Accelerating consensus in high-order leader-follower networks

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***Abstract*—We consider the problem of accelerating the convergence to consensus of a network of homogeneous high-order agents, through the injection of an additional control input performed by a *leader*. After a brief set-up description, we derive the characteristic polynomial of the resulting system under the leader's control. We introduce the concept of leader controlled distributed consensus, by imposing that the leader's action improves the convergence speed but does not affect the consensus value, except possibly for a scaling factor. Finally, we prove that, under certain assumptions, consensus can be achieved with arbitrary speed.**

Keywords – **Multi-agent systems, consensus networks, convergence speed.**

I. Introduction

Consensus is a key tool for biological as well as technological distributed collaborative systems. Collective decisions and cooperative behaviors of biological networks are frequently the results of repeated local interactions, and indeed consensus decisions play a fundamental role in the lives of social animals [1]. Inspired by this, there has been an impressive amount of research on iterative algorithms, based on local data, which make the group evolution reach an *agreement* or *consensus* on a variable [2], [3], [4]. In many contexts, seeking consensus is an enabling condition for the prosecution of the group toward a common goal [3], [4]. In this respect, the convergence rate toward consensus is a fundamental aspect [4], and standard consensus strategies may be unsatisfactory [5]. The problem of accelerating consensus has been the subject of a good number of papers. Several methods have been proposed, ranging from the optimal design of the coupling coefficients [6] to the use of additional memory slots [7], [8]. However, most of the results deal with the simple case when agents are described by integrators and a few with higher-order integrators.

In this paper, we consider networks of *homogeneous* high-order agents, that is networks of nodes with identical and generic dynamics, which is the most common in nature (team members usually belong to the same species), it is often used in applications [9], [10] and has been intensively studied in the last decade [11], [12], [13]. Consensus for high-order multi-agent systems has been investigated from different perspectives, and different definitions are possible. We consider here consensus on a constant function, also called *stationary consensus* [14], since a wide number of applications have been developed in diverse technological fields based on this simple peculiar condition (e.g. the rendez-vous problem for a team of robots [14]). Specifically, we address the following problem: we assume that a multi-agent system of homogeneous high-order agents has been designed to achieve stationary consensus. In this set-up, a single agent takes the leader's role with the goal of increasing the convergence speed without affecting the final consensus value $\boldsymbol{\nu}(\mathbf{x}_0)$ (that depends on the initial conditions of the whole group), except possibly for a scaling factor. This situation may often arise in practice, for instance in multirobot systems [9], when a single robots has complicated sensory ability, such as GPS, and powerful computation ability to plan the trajectory, while followers have limited computational resources. To achieve this goal, the leader injects an additional control input based only on its own state trajectory. The results of this paper represent the natural evolution of those in [15], [16], [17] for simple integrators, but the complexity of the agents' description demands for a more complex analysis. To the best of our knowledge, this is the first paper exploring acceleration schemes for high order multi-agent systems. Our main contribution is to prove that under reasonable assumptions it is possible to allocate all the eigenvalues of the resulting closed-loop system matrix and hence to freely choose the convergence speed, a goal that could not be achieved by simply acting on the state-feedback matrix that implements the original consensus protocol (see Section 2). It must be however remarked that Theorem 1, concluding the paper, represents a result about theoretical feasibility, rather than a practical method, as the size of the control algorithm may be very large and hence impractical. However the result is of interest as it represents the first step toward a complete analysis of which convergence speeds can be obtained through the action of a leader. The best speed achievable with a fixed complexity is the object of future research.

The note is organized as follows. In Section II we describe the notation and summarize some background material. In Section III the problem statement is described and the leader's control protocol is introduced. In Section IV the characteristic polynomial of the whole system is derived. Section V provides the definition of leader-controlled distributed consensus and conditions on the system that allow to design a leader's control protocol achieving it.

II. Notation and background material

Given $p \in \mathbb{Z}, p > 0$, we set $[1, p] := \{1, 2, \ldots, p\}$. $\mathbf{e}_i$ is the ith *canonical vector* in $\mathbb{R}^N$, where N is clear from the context. The N-dimensional vector with all entries equal to 1

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is $\mathbf{1}_N$. Given $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ the *spectrum* of A, i.e., the set of its eigenvalues. A is *Hurwitz* if $\lambda \in \sigma(A)$ implies $\mathrm{Re}(\lambda) < 0$. The *Kronecker product* of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is $A \otimes B := [a_{ij}B]_{i,j} \in \mathbb{R}^{mp\times nq}$. An *undirected, weighted graph* is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}^{N\times N}$ is the (entrywise nonnegative) *adjacency matrix* of $\mathcal{G}$. In this paper we assume that $\mathcal{G}$ has no self-loops, namely each diagonal entry $[\mathcal{A}]_{ii}, i \in [1, N]$, is zero. A sequence of edges $(j_\ell, j_{\ell+1}) \in \mathcal{E}$, with $\ell \in [1, k]$, is a *path* of length k connecting vertex j_1 with j_{k+1}. A graph is said to be *connected* if for every pair of distinct vertices $i, j \in \mathcal{V}$ there is a path connecting i and j. This is equivalent to the fact that $\mathcal{A}$ is an *irreducible* matrix [18]. The *Laplacian matrix* $\mathcal{L} = \mathcal{L}^\top = [\ell_{ij}]_{i,j\in[1,N]} \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ is

$$\mathcal{L} \;:=\; \begin{bmatrix} \sum_{j=1}^N [\mathcal{A}]_{1j} & -[\mathcal{A}]_{12} & \dots & -[\mathcal{A}]_{1N} \\ -[\mathcal{A}]_{21} & \sum_{j=1}^N [\mathcal{A}]_{2j} & \dots & -[\mathcal{A}]_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -[\mathcal{A}]_{N1} & -[\mathcal{A}]_{2N} & \dots & \sum_{j=1}^N [\mathcal{A}]_{Nj} \end{bmatrix}.$$

As all rows and columns of $\mathcal{L}$ sum up to 0, $\mathbf{1}_N$ is always both a right and a left eigenvector of $\mathcal{L}$ corresponding to the eigenvalue 0. The following lemma states a useful and well-known result about Laplacian matrices of undirected graphs.

Lemma 1. *[11], [18] If the undirected, weighted graph $\mathcal{G}$ is connected, then $\mathcal{L}$ is a positive semidefinite matrix, and its eigenvalues λ_i, $i \in [1, N]$, satisfy*

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N. \tag{1}$$

A matrix $A(z) \in \mathbb{R}[z]^{p\times m}$ is said to be *right prime* (*left prime*) if $\mathrm{rank}A(\lambda) = m$ ($\mathrm{rank}A(\lambda) = p$) for every $\lambda \in \mathbb{C}$. Given $G(z) \in \mathbb{R}(z)^{p\times m}$, we say that a pair $(N_R(z), D_R(z)) \in \mathbb{R}[z]^{p\times m} \times \mathbb{R}[z]^{m\times m}$ provides a *right matrix fraction description (rMFD)* of $G(z)$ if $\det D_R(z) \neq 0$ and $N_R(z)D_R^{-1}(z) = G(z)$. If $\begin{bmatrix} N_R(z) \\ D_R(z) \end{bmatrix}$ is right prime, $N_R(z)D_R^{-1}(z)$ is a *right coprime matrix fraction description (rcMFD)* of $G(z)$. Left (coprime) matrix fraction descriptions (lMFD and lcMFD, respectively) are analogously defined.

III. Problem Statement

Consider N agents, each of them described by the same n-dimensional, continuous-time, single-input system:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t), \qquad t \in \mathbb{R}_+, \tag{2}$$

where $\mathbf{x}_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$ are the state vector and the input of the ith agent, respectively. **Assumptions: (A1)** The communication among the N agents is described by an undirected, weighted, and connected graph with adjacency matrix $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}^{N\times N}$ and Laplacian matrix $\mathcal{L} = \mathcal{L}^\top$, (both of them irreducible), with eigenvalues $\lambda_i, i = 1, 2, \dots, N$, ordered as in (1). Each ith agent adopts the following DeGroot type control law [11]:

$$u_i(t) = K \sum_{j=1}^N [\mathcal{A}]_{ij}[\mathbf{x}_j(t) - \mathbf{x}_i(t)]. \tag{3}$$

where $K \in \mathbb{R}^{1\times n}$ is a fixed feedback matrix. If we denote by $\mathbf{x}(t) \in \mathbb{R}^{nN}$ and $\mathbf{u}(t) \in \mathbb{R}^N$ the state vector and the input vector, respectively, of the multi-agent system, i.e.

$$\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1^\top(t) & \dots & \mathbf{x}_N^\top(t) \end{bmatrix}^\top \mathbf{u}(t) := \begin{bmatrix} u_1(t) & \dots & u_N(t) \end{bmatrix}^\top$$

the multi-agent system is described [11] by:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (I_N \otimes A)\mathbf{x}(t) + (I_N \otimes B)\mathbf{u}(t) \\ \mathbf{u}(t) &= -(\mathcal{L} \otimes K)\mathbf{x}(t), \end{aligned}$$

or, using elementary properties of the Kronecker product, by:

$$\dot{\mathbf{x}}(t) \;=\; [(I_N \otimes A) - (\mathcal{L} \otimes BK)]\mathbf{x}(t). \tag{4}$$

(A2) The state-feedback matrix K has been chosen to ensure *state consensus* of the multi-agent system, i.e. for every initial state $\mathbf{x}(0) \in \mathbb{R}^{nN}$ there exists a suitable vector $\boldsymbol{\nu}(\mathbf{x}(0)) \in \mathbb{R}^n$ (depending on the initial conditions) such that

$$\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{1}_N \otimes \boldsymbol{\nu}(\mathbf{x}(0)). \tag{5}$$

The spectrum of $(I_N \otimes A) - (\mathcal{L} \otimes BK)$ coincides with $\sigma(A) \cup \sigma(A - \lambda_2 BK) \cup \dots \cup \sigma(A - \lambda_N BK)$, and consensus is achieved if and only if $A \in \mathbb{R}^{n\times n}$ is a simply stable matrix, with a simple and strictly dominant eigenvalue in 0, and all matrices $A - \lambda_i BK, i \in [2, N]$, are Hurwitz [11]. This shows that the speed of convergence is always constrained by the position of the subdominant eigenvalues of A, and hence it cannot be freely modified with K. We let $\mathbf{v}_A$ ($\mathbf{w}_A$) denote a right (left) eigenvector of A both relative to the zero eigenvalue and related by condition $\mathbf{w}_A^\top \mathbf{v}_A = 1$. Consequently, upon denoting by $\mathbf{w}_\mathcal{L}$ the left eigenvector of $\mathcal{L}$ associated with the zero eigenvalue and satisfying $\mathbf{w}_\mathcal{L}^\top \mathbf{1}_N = 1$, i.e., $\mathbf{w}_\mathcal{L} = \frac{1}{N}\mathbf{1}_N$, we observe that $\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A$ is a left eigenvector of $(I_N \otimes A) - (\mathcal{L} \otimes BK)$ associated with $\lambda = 0$. Also, by the assumptions $\mathbf{w}_\mathcal{L}^\top \mathbf{1}_N = 1$ and $\mathbf{w}_A^\top \mathbf{v}_A = 1$, it follows $[\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A]^\top [\mathbf{1}_N \otimes \mathbf{v}_A] = 1$. It is a matter of simple computations (see, also, [11]) to show that

$$\boldsymbol{\nu}(\mathbf{x}(0)) = \left(\sum_{i=1}^N \frac{1}{N} \mathbf{w}_A^\top \mathbf{x}_i(0)\right)\mathbf{v}_A = \left[(\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A)^\top \mathbf{x}(0)\right]\mathbf{v}_A. \tag{6}$$

Assume now that, in the current set-up one of the N agents takes the role of leader, and wants to influence the group dynamics by making use of a control signal v based on its own state evolution. The purpose of the leader's action is to improve the performance of the group dynamics, by accelerating consensus. The multi-agent system with consensus protocol (4) and one leader exerting an additional control action becomes:

$$\dot{\mathbf{x}}(t) = [(I_N \otimes A) - (\mathcal{L} \otimes BK)]\mathbf{x}(t) + \mathbf{e}_1 \otimes Bv(t) \tag{7}$$

where we assumed, without loss of generality, that the leader is the first agent. The leader generates the scalar control input $v(t)$ by elaborating its own state $\mathbf{x}_1(t)$ according to the following continuous-time μ-dimensional state model, with state variable $\boldsymbol{\varepsilon}(t)$:

$$\dot{\boldsymbol{\varepsilon}}(t) = F\boldsymbol{\varepsilon}(t) + G\mathbf{x}_1(t), \tag{8a}$$

$$v(t) = H\boldsymbol{\varepsilon}(t) + J\mathbf{x}_1(t), \tag{8b}$$

where $F \in \mathbb{R}^{\mu\times\mu}$, $G \in \mathbb{R}^{\mu\times n}$, $H \in \mathbb{R}^{1\times\mu}$ and $J \in \mathbb{R}^{1\times n}$. Introduce the augmented state vector

$$\boldsymbol{\chi}(t) := \begin{bmatrix}\mathbf{x}(t)\\ \boldsymbol{\varepsilon}(t)\end{bmatrix} \in \mathbb{R}^{nN+\mu}. \tag{9}$$

System (7) under the leader action (8) can be written as

$$\dot{\boldsymbol{\chi}}(t) = \mathcal{M}\boldsymbol{\chi}(t), \tag{10}$$

where $\mathcal{M} \in \mathbb{R}^{(nN+\mu)\times(nN+\mu)}$ takes the form

$$\mathcal{M} := \left[\begin{array}{c|c}(I_N\otimes A)-(\mathcal{L}\otimes BK)+(\mathbf{e}_1\mathbf{e}_1^\top\otimes BJ) & \mathbf{e}_1\otimes BH\\ \hline \mathbf{e}_1^\top\otimes G & F\end{array}\right] \tag{11}$$

Assume, now, to partition $\mathcal{L}$ as follows:

$$\mathcal{L} = \begin{bmatrix}\ell_{11} & \lambda^\top\\ \lambda & \hat{\mathcal{L}}\end{bmatrix}, \quad \lambda \in \mathbb{R}^{N-1}, \hat{\mathcal{L}} \in \mathbb{R}^{(N-1)\times(N-1)}. \tag{12}$$

Then

$$\mathcal{M} := \left[\begin{array}{cc|c}A-\ell_{11}BK+BJ & -\lambda^\top\otimes BK & BH\\ -\lambda\otimes BK & (I_{N-1}\otimes A)-(\hat{\mathcal{L}}\otimes BK) & 0\\ \hline G & 0 & F\end{array}\right] \tag{13}$$

Additional assumptions: (**A3**) The pair (A, B) in the agents' description (2) is reachable and in controllability canonical form.

(**A4**) The quadruple (F, G, H, J) in (8) is a minimal realization of its tranfer matrix $W(s) := H(sI - F)^{-1}G + J \in \mathbb{R}(s)^{1\times n}$. So, if $x(s)^{-1}\begin{bmatrix}y_1(s) & y_2(s) & \dots & y_n(s)\end{bmatrix}$, where $x(s), y_i(s) \in \mathbb{R}[z]$, is the (unique) lcMFD of $W(s)$ with monic denominator $x(s)$, then $\det(sI_\mu - F) = x(s)$.

(**A5**) Matrix $K = \begin{bmatrix}k_0 & k_1 & \dots & k_{n-1}\end{bmatrix}$ in (3) not only ensures consensus, but it also guarantees that

$$\kappa(s) := \sum_{i=0}^{n-1} k_i s^i \tag{14}$$

is coprime with $\Delta_A(s) := \det(sI_n - A)$.

(**A6**) The eigenvalues of matrix $\hat{\mathcal{L}}$ in (12) are distinct, namely they are simple zeros of the characteristic polynomial of $\hat{\mathcal{L}}$.

Remark 2. *We would like to briefly comment on the previously introduced assumptions (see also Remark 5).*

*- (**A1**) is admittedly a restrictive hypothesis, but nonetheless a realistic one when dealing with consensus problems for networks whose nodes have fixed positions. For instance, consensus strategies have been used to implement distributed architectures in several smart grid applications [19].*

*- The stabilizability of the pair (A, B) is a necessary condition for consensus of high-order multi-agent systems [11]. The reachability assumption (**A3**) is a slightly stronger assumption that makes the analysis simpler, since it does not require to keep into account the eigenvalues of A that cannot be allocated through state-feedback. The approach adopted in this paper would allow to account for these eigenvalues, and hence for the constraints they impose on the convergence speed, but the presentation would become more involved. The controllability canonical form, under the reachability assumption, only requires a change of coordinates.*

*- Condition (**A4**) entails no loss of generality, since we can always choose the matrices in (8) so that (**A4**) holds.*

*- Assumption (**A5**) (together with (**A3**)), ensures that $\det(sI_n - A + \lambda BK) = \det(sI_n - A) + \lambda\kappa(z)$ is necessarily coprime with $\Delta_A(s)$ for any $\lambda \in \mathbb{C}, \lambda \neq 0$, and hence $\sigma(A) \cap \sigma(A - \lambda BK) = \emptyset$. This is in turn a mild constraint on the choice of K that allows to simplify the analysis, but in turn could be easily omitted.*

*- Finally, with regards to (**A6**), it is easy to see that $\hat{\mathcal{L}}$ is a symmetric positive definite matrix, and hence its positive real eigenvalues can be sorted in such a way that $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_{N-1}$. The assumption that they are all distinct allows to say, by the same reasoning adopted to comment on (**A5**), that for every $i \neq j, \sigma(A - \mu_i BK) \cap \sigma(A - \mu_j BK) = \emptyset$. Again, this is only required to simplify the analysis.*

IV. Preliminary analysis

We first determine the expression of the characteristic polynomial of $\mathcal{M}$, by making use of the following lemma.

Lemma 3. *[20] Given a matrix $\mathcal{M} = \begin{bmatrix}R & S\\ P & Q\end{bmatrix}$, with $R \in \mathbb{R}^{N\times N}$, $S \in \mathbb{R}^{N\times\mu}$, $P \in \mathbb{R}^{\mu\times N}$, $Q \in \mathbb{R}^{\mu\times\mu}$ and Q nonsingular, one has $\det\mathcal{M} = \det Q \cdot \det\left(R - SQ^{-1}P\right)$.*

Proposition 4. *Consider the multi-agent system (7) under the leader control action (8). Under the previous Assumptions* **(A1)-(A6)***, the characteristic polynomial $p_\mathcal{M}(s) := \det(sI_{nN+\mu} - \mathcal{M})$ of the state matrix $\mathcal{M}$ is equal to*

$$\begin{aligned} p_\mathcal{M}(s) &= [\Delta_A(s)d_\Psi(s) - n_\Psi(s)\kappa(s)]x(s) \\ &- d_\Psi(s)(\sum_{i=0}^{n-1} y_{i+1}(s)s^i), \end{aligned} \tag{15}$$

where $x(s)^{-1}\begin{bmatrix}y_1(s) & \dots y_n(s)\end{bmatrix}$ is the lcMFD of $W(s)$ given in **(A4)***, $\frac{n_\Psi(s)}{d_\Psi(s)}$ is the irreducible representation of $\Psi(s)$ given in Lemma 10 (see Appendix) and $\kappa(s)$ is given in (14).*

Proof. By making use of (11) and Lemma 3 we obtain

$$\begin{aligned} p_\mathcal{M}(z) = \det(sI_\mu - F)\det[&sI_{nN} - I_N\otimes A + \mathcal{L}\otimes BK \\ &-\mathbf{e}_1\mathbf{e}_1^\top\otimes BJ - (\mathbf{e}_1\otimes BH)(sI-F)^{-1}(\mathbf{e}_1^\top\otimes G)] \end{aligned}$$

which can be rewritten as $p_\mathcal{M}(z) = \det(sI_\mu - F)\det[sI_{nN} - I_N\otimes A + \mathcal{L}\otimes BK - \mathbf{e}_1\mathbf{e}_1^\top \otimes BW(s)]$. By now referring to (13) and by making use, again, of Lemma 3 we obtain

$$\begin{aligned} p_\mathcal{M}(z) = &\det(sI_\mu - F)\det[sI_{n(N-1)} - I_{N-1}\otimes A + \hat{\mathcal{L}}\otimes BK] \\ &\cdot \det[sI_n - A + \ell_{11}BK - BW(s) - (\lambda^\top\otimes BK) \\ &\cdot [sI_{n(N-1)} - I_{N-1}\otimes A + \hat{\mathcal{L}}\otimes BK]^{-1}(\lambda\otimes BK)]. \end{aligned}$$

Upon defining the *leader to followers transfer matrix*

$$\begin{aligned} W_{LF}(s) := &(\lambda^\top\otimes BK)[sI_{n(N-1)} - I_{N-1}\otimes A + \hat{\mathcal{L}}\otimes BK]^{-1} \\ &\cdot (\lambda\otimes BK) + A - \ell_{11}BK \in \mathbb{R}(s)^{n\times n}, \end{aligned}$$

the above expression can be written as

$$\begin{aligned} p_\mathcal{M}(z) = &\det(sI_\mu - F)\det[sI_{n(N-1)} - I_{N-1}\otimes A + \hat{\mathcal{L}}\otimes BK] \\ &\cdot \det[sI_n - W_{LF}(s) - BW(s)]. \end{aligned}$$

Using, again, Lemma 10 in the Appendix, and the aforementioned lcMFD of $W(s)$, we obtain $\det[sI_n - W_{LF}(s) - BW(s)] = \det\left[sI_n - A - \Psi(s)BK - B\frac{1}{x(s)}\begin{bmatrix} y_1(s) & \dots y_n(s)\end{bmatrix}\right]$. If we introduce the irreducible representation $\frac{n_\Psi(s)}{d_\Psi(s)}$ of $\Psi(s)$ (see Lemma 10), elementary matrix manipulations allow to say that $\det\left[sI_n - A - \Psi(s)BK - B\frac{1}{x(s)}\begin{bmatrix} y_1(s) & \dots y_n(s)\end{bmatrix}\right] = \Delta_A(s) - \left[\frac{n_\Psi(s)}{d_\Psi(s)}K + \frac{1}{x(s)}\begin{bmatrix} y_1(s) & \dots y_n(s)\end{bmatrix}\right]\mathrm{adj}(sI_n - A)B = \Delta_A(s) - \frac{n_\Psi(s)}{d_\Psi(s)}\kappa(s) - \frac{1}{x(s)}(\sum_{i=0}^{n-1} y_{i+1}(s)s^i)$, where we used the fact that if (A,B) is in controllability canonical form, then $\mathrm{adj}(sI_n - A)B = \begin{bmatrix} 1 & s & \dots & s^{n-1}\end{bmatrix}^\top$. Lemma 10 and Assumption **(A4)**, finally, lead to $p_{\mathcal{M}}(s) = \Delta_A(s)x(s)d_\Psi(s) - n_\Psi(s)\kappa(s)x(s) - d_\Psi(s)(\sum_{i=0}^{n-1} y_{i+1}(s)s^i)$, which can be rewritten as in (15). □

Remark 5. *The expression of $p_{\mathcal{M}}(s)$ provided in (15) strongly depends on Assumptions* **(A3)**, **(A5)** *and* **(A6)**. *Indeed, if any of them were not satisfied, $p_{\mathcal{M}}(s)$ would be the multiple of some fixed polynomial related either to the not reachable part of the pair (A,B), or to the common divisor of $\Delta_A(s)$ and $\kappa(z)$, or to the nontrivial polynomial relating $d_\Psi(s)$ and $\det[I_{N-1}\otimes(sI_n - A) + \hat{\mathcal{L}}\otimes BK]$ (see Lemma 10). This would not alter the analysis but would limit the possibility of freely allocating the eigenvalues of $\mathcal{M}$ and hence would constrain the best achievable speed in case of consensus.*

V. Leader-controlled distributed consensus

In this section, we first introduce the definition of leader-controlled distributed consensus (see [16] and [17]).

Definition 1 (Leader-controlled distributed consensus)**.** *Consider a multi-agent system with a single leader described as in (7). We say that the control (8) leads the system to* leader-controlled distributed consensus*, under the leader's action $v(t)$, if for every initial state $\mathbf{x}(0)$ (assuming $\boldsymbol{\varepsilon}(0) = 0$), there is a vector $\tilde{\boldsymbol{\nu}} = \tilde{\boldsymbol{\nu}}(\mathbf{x}(0)) \in \mathbb{R}^n$ such that*

$$\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{1}_N \otimes \tilde{\boldsymbol{\nu}}(\mathbf{x}(0)). \quad (16)$$

Furthermore, to make the achieved leader-controlled distributed consensus consistent with what we would achieve without the auxiliary control protocol, we impose the additional constraint that there exists a nonzero scaling factor C such that, for every $\mathbf{x}(0) \in \mathbb{R}^n$,

$$\tilde{\boldsymbol{\nu}}(\mathbf{x}(0)) = C \cdot \boldsymbol{\nu}(\mathbf{x}(0)). \quad (17)$$

We now determine conditions guaranteeing that a given state-space model $\Sigma_c = (F,G,H,J)$, implementing the leader's control protocol (8) leads the multi-agent system to leader-controlled distributed consensus.

Proposition 6. *Consider a multi-agent system with a single leader described as in (7), exerting the control protocol (8), for given matrices (F,G,H,J). Assume that Assumptions* **(A1)**-**(A6)** *and the following conditions hold:*

i) $W(0) = 0$;
ii) *$\mathcal{M}$ has a simple and strictly dominant eigenvalue in 0.*

Then
(A) $\exists\ \mathbf{z} \in \mathbb{R}^\mu$ such that $\begin{bmatrix}(\mathbf{1}_N\otimes\mathbf{v}_A)^\top & \mathbf{z}^\top\end{bmatrix}^\top$ is a right eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0.
(B) $\exists\ \mathbf{w}_{ext} \in \mathbb{R}^\mu$ such that $\begin{bmatrix}(\mathbf{w}_{\mathcal{L}}\otimes\mathbf{w}_A)^\top & \mathbf{w}_{ext}^\top\end{bmatrix}^\top$ is a left eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0.
(C) The control (8) leads the system to leader-controlled distributed consensus, under the leader's action $v(t)$.
(D) Condition (16) holds for

$$\tilde{\boldsymbol{\nu}}(\mathbf{x}(0)) = \frac{(\mathbf{w}_{\mathcal{L}}\otimes\mathbf{w}_A)^\top\mathbf{x}(0)}{1+\mathbf{w}_{ext}^\top\mathbf{z}}\ \mathbf{v}_A, \quad (18)$$

and hence the scaling factor C in (17) is $C = [1+\mathbf{w}_{ext}^\top\mathbf{z}]^{-1}$.

Proof. We first note that since $W(0) = 0$ then all polynomials $y_i(s)$ in the lcMFD of $W(s)$ have a common zero in 0. This also implies, by the coprimality of $x(s), y_1(s), \dots, y_n(s)$, that $x(0) \neq 0$ and hence, by Assumption **(A4)**, F is nonsingular. Condition (A) holds if and only if

$$\begin{bmatrix}(I_N\otimes A) - (\mathcal{L}\otimes BK) + (\mathbf{e}_1\mathbf{e}_1^\top\otimes BJ) & \mathbf{e}_1\otimes BH \\ \mathbf{e}_1^\top\otimes G & F\end{bmatrix}\cdot \begin{bmatrix}\mathbf{1}_N\otimes\mathbf{v}_A \\ \mathbf{z}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

which is equivalent to saying that

$$[(I_N\otimes A) - (\mathcal{L}\otimes BK) + (\mathbf{e}_1\mathbf{e}_1^\top\otimes BJ)](\mathbf{1}_N\otimes\mathbf{v}_A) + +(\mathbf{e}_1\otimes BH)\mathbf{z} = 0 \quad (19)$$

$$(\mathbf{e}_1^\top\otimes G)(\mathbf{1}_N\otimes\mathbf{v}_A) + F\mathbf{z} = 0 \quad (20)$$

On the other hand, as F is nonsingular, from (20) we get $\mathbf{z} = -F^{-1}(\mathbf{e}_1^\top\otimes G)(\mathbf{1}_N\otimes\mathbf{v}_A)$, that once replaced in (19) gives $[(I_N\otimes A) - (\mathcal{L}\otimes BK) + (\mathbf{e}_1\mathbf{e}_1^\top\otimes B(J - HF^{-1}G))](\mathbf{1}_N\otimes\mathbf{v}_A) = 0$. Since $[(I_N\otimes A) - (\mathcal{L}\otimes BK)](\mathbf{1}_N\otimes\mathbf{v}_A) = 0$, because, by assumption, the original multi-agent system was achieving (standard) consensus, it follows that the previous identity holds if and only if $(\mathbf{e}_1\mathbf{e}_1^\top\otimes B(J - HF^{-1}G))(\mathbf{1}_N\otimes\mathbf{v}_A) = 0$, namely $(J - HF^{-1}G)\mathbf{v}_A = 0$. As $W(0) = J - HF^{-1}G = 0$ by condition i), this ensures that (A) holds.

The proof of (B) is very similar to the one of (A), and hence we give a sketch of it. To prove that $\exists\ \mathbf{w}_{ext} \in \mathbb{R}^\mu$ such that $\begin{bmatrix}(\mathbf{w}_{\mathcal{L}}\otimes\mathbf{w}_A)^\top & \mathbf{w}_{ext}^\top\end{bmatrix}^\top$ is a left eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0, we note that $\begin{bmatrix}(\mathbf{w}_{\mathcal{L}}\otimes\mathbf{w}_A)^\top & \mathbf{w}_{ext}^\top\end{bmatrix}^\top\mathcal{M} = \begin{bmatrix}0^\top & 0^\top\end{bmatrix}$ is equivalent to

$$(\mathbf{w}_{\mathcal{L}}\otimes\mathbf{w}_A)^\top(\mathbf{e}_1\mathbf{e}_1^\top\otimes BJ) + \mathbf{w}_{ext}^\top(\mathbf{e}_1^\top\otimes G) = 0^\top \quad (21)$$

$$(\mathbf{w}_{\mathcal{L}}\otimes\mathbf{w}_A)^\top(\mathbf{e}_1\otimes BH) + \mathbf{w}_{ext}^\top F = 0^\top \quad (22)$$

where we used the fact that $\mathbf{w}_{\mathcal{L}}\otimes\mathbf{w}_A$ is a left eigenvector of $(I_N\otimes A) - (\mathcal{L}\otimes BK)$ corresponding to the zero eigenvalue, By using, again, the nonsingularity of F, we show that the previous conditions hold if and only if

$$\frac{1}{N}\mathbf{w}_A^\top B(J - HF^{-1}G) = \frac{1}{N}\mathbf{w}_A^\top BW(0) = 0^\top.$$

This identity holds because of i), and hence (B) holds.

We now show that, by making use of condition ii) and point (A) we can prove (C). Consider a basis of $\mathbb{R}^{nN+\mu}$ consisting of (generalised) eigenvectors of $\mathcal{M}$, having $\hat{\mathbf{z}} := \begin{bmatrix}(\mathbf{1}_N\otimes\mathbf{v}_A)^\top & \mathbf{z}^\top\end{bmatrix}^\top$ as its first element. Every $\boldsymbol{\chi}(0) = \begin{bmatrix}\mathbf{x}(0)^\top & \mathbf{0}^\top\end{bmatrix}^\top$ can be expressed as a linear combination of such eigenvectors. Let α be the coefficient weighting the eigenvector $\hat{\mathbf{z}}$ in the expression of $\boldsymbol{\chi}(0)$. The state evolution of system (10) asymptotically converges to $\alpha \left[(\mathbf{1}_N \otimes \mathbf{v}_A)^\top \quad \mathbf{z}^\top \right]^\top$ and hence $\mathbf{x}(t) \to \alpha(\mathbf{1}_N \otimes \mathbf{v}_A)$ for $t \to +\infty$. To determine the value of α, consider the previously determined left eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0 (see point (B)). Note that for every $t \geq 0$

$$\begin{bmatrix} (\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A) \\ \mathbf{w}_{ext} \end{bmatrix}^\top \dot{\boldsymbol{\chi}}(t) = \begin{bmatrix} (\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A) \\ \mathbf{w}_{ext} \end{bmatrix}^\top \mathcal{M}\boldsymbol{\chi}(t) = 0,$$

so that

$$\begin{bmatrix} (\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A) \\ \mathbf{w}_{ext} \end{bmatrix}^\top \boldsymbol{\chi}(t) = \begin{bmatrix} (\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A) \\ \mathbf{w}_{ext} \end{bmatrix}^\top \boldsymbol{\chi}(0) \qquad \forall t \geq 0$$

and, since $\boldsymbol{\varepsilon}(0) = 0$, the right term of the previous identity can be rewritten as

$$\begin{bmatrix} (\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A) \\ \mathbf{w}_{ext} \end{bmatrix}^\top \boldsymbol{\chi}(0) = (\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A)^\top \mathbf{x}(0).$$

On the other hand, $\lim_{t\to+\infty} \begin{bmatrix} (\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A) \\ \mathbf{w}_{ext} \end{bmatrix}^\top \boldsymbol{\chi}(t) = \begin{bmatrix} (\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A) \\ \mathbf{w}_{ext} \end{bmatrix}^\top \alpha \begin{bmatrix} (\mathbf{1}_N \otimes \mathbf{v}_A) \\ \mathbf{z} \end{bmatrix} = \alpha[(\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A)^\top (\mathbf{1}_N \otimes \mathbf{v}_A) + \mathbf{w}_{ext}^\top \mathbf{z}]$. This implies that

$$\alpha = \frac{(\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A)^\top \mathbf{x}(0)}{(\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A)^\top (\mathbf{1}_N \otimes \mathbf{v}_A) + \mathbf{w}_{ext}^\top \mathbf{z}} = \frac{(\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A)^\top \mathbf{x}(0)}{1 + \mathbf{w}_{ext}^\top \mathbf{z}}$$

where we used the identity $(\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A)^\top (\mathbf{1}_N \otimes \mathbf{v}_A) = 1$, and hence (16) holds for

$$\tilde{\boldsymbol{\nu}}(\mathbf{x}(0)) = \frac{(\mathbf{w}_\mathcal{L} \otimes \mathbf{w}_A)^\top \mathbf{x}(0)}{1 + \mathbf{w}_{ext}^\top \mathbf{z}} \mathbf{v}_A = \frac{\boldsymbol{\nu}(\mathbf{x}(0))}{1 + \mathbf{w}_{ext}^\top \mathbf{z}}.$$

This ensures leader-controlled distributed consensus and proves at the same time (C) and (D). □

Propositions 4 and 6 allow to derive, under Assumptions **(A1)-(A6)**, the following concluding result.

