. Condition

$$\begin{bmatrix} \lambda I_{2N} - \Delta \\ \Gamma_{p,obs} \end{bmatrix} \mathbf{w} = 0$$

is equivalent, upon block-partitioning $\mathbf{w}$ conformably with the block partition of the matrices involved, to the fact that the vector

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_1^\top & \mathbf{w}_2^\top & w_3 & \mathbf{w}_4^\top & \mathbf{w}_5^\top & w_6 \end{bmatrix}^\top,$$

belongs to the kernel of

$$\left[\begin{array}{cccccc} \lambda I - A_{11} & -A_{12} & \kappa\boldsymbol{\mu}_{13} & 0 & 0 & 0 \\ -A_{12}^\top & \lambda I - A_{22} & \kappa\boldsymbol{\mu}_{23} & 0 & 0 & 0 \\ \kappa\boldsymbol{\mu}_{13}^\top & \kappa\boldsymbol{\mu}_{23}^\top & \lambda - 1 + \kappa\ell_{NN} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda I - \bar{A}_{11} & -A_{12} & 0 \\ 0 & 0 & 0 & -A_{12}^\top & \lambda I - \bar{A}_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 \\ \hline 0 & I_{p-1} & 0 & 0 & -I_{p-1} & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right].$$

Since $\lambda \neq 1$, it must be $w_6 = 0$ which implies $w_3 = w_6 = 0$. On the other hand, $\mathbf{w}_5 = \mathbf{w}_2$, therefore the previous condition can be rewritten as

$$\begin{bmatrix} \lambda I - A_{11} & -A_{12} & 0 & 0 \\ -A_{12}^\top & \lambda I - A_{22} & 0 & 0 \\ \kappa\boldsymbol{\mu}_{13}^\top & \kappa\boldsymbol{\mu}_{23}^\top & 0 & 0 \\ 0 & 0 & \lambda I - \bar{A}_{11} & -A_{12} \\ 0 & 0 & -A_{12}^\top & \lambda I - \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_4 \\ \mathbf{w}_2 \end{bmatrix} = 0.$$

By making use of elementary transformations on the rows and columns of the previous expression, we easily obtain the equivalent condition

$$\left[\begin{array}{ccc} \lambda I - A_{11} & -A_{12} & 0 \\ -A_{12}^\top & \lambda I - A_{22} & 0 \\ 0 & -A_{12} & \lambda I - \bar{A}_{11} \\ \hline \boldsymbol{\mu}_{13}^\top & \boldsymbol{\mu}_{23}^\top & 0 \\ -A_{12}^\top & \kappa \text{ diag}(\boldsymbol{\mu}_{23}) & A_{12}^\top \end{array}\right] \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_4 \end{bmatrix} = 0.$$

So, condition ii) holds if and only if the previous PBH observability matrix is of full column rank for every $\lambda \neq 1$.
iii) $\Leftrightarrow$ iv) Condition iii) is equivalent to saying that the pair (Φ, H) has not unobservable eigenvectors corresponding to $\lambda \neq 1$. On the other hand, it is immediate to see that $\sigma(\Phi) = \sigma(B) \cup \sigma(\bar{A}_{11})$. Both B and $\bar{A}_{11}$ are principal submatrices of nonnegative irreducible matrices (A and $\bar{A}$, respectively) having unitary spectral radius. Therefore [14], ρ_B and $\rho_{\bar{A}_{11}}$ are both smaller than 1. Consequently, $\lambda = 1$ cannot be an unobservable eigenvalue of the pair and hence condition iii) is equivalent to the observability of (Φ, H). □

D. Case 2: Discernibility from the observation of the first p agents

In this subsection, we will assume that the state-to-output matrix M is described as in (26). To investigate discernibility, also in this case we assume that A and $\bar{A}^{ext}$ are block-partitioned as in (28) and (29), however the sizes of the various blocks are now different, since $A_{11} \in \mathbb{R}_+^{p\times p}$, $A_{22} \in \mathbb{R}_+^{(N-1-p)\times(N-1-p)}$, $\boldsymbol{\mu}_{12} \in \mathbb{R}^p$ and $\boldsymbol{\mu}_{23} \in \mathbb{R}^{N-1-p}$. We also to introduce the matrix pair $(\Delta, \Gamma_{p,nobs})$:

$$\begin{aligned} \Delta &:= \begin{bmatrix} A & 0 \\ 0 & \bar{A}^{ext} \end{bmatrix} \in \mathbb{R}^{2N\times 2N} \\ \Gamma_{p,nobs} &:= \begin{bmatrix} I_p & 0 & -I_p & 0 \end{bmatrix} \in \mathbb{R}^{p\times 2N}, \end{aligned} \tag{30}$$

but this is not the appropriate matrix pair through which we may characterize the discernibility property from the measurements of the first p agents. Indeed, as the set of measured states does not include the state of the disconnected node N, we easily see that in the faulty system the N-th entry of the state has no impact on the output sequence, namely $\lambda = 1$ is always an eigenvalue of the unobservable subsystem of $(\Delta, \Gamma_{p,nobs})$ and $\mathbf{e}_N$ is an eigenvector of Δ that belongs to the unobservable subspace of the pair. Consequently, all the states in $\langle \begin{bmatrix} \mathbf{1}_{2N-1} \\ 0 \end{bmatrix}, \mathbf{e}_{2N} \rangle$ belong to the unobservable subspace of the pair $(\Delta, \Gamma_{p,nobs})$. So, to investigate the discernibility from the first p outputs, it makes sense to focus on the unobservable states of the matrix pair $(\bar{\Delta}, \bar{\Gamma}_{p,nobs})$, where

$$\begin{aligned} \bar{\Delta} &:= \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix} \in \mathbb{R}^{(2N-1)\times(2N-1)} \\ \bar{\Gamma}_{p,nobs} &:= \begin{bmatrix} I_p & 0 & -I_p & 0 \end{bmatrix} \in \mathbb{R}^{p\times 2N-1}. \end{aligned} \tag{31}$$

Accordingly, we introduce the following:

Assumption 5N. The pair $(\bar{A}, \begin{bmatrix} I_p & 0 \end{bmatrix})$ is observable.

Proposition 10. *Consider the networks (2) and (17), this latter obtained from the former after the disconnection of the node N. Assume that the matrices A and $\bar{A}$ are described as in (28) and (29), respectively, with $A_{11} \in \mathbb{R}_+^{p\times p}, A_{22} \in \mathbb{R}_+^{(N-1-p)\times(N-1-p)}, \boldsymbol{\mu}_{12} \in \mathbb{R}^p$ and $\boldsymbol{\mu}_{23} \in \mathbb{R}^{N-1-p}$. Finally, set $\bar{A}_{11} = A_{11} - \kappa \operatorname{diag}(\boldsymbol{\mu}_{13})$ and $\bar{A}_{22} = A_{22} - \kappa \operatorname{diag}(\boldsymbol{\mu}_{23})$. Under Assumptions 1, 2N, 3N and 5N, the following facts are equivalent:*

i) *the networks (2) and (17) are discernible from the observation of the first p nodes;*
ii) *the unobservable states of the pair $(\bar{\Delta}, \bar{\Gamma}_{p,nobs})$ are those in $\langle \mathbf{1}_{2N-1} \rangle$;*
iii) *the PBH matrix*

$$\left[\begin{array}{cccc} \lambda I - A_{11} & -A_{12} & \kappa\boldsymbol{\mu}_{13} & 0 \\ -A_{12}^\top & \lambda I - A_{22} & \kappa\boldsymbol{\mu}_{23} & 0 \\ \kappa\boldsymbol{\mu}_{13}^\top & \kappa\boldsymbol{\mu}_{23}^\top & \lambda - 1 + \kappa\ell_{NN} & 0 \\ -A_{12}^\top & 0 & 0 & \lambda I - \bar{A}_{22} \\ \hline \kappa \operatorname{diag}(\boldsymbol{\mu}_{13}) & A_{12} & -\kappa\boldsymbol{\mu}_{13} & -A_{12} \end{array}\right]$$

is of full column rank for every $\lambda \neq 1$;
iv) *the unobservable states of the pair (Ξ, C), where*

$$\Xi := \begin{bmatrix} A_{11} & A_{12} & -\kappa\boldsymbol{\mu}_{13} & 0 \\ A_{12}^\top & A_{22} & -\kappa\boldsymbol{\mu}_{23} & 0 \\ -\kappa\boldsymbol{\mu}_{13}^\top & -\kappa\boldsymbol{\mu}_{23}^\top & 1 - \kappa\ell_{NN} & 0 \\ A_{12}^\top & 0 & 0 & \bar{A}_{22} \end{bmatrix}$$

and

$$C := \begin{bmatrix} \kappa \operatorname{diag}(\boldsymbol{\mu}_{13}) & A_{12} & -\kappa\boldsymbol{\mu}_{13} & -A_{12} \end{bmatrix}$$

are those in $\langle \mathbf{1}_{2N-1-p} \rangle$.

Proof. i) $\Leftrightarrow$ ii). It is very similar to the proof of i) $\Leftrightarrow$ ii) in the previous proposition and hence it is omitted.
ii) $\Leftrightarrow$ iii). Consider the PBH observability matrix associated with the pair (31). If $\lambda = 1$, then the only right eigenvector of A corresponding to $\lambda = 1$ is $\mathbf{1}_N$, therefore the only vector $\mathbf{v} \in \mathbb{R}^{N-1}$ such that

$$\bar{A}\mathbf{v} = \mathbf{v} \qquad \begin{bmatrix} I_p & 0 \end{bmatrix} \mathbf{v} = \mathbf{1}_p$$

is necessarily (see Remark 7) the vector $\mathbf{1}_N$. So, the only nonobservable eigenvector of the pair $(\bar{\Delta}, \bar{\Gamma}_{p,nobs})$ corresponding to $\lambda = 1$ is $\mathbf{1}_{2N-1}$. Now, suppose that $\lambda \neq 1$. Condition

$$\begin{bmatrix} \lambda I_{2N-1} - \bar{\Delta} \\ \bar{\Gamma}_{p,nobs} \end{bmatrix} \mathbf{w} = 0$$

is equivalent, upon block-partitioning $\mathbf{w}$ conformably with the block partition of the matrices involved, to the fact that the vector

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_1^\top & \mathbf{w}_2^\top & w_3 & \mathbf{w}_4^\top & \mathbf{w}_5^\top \end{bmatrix}^\top,$$

belongs to the kernel of

$$\left[\begin{array}{ccccc} \lambda I - A_{11} & -A_{12} & \kappa\boldsymbol{\mu}_{13} & 0 & 0 \\ -A_{12}^\top & \lambda I - A_{22} & \kappa\boldsymbol{\mu}_{23} & 0 & 0 \\ \kappa\boldsymbol{\mu}_{13}^\top & \kappa\boldsymbol{\mu}_{23}^\top & \lambda - 1 + \kappa\ell_{NN} & 0 & 0 \\ 0 & 0 & 0 & \lambda I - \bar{A}_{11} & -A_{12} \\ 0 & 0 & 0 & -A_{12}^\top & \lambda I - \bar{A}_{22} \\ \hline I_p & 0 & 0 & -I_p & 0 \end{array}\right].$$

Since $\mathbf{w}_4 = \mathbf{w}_1$, the previous condition can be rewritten as

$$\begin{bmatrix} \lambda I - A_{11} & -A_{12} & \kappa\boldsymbol{\mu}_{13} & 0 & 0 \\ -A_{12}^\top & \lambda I - A_{22} & \kappa\boldsymbol{\mu}_{23} & 0 & 0 \\ \kappa\boldsymbol{\mu}_{13}^\top & \kappa\boldsymbol{\mu}_{23}^\top & \lambda - 1 + \kappa\ell_{NN} & 0 & 0 \\ 0 & 0 & 0 & \lambda I - \bar{A}_{11} & -A_{12} \\ 0 & 0 & 0 & -A_{12}^\top & \lambda I - \bar{A}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ w_3 \\ \mathbf{w}_1 \\ \mathbf{w}_5 \end{bmatrix} = 0$$

and hence as

$$\begin{bmatrix} \lambda I - A_{11} & -A_{12} & \kappa\boldsymbol{\mu}_{13} & 0 \\ -A_{12}^\top & \lambda I - A_{22} & \kappa\boldsymbol{\mu}_{23} & 0 \\ \kappa\boldsymbol{\mu}_{13}^\top & \kappa\boldsymbol{\mu}_{23}^\top & \lambda - 1 + \kappa\ell_{NN} & 0 \\ \lambda I - \bar{A}_{11} & 0 & 0 & -A_{12} \\ -A_{12}^\top & 0 & 0 & \lambda I - \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ w_3 \\ \mathbf{w}_5 \end{bmatrix} = 0$$

By making use of elementary transformations on the rows and columns of the previous expression, we easily obtain the equivalent condition

$$\left[\begin{array}{cccc} \lambda I - A_{11} & -A_{12} & \kappa\boldsymbol{\mu}_{13} & 0 \\ -A_{12}^\top & \lambda I - A_{22} & \kappa\boldsymbol{\mu}_{23} & 0 \\ \kappa\boldsymbol{\mu}_{13}^\top & \kappa\boldsymbol{\mu}_{23}^\top & \lambda - 1 + \kappa\ell_{NN} & 0 \\ -A_{12}^\top & 0 & 0 & \lambda I - \bar{A}_{22} \\ \hline \kappa \operatorname{diag}(\boldsymbol{\mu}_{13}) & A_{12} & -\kappa\boldsymbol{\mu}_{13} & -A_{12} \end{array}\right] \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ w_3 \\ \mathbf{w}_5 \end{bmatrix} = 0.$$

So, condition ii) holds if and only if the previous PBH observability matrix is of full column rank for every $\lambda \neq 1$.
iii) $\Leftrightarrow$ iv) It is easy to check that $\mathbf{1}_{2N-1-p}$ belongs to the unobservable space of the pair. The rest is obvious. ☐

VII. How to Restore the Consensus Value in Case of a Node Disconnection

As we already observed in Section V, the disconnection of a node in an undirected graph in general leads, even when the remaining subgraph is still connected, to a consensus value that is different from the original one. Indeed, if the disconnection of the node N, for instance, happens at $t = \tau$, then from that moment onward the reduced multi-agent system, consisting of the first $N-1$ agents, will necessarily achieve consensus on the value

$$\begin{aligned}\bar{\mathbf{w}}_{\bar{A}}^{\top}\bar{\mathbf{x}}(\tau) &= \frac{1}{N-1}\mathbf{1}_{N-1}^{\top}S_N^{\top}\mathbf{x}(\tau) \\ &= \frac{1}{N}\mathbf{1}_N^{\top}\mathbf{x}(0) - \sum_{i=2}^{N}\alpha_i\lambda_i^{\tau}\frac{[\mathbf{v}_i]_N}{N-1} \\ &\neq \frac{1}{N}\mathbf{1}_N^{\top}\mathbf{x}(0),\end{aligned}$$

while the state of the N-th agent will be stuck to the value it had before disconnection. Since the consensus value after the disconnection depends only on the left dominant eigenvector $\bar{\mathbf{w}}_{\bar{A}} = \frac{1}{N-1}\mathbf{1}_{N-1}$ of $\bar{A}$, and on the values of the first $N-1$ entries of the state at $t=\tau$, any modification of the graph weights will be irrelevant, since it may affect the convergence speed but not the consensus value.

If we assume that the node disconnection at $t=\tau$ can be communicated to all the nodes (possibly in a centralised way), and not only detected by the neighbours of node N, then the system can react in a distributed way to ensure that the information regarding the disconnected node is preserved and hence the final consensus value remains identical to the original one. In the following we illustrate a possible correcting scheme. Other solutions have been proposed in [10]–[12]. If at $t=\tau$ the node N becomes disconnected, and hence its state is no longer available, then correction is possible if each of the remaining $N-1$ nodes updates its state value based only on its local information in order to guarantee that

$$\frac{1}{N-1}\mathbf{1}_{N-1}^{\top}\bar{\mathbf{x}}(\tau+1) = \frac{1}{N}\mathbf{1}_N^{\top}\mathbf{x}(\tau).$$

We can obtain this result by simply injecting a correction input at $t=\tau$ (and only at that time instant, so it is an instantaneous contribution):

$$\bar{\mathbf{x}}(\tau+1) = \bar{A}[\bar{\mathbf{x}}(\tau)+\bar{\mathbf{u}}(\tau)],$$

where $\bar{\mathbf{u}}(\tau)$ is such that

$$\frac{1}{N-1}\mathbf{1}_{N-1}^{\top}[\bar{\mathbf{x}}(\tau)+\bar{\mathbf{u}}(\tau)] = \frac{1}{N}\mathbf{1}_N^{\top}\mathbf{x}(\tau).$$

Note that the N-th entry of $\mathbf{x}(\tau)$, $x_N(\tau)$, may not be directly measured, since the node N is no longer communicating, but it can be deduced from the information available up to $t=\tau-1$, when the node was still active. By imposing

$$\frac{1}{N-1}\mathbf{1}_{N-1}^{\top}\left[\bar{\mathbf{x}}(\tau)+\bar{\mathbf{u}}(\tau)\right] = \frac{1}{N}\mathbf{1}_N^{\top}\begin{bmatrix}\bar{\mathbf{x}}(\tau)\\ x_N(\tau)\end{bmatrix}$$

we obtain

$$\frac{1}{N-1}\sum_{i\neq N}x_i(\tau) + \frac{1}{N-1}\mathbf{1}_{N-1}^{\top}\bar{\mathbf{u}}(\tau) = \frac{1}{N}\sum_{i\neq N}x_i(\tau) + \frac{1}{N}\sum_{j=1}^{N}[A]_{Nj}x_j(\tau-1),$$

and hence

$$\mathbf{1}_{N-1}^{\top}\bar{\mathbf{u}}(\tau) = -\frac{1}{N}\sum_{i\neq N}x_i(\tau) + \frac{N-1}{N}\sum_{j=1}^{N}[A]_{Nj}x_j(\tau-1).$$

We can obtain this goal as a result of a distributed control action by assuming:

$$\bar{\mathbf{u}}(\tau) = -\frac{1}{N}\bar{\mathbf{x}}(\tau) + \frac{N-1}{N}\mathrm{diag}(A\mathbf{e}_N)\ \mathbf{x}(\tau-1),$$

where we exploited the symmetry of A.

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