Leader-controlled protocols to accelerate convergence in consensus networks

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Abstract

In this paper we consider a discrete-time consensus network, and assume that one of the agents acts as a leader and injects an input signal to improve the overall system performance, in particular to increase the speed of convergence to consensus or to achieve finite-time consensus. Two possible control protocols are proposed and the characteristic polynomials of the resulting closed-loop systems are determined. These results allow to investigate consensus and finite-time consensus of the overall systems. Open problems and future research directions conclude the paper.

Keywords Multi-agent systems, consensus networks, convergence speed, finite-time consensus.

I. Introduction

The interest in consensus problems originated in Statistics and Computer Science in the Sixties and since then it stimulated quite an impressive research activity, strongly encouraged by the application areas where consensus finds an immediate application, e.g., formation control, distributed optimization, agreement in social networks or synchronisation, to mention a few [1], [2], [3]. Consensus is all about converging to a common decision, by exchanging information with neighbouring agents/nodes (representing sensors, robots, birds, fishes, ...), and such an agreement should be reached in the most efficient way, namely with the highest possible speed and possibly in finite time. A major role in determining the speed of convergence, both in case of agents described as simple or double integrators and in case of higher-order models for the agents' description, is played by the Laplacian(s) of the communication graph(s). In the simplest case, if we assume that the communication topology is fixed, the speed of convergence

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is determined by the second smallest eigenvalue of the Laplacian [4], and sometimes it may be inadequate to meet the expected system performance [5].

To improve the multi-agent system performance, "second order schedules", based on shift register or accelerated gradient methods and making use of the agents' states in the two previous time instants, have been proposed. In [6] distributed averaging is considered. The performance benefits of adding extra states to distributed averaging iterations is explored. Conditions for convergence and possible ways of optimizing the convergence rates are discussed, and illustrated by means of numerical examples. To improve convergence, as well as to cope with interrupted communications with neighbouring agents, in [7] the case when in addition to the standard weighted average consensus protocol each agent performs some correction, based on the value of its own state at the previous time step, has been considered. The accelerated gradient method is used for distributed resource allocation and consensus problems, and accelerated gradient algorithms are proved to outperform the existing methods available in the literature. In [8] the first theoretical demonstration that adding a local prediction component to the update rule can significantly improve the convergence rate of distributed averaging algorithms is provided. The problem of maximizing the convergence speed is posed as an optimization problem, and an expression of the best possible solution is provided. In [9] gossip algorithms using two-step local memory for each node are studied in order to accelerate distributed averaging and more general multi-step memory are discussed. The paper exploits the same over-relaxation technique as in the previous augmented broadcast algorithms, but the accelerated gossip algorithms discussed here admit probabilistic time-varying systems in which the update matrix depends on time, while previous references considered only the deterministic case. This paper provides theoretical evidence of the fact that shift registers can help accelerate convergence even in the more general context of probabilistic time-varying multi-agent systems. If we assume a discrete-time model for the agents' dynamics and the only information we have about the (symmetric) Laplacian associated with the communication graph is the interval to which the non-trivial eigenvalues of the Laplacian belong, adding one memory slot to each agent allows to improve the (guaranteed) convergence speed. However, in this set-up there is no significant benefit in adding more memory slots [10]. Alternatively, polynomial filters, acting only on the coupling strength (see Section II of the paper) and determining updating algorithm for such a parameter in order to improve the speed of convergence, have been proposed [11]. For a complete overview of the acceleration methods, see [11], [12].

A common trait of the previous references, dealing with control schemes to accelerate consensus, is that all agents have the possibility of storing and elaborating local data. This is not necessarily the most natural solution: there are cases when selecting a subset of nodes with these capabilities is more robust and economically beneficial than attributing these same features, even at a lower level, to all nodes. This is often the case when dealing with electric grids or building automation systems, where most of the nodes have very basic features and only a few of them can perform more advanced data processing. In fact, in a lot of multi-agent systems the agents are not all peers, and consensus is achieved under the leadership of a few of them, named *leaders*, that overview and coordinate the activity of the remaining agents (*followers*), leading them to a common decision. The idea of selecting one or more leaders within a set of agents, with the final goal of improving the converge speed to consensus of the overall multi-agent system, has been previously explored in the literature.