Theorem 7. *Consider the multi-agent system with a single leader as in* (7)*. Under Assumptions* **(A1)-(A6)***, there exist* $\mu \in \mathbb{Z}_+, \mu > 0$*, and matrices* (F, G, H, J)*, with* $F \in \mathbb{R}^{\mu\times\mu}, G \in \mathbb{R}^{\mu\times n}, H \in \mathbb{R}^{1\times\mu}$*,* $J \in \mathbb{R}^{1\times n}$*, such that system* (7)*, under the leader control action* (8)*, achieves consensus with arbitrary convergence speed.*

Proof. By Proposition 6, we need to show that under Assumptions **(A1)-(A6)** we can find $\mu \in \mathbb{Z}_+$ and matrices (F, G, H, J), with F nonsingular, such that $W(0) = J - HF^{-1}G = 0$ and the characteristic polynomial $p_\mathcal{M}(s)$ of the resulting matrix $\mathcal{M}$ takes the form $p_\mathcal{M}(s) = s \cdot \hat{p}_\mathcal{M}(s)$, where $\hat{p}_\mathcal{M}(s)$ is a Hurwitz polynomial with (essentially) arbitrarily chosen zeroes. By Proposition 4, we have

$$p_\mathcal{M}(s) = q_\Psi(s)x(s) - d_\Psi(s)(\sum_{i=0}^{n-1} y_{i+1}(s)s^i), \tag{23}$$

where $q_\Psi(s) = \Delta_A(s)d_\Psi(s) - n_\Psi(s)\kappa(s)$, $x(s)^{-1}\left[y_1(s) \quad \dots y_n(s)\right]$ is an lcMFD of $W(s)$ and $\frac{n_\Psi(s)}{d_\Psi(s)}$ is the irreducible representation of $\Psi(s)$ given in Lemma 10. We first observe that the pair $(q_\Psi(s), d_\Psi(s))$ is coprime. Indeed, suppose that $\alpha \in \mathbb{C}$ is a zero of $d_\Psi(s)$. This amounts to saying (see Lemma 10) that there exists $i \in [1, N-1]$ such that α is a zero of $\det(sI_n - A + \mu_i BK) = \Delta_A(s) + \mu_i\kappa(s)$. On the other hand, if $0 = q_\Psi(\alpha) = \Delta_A(\alpha)d_\Psi(\alpha) - n_\Psi(\alpha)\kappa(\alpha) = -n_\Psi(\alpha)\kappa(\alpha)$, then by the coprimality of $(n_\Psi(s), d_\Psi(s))$ it follows that $\kappa(\alpha) = 0$. But if α is a common zero of $\det(sI_n - A + \mu_i BK) = \Delta_A(s) + \mu_i\kappa(s)$ and $\kappa(s)$, then it is a common zero of $\Delta_A(s)$ and $\kappa(s)$, thus contradicting **(A5)**. Secondly, we note that $q_\Psi(0) = 0$, since $\Delta_A(0) = 0$ and $n_\Psi(0) = 0$ (see Lemma 10). So, we assume that $q_\Psi(s) = s \cdot \hat{q}_\Psi(s)$. Finally, we assume that $y_i(s) = s \cdot \hat{y}_i(s)$ for every $i \in [1, N-1]$. This ensures that if the polynomials $x(s), y_1(s), \dots, y_n(s)$ are coprime then $W(0) = \frac{1}{x(0)}\left[y_1(0) \quad \dots \quad y_n(0)\right] = 0$. Moreover, the polynomial Diophantine equation (23) becomes

$$\hat{p}_\mathcal{M}(s) = \hat{q}_\Psi(s)x(s) - d_\Psi(s)(\sum_{i=0}^{n-1} \hat{y}_{i+1}(s)s^i), \tag{24}$$

with $(\hat{q}_\Psi(s), d_\Psi(s))$ coprime. Note that $\deg d_\Psi(s) = (N-1)n$, while $\deg \hat{q}_\Psi(s) = Nn - 1$. The coprimality of $\hat{q}_\Psi(s)$ and $d_\Psi(s)$ ensures that

$$\hat{p}_\mathcal{M}(s) = \hat{q}_\Psi(s)x(s) - d_\Psi(s)t(s), \tag{25}$$

has a solution $(x(s), t(s))$ for every choice of $\hat{p}_\mathcal{M}(s)$. On the other hand, we can always ensure [21] that there exists a solution $(x(s), t(s))$ with $\deg t(s) \leq Nn - 2$. So, if $\deg \hat{p}_\mathcal{M}(s) = (N-1)n + Nn - 1$ (which amounts to assuming $\mu = (N-1)n$), then $\deg \hat{p}_\mathcal{M}(s) = \deg[\hat{q}_\Psi(s)x(s)] = Nn - 1 + \deg x(s)$, and therefore $\deg x(s) = (N-1)n$. We can always ensure that $x(0) \neq 0$, possibly by introducing a small perturbation of the coefficients of $\hat{p}_\mathcal{M}(s)$ that does not significantly affect its zeros. On the other hand, if we represent $t(s)$ as $t(s) = \sum_{i=0}^{Nm-2} t_i s^i$, and we set

$$\begin{aligned} \hat{y}_n(s) &:= t_{n-1} + t_n s + \dots t_{Nn-2} s^{(N-1)n-1} \\ \hat{y}_{n-1}(s) &:= t_{n-2} \\ &\vdots \\ \hat{y}_1(s) &:= t_0, \end{aligned}$$

then (24) holds for the given $x(s)$ and $\hat{y}_i(s), i \in [1, N-1]$. Since, for every $i \in [1, N-1]$, $\deg \hat{y}_i(s) \leq (N-1)n - 1$, then $\deg y_i(s) \leq (N-1)n = \deg x(s)$, so $W(s) = \frac{1}{x(s)}\left[y_1(s) \quad \dots \quad y_n(s)\right]$ is a proper rational matrix that admits a minimal realization $\Sigma = (F, G, H, J)$, of dimension μ, with F nonsingular. □

Remark 8. *Theorem 7 has shown that by assuming* $\mu = (N-1)n$ *we can always guarantee that the resulting multi-agent system reaches consensus with arbitrary convergence speed. As a matter of fact, and as extensively discussed in [16], this is a conservative result, and* (25) *may be solved for much lower values of* μ*, by suitably choosing the zeros of the Hurwitz polynomial* $\hat{p}_\mathcal{M}(s)$*. A detailed analysis of this issue and of the search for the best way to allocate the eigenvalues of* $\mathcal{M}$*, assuming* μ *fixed, is the subject of ongoing research.*

Remark 9. *The scaling factor C, introduced in (17), is a consequence of the fact that the state variables of the overall controlled system include the μ auxiliary state variables ε for which no alignment is required and for which the initial value is set to zero (otherwise they would influence the consensus value). Such a scaling factor is fixed, once the control protocol (8) has been designed, and hence can be easily accounted for (but this requires the leader to provide this information to the other agents). Alternatively, one can ensure $C = 1$ by imposing in the proof of the previous theorem $y_i(s) = s^2\hat{y}_i(s)$. Indeed, it is a matter of simple calculations to show that if this is the case then $HF^{-2}G = \left[\frac{d}{ds}W(s)\right]_{s=0} = 0$ and this ensures $\mathbf{w}_{ext}^\top \mathbf{z} = 0$, i.e., $C = 1$.*

APPENDIX

Lemma 10. *Let $\mathcal{L} \in \mathbb{R}^{N\times N}$ be the Laplacian matrix of an undirected, weighted and connected graph, partitioned as in (12), with $\hat{\mathcal{L}}$ positive definite. Let $T \in \mathbb{R}^{(N-1)\times(N-1)}$ be a nonsingular orthonormal matrix such that $T^\top \hat{\mathcal{L}} T = \mathrm{diag}\{\mu_1, \mu_2, \dots, \mu_{N-1}\} =: \hat{\Lambda}$, with $\mu_i > 0$ for every $i \in [1, N-1]$. Set $\alpha := T^\top \lambda \in \mathbb{R}^{N-1}$. Then $W_{LF}(s) := (\lambda^\top \otimes BK)\cdot[I_{N-1}\otimes(sI_n - A) + \hat{\mathcal{L}}\otimes BK]^{-1}(\lambda\otimes BK) + A - \ell_{11}BK \in \mathbb{R}(s)^{n\times n}$ can be expressed as follows:*

$$W_{LF}(s) = A - \Psi(s)BK, \qquad (26)$$

with $\Psi(s) := \ell_{11} - \left[\sum_{i=1}^{N-1} \alpha_i^2 K(sI_n - A + \mu_i BK)^{-1}B\right]$. Moreover, under Assumptions **(A5)** *and* **(A6)**, *$\Psi(s)$ has an irreducible representation $\frac{n_\Psi(s)}{d_\Psi(s)}$, with $d_\Psi(s) = \prod_{i=1}^{N-1}\det(sI_n - A + \mu_i BK) = \det[I_{N-1}\otimes(sI_n - A) + \hat{\mathcal{L}}\otimes BK]$ and $n_\Psi(s)$ satisfies $n_\Psi(0) = 0$.*

Proof. Set $S := T\otimes I_n$, so that $S^\top = T^\top \otimes I_n = T^{-1}\otimes I_n = S^{-1}$. Set $A_{LF} := I_{N-1}\otimes A - \hat{\mathcal{L}}\otimes BK$, $B_{LF} := \lambda\otimes BK$, and $C_{LF} := \lambda^\top \otimes BK$. It is easy to see that

$$\begin{aligned} S^\top A_{LF} S &= I_{N-1}\otimes A - \hat{\Lambda}\otimes BK, \\ S^\top B_{LF} &= (T^\top\lambda)\otimes BK = \alpha\otimes BK, \\ C_{LF} S &= (\lambda^\top T)\otimes BK = \alpha^\top \otimes BK. \end{aligned}$$

Therefore

$$\begin{aligned} &W_{LF}(s) = C_{LF} S\left[I_{N-1}\otimes(sI_n - A) + \hat{\Lambda}\otimes BK\right]^{-1} \\ &\cdot S^\top B_{LF} + A - \ell_{11}BK \\ &= (\alpha^\top \otimes BK)\left[I_{N-1}\otimes(sI_n - A) + \hat{\Lambda}\otimes BK\right]^{-1} \\ &\cdot(\alpha\otimes BK) + A - \ell_{11}BK \\ &= \left[\sum_{i=1}^{N-1}\alpha_i^2 BK(sI_n - A + \mu_i BK)^{-1}BK\right] + A - \ell_{11}BK. \end{aligned}$$

Since each $K(sI_n - A + \mu_i BK)^{-1}B$ is a scalar rational function, the previous expression can be rewritten as in (26). As far as the last statement is concerned, we have just proved that $\det[I_{N-1}\otimes(sI_n - A) + \hat{\mathcal{L}}\otimes BK] = \prod_{i=1}^{N-1}\det(sI_n - A + \mu_i BK)$. Set $n_i(s) := K\mathrm{adj}(sI_n - A + \mu_i BK)B$ and $d_i(s) := \det(sI_n - A + \mu_i BK)$. Since (A, B) is in controllability canonical form, it is easy to see that $n_i(s) = \kappa(s)$ while $d_i(s) = \Delta_A(s) + \mu_i\kappa(s)$, and by **(A5)** this is an irreducible representation for every i. On the other hand, by **(A6)**, all μ_i's are distinct and this ensures that for each $i \neq j$, $d_i(s)$ and $d_j(s)$ are coprime. It is immediate to see that the least common multiple of the $d_i(s), i\in[1, N-1]$, is $d_\Psi(s)$ and

$$\begin{aligned} \sum_{i=1}^{N-1}\alpha_i^2 K(sI_n - A + \mu_i BK)^{-1}B &= \sum_{i=1}^{N-1}\alpha_i^2\frac{n_i(s)}{d_i(s)} \\ &= \frac{\sum_{i=1}^{N-1}\alpha_i^2 n_i(s)\prod_{j\neq i}d_j(s)}{d_\Psi(s)} \end{aligned}$$

is irreducible. Finally, note that $n_\Psi(0) = 0$ is equivalent to $\Psi(0) = 0$ and hence to $0 = \ell_{11} - \sum_{i=1}^{N-1}\alpha_i^2\frac{\kappa(0)}{\Delta_A(0)+\mu_i\kappa(0)}$, and since $\Delta_A(0) = 0$ this is equivalent in turn to $\ell_{11} = \sum_{i=1}^{N-1}\frac{\alpha_i^2}{\mu_i}$. By the definitions of α and $\mu_i, i\in[1, N-1]$, the identity can be rewritten as $\ell_{11} = \lambda^\top\hat{\mathcal{L}}^{-1}\lambda$, and this identity is true because of the properties of the Laplacian. ☐

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