In [13] discrete-time multi-agent systems whose dynamics is either a simple or a double-integrator are considered, and a model predictive control protocol is adopted by the so-called pinning nodes, namely nodes that inject some additional input signal with the purpose of accelerating convergence to consensus. It is worth noticing that the communication constraint on the pinning nodes introduced in [13] is quite strong, since it does not simply require pinning nodes, regarded as leaders to be "connected" with all the followers, but to have direct access to the information of all the networks agents, including the other leaders. In the case of a single leader, the results of [13] would require the communication graph to include a star graph centred in the leader's node. More recently, pinning consensus control has been investigated by imposing weaker assumptions on the communication structure. In particular, in [14] the case of a single pinning agent that is the root of a spanning tree included in the communication graph is considered. By adopting a rather different model for the overall multi-agent system with respect to the one we consider in this paper, it is shown that dynamic pinning control performs at least as well as the static one in increasing the speed of convergence (but no detailed results about the achievable speed of convergence are provided). In [15] it is shown how consensus on a prescribed (and hence a priori known) value can be achieved, by ensuring that a small fraction of the agents introduce a corrective feedback control mechanism, based on the consensus error. In [16], the case of multi-agent networks with communication delays is considered. If the network topology is fixed a multi-hop relay scheme is adopted to ensure rapid consensus. Each agent can receive information from its multi-hop neighbors with a certain delay and the number of hops is optimized to maximize the convergence speed. On the other hand if the communication topology can be used as a design parameter, also a strategy to select the optimal network topology is proposed. In [17], multi-agent consensus in a network where there is an active leader and variable interconnection topology is considered. The consensus to the leader's value is achieved by all the followers through neighbor-based local controller and state-estimation, by assuming that the (acceleration) input of the leader is available.

In all previous references, where the target of accelerating convergence to consensus is led by the action of a group of leaders, either restrictive conditions on the communication graph are introduced or somewhat weak results about the achievable results are derived. Even more, the leaders typically tend to impose the consensus value rather than simply guiding all the agents to an agreement. In this paper we consider cooperative multi-agent systems described as simple (discrete-time) integrators and adopting the standard nearest neighbor linear protocol. The communication topology is fixed and described by a strongly connected directed graph. We assume that a single node acts as a leader and injects in the overall consensus network an input signal, which is locally elaborated based on the past values of the leader's status, with the target of modifying the dynamics of the overall closed-loop system (by allocating its eigenvalues), of increasing the convergence speed to consensus or even ensuring finite-time consensus. As we will see, in this set-up, which may be also preferable for practical reasons, it is possible to achieve better performance in terms of achievable speed of convergence than the commonly used strategy of endowing each agent with some memory storage to update the local state (see Remark 15, below). So, even if surely there are networks for which attributing different computational capabilities to the nodes is not an option, we will show that when this solution is possible, the obtained performance is definitely better. Differently from previous works dealing with methods to accelerate consensus, the leader acts as a coordinator that tries to improve the system performance but does not impose the consensus value (or the tracking of some reference signal). On the contrary, the consensus value is a weighted sum of the agents and the leader's initial values, as in the leaderless consensus case. To achieve this goal, two control protocols are proposed, whose structures are different from those already investigated in the literature. In the first protocol the signal injected by the leader is obtained through a moving average model that makes use of the past values of the leader's state. The second protocol is more elaborated since the injected input is obtained through a dynamic system fed by the leader's state values and making use of a special kind of static output feedback. The first protocol has limited performance and a complete investigation of its capabilities in terms of increased speed of convergence is still open. On the contrary, the second protocol always guarantees, under certain natural assumptions on the relationship between leader and followers that can be formalized in terms of reachability and observability properties, consensus at the desired speed and even finite time consensus. Finite-time consensus represents a quite desirable condition for a discrete-time multi-agent system and it has been the object of intensive research in the last years (see, e.g., [18], [19], [20], [21], [22]). In this paper to achieve finite-time consensus no change is introduced in the communication structure (an aspect that not always represents a design parameter but rather a constraint in the convergence speed optimization problem); the price to pay is the increased size of the controller. Note that in [19], [21], finite-time consensus is achieved through a time-varying control law that requires to modify the coupling strength based on the value of the nonzero eigenvalues of the Laplacian. This is equivalent to designing a sequence of step-sizes so that exact average consensus is achieved in finite-time. The algorithm proposed in this paper is time-invariant. Also, in [18], [20], [22] the main goal is to provide algorithms for an agent to estimate (in a finite number of steps) the asymptotic consensus value based on the values of its own state (or of the local states) on a sufficiently large time window (possibly in the presence of bounded delays in the communication links), while in this paper the leader's control action actually drives the agents' states to the consensus value in finite time, and not only predicts the consensus value.

The paper is organized as follows. In Section II the problem set-up is introduced, and the consensus network with a single leader injecting a control input is presented. In Section III two protocols to synthesize the control input are introduced, and the characteristic polynomials of the resulting systems are derived in Section IV. This immediately allows to determine the conditions for the eigenvalues allocation of the closed-loop systems. In Section V necessary and sufficient conditions for the two control protocols to ensure that the corresponding closed-loop system achieves "leader-controlled distributed consensus" are derived as spectral conditions on the resulting system matrices. As a result, necessary and sufficient conditions for the existence of such leader-controlled distributed consensus control protocols are determined (Section VI). Finite-time leader-controlled distributed consensus is finally investigated in Section VII.

A preliminary version of this paper was presented at the 2017 IFAC World Congress [23]. In [23] the two control protocols have been introduced and their characteristic polynomials have been derived. Preliminary results on consensus and finite-time consensus have been proposed.

The current manuscript provides more in depth analytic results enlightening the role of certain hypotheses, refines the preliminary characterizations presented in [23] and provides them with detailed proofs. Also, counter-examples showing when the sufficient conditions fail to be necessary are discussed.

Notation. $\mathbb{Z}_+$ and $\mathbb{R}$ denote the set of nonnegative integers and the set of real numbers, respectively. We let $\mathbf{e}_i$ be the i-th element of the canonical basis in $\mathbb{R}^n$ (n being clear from the context), with all entries equal to zero except for the i-th one which is unitary. We let $\mathbf{1}_k$ and $\mathbf{0}_k$ denote the k-dimensional real vectors whose entries are all 1 or all 0, respectively. Given a matrix $A \in \mathbb{R}^{n\times n}$, the (i,j)-th entry of A is denoted either by a_{ij} or by $[A]_{ij}$. The *spectrum* of A, denoted by $\sigma(A)$, is the set of its eigenvalues. Given $\alpha_1,\dots,\alpha_n \in \mathbb{R}$, the symbol $\mathrm{diag}\{\alpha_1,\dots,\alpha_n\}$ denotes the n-dimensional diagonal matrix whose (i,i)-th entry is equal to α_i.

We denote by $\mathbb{R}[z]$ the ring of polynomials in the indeterminate z with coefficients in $\mathbb{R}$. Given $A \in \mathbb{R}^{n\times n}$, $p_A(z) := \det(zI_n - A) \in \mathbb{R}[z]$ denotes the *characteristic polynomial* of A. A polynomial $a(z) \in \mathbb{R}[z]$ is said to be *Schur* if $a(\lambda) = 0$ for some $\lambda \in \mathbb{C}$ implies $|\lambda| < 1$. Given two polynomials $a(z)$ and $b(z)$, the expression ∂a denotes the degree of $a(z)$, $a(z)|b(z)$ means that $a(z)$ divides $b(z)$, and $\mathrm{GCD}(a(z), b(z))$ represents the greatest common divisor of $a(z)$ and $b(z)$. Given $a(z)$, $b(z), c(z) \in \mathbb{R}[z]$, the equation $a(z)x(z) + b(z)y(z) = c(z)$, in the unknown polynomials $x(z)$ and $y(z)$ belonging to $\mathbb{R}[z]$, is called *Diophantine equation* (in the ring of polynomials). We refer to [24] for the main results about the existence and paramerization of all solutions $(x(z), y(z))$, and for the uniqueness of the two, in general distinct, “minimal degree solutions”, by this meaning either $(\bar{x}_{min}(z), \bar{y}(z))$ such that $\partial \bar{x}_{min} < \partial x$ for any other solution $(x(z), y(z))$, or $(\bar{x}(z), \bar{y}_{min}(z))$ such that $\partial \bar{y}_{min} < \partial y$ for any other solution $(x(z), y(z))$.

II. Problem setup

Consider a multi-agent system of N agents, each of them indexed in the integer set $\{1, .., N\}$. The state of the i-th agent is described by the scalar variable x_i that updates according to the following discrete-time linear state-space model [4]:

$$x_i(t+1) = x_i(t) + v_i(t), \qquad t \in \mathbb{Z}_+,$$

where v_i is the input signal of the i-th agent. The communication among the N agents is described by a fixed directed graph with *adjacency matrix* $\mathcal{A} \in \mathbb{R}^{N\times N}$. The (i,j)-th entry of $\mathcal{A}$ is positive, i.e., $[\mathcal{A}]_{ij} > 0$, if the information flows from agent j to agent i and $[\mathcal{A}]_{ij} = 0$

otherwise. We assume $[\mathcal{A}]_{ii} = 0$, $\forall i \in \{1, \dots, N\}$. The time-invariance assumption on the communication topology is a restrictive hypothesis, but nonetheless a quite realistic one when dealing with consensus problems for networks whose nodes have fixed positions. For instance, under this assumption consensus algorithms have been effectively applied to provide distributed solutions to fundamental issues for an electric grid, such as state estimation, economic dispatch and optimal power flow [25], thus providing important tools for its evolution to a smart grid. A related recent application field is heat and energy distribution in building automation systems [26]. Each agent adopts the (nearest neighbor linear) consensus protocol [4] which amounts to saying that the input v_i takes the form:

$$v_i(t) = \kappa \sum_{j=1}^{N} [\mathcal{A}]_{ij}(x_j(t) - x_i(t)), \tag{1}$$

where $\kappa > 0$ is a given real parameter known as *coupling strength*. If we stack the states of the agents in a single state vector $\mathbf{x} \in \mathbb{R}^N$, the overall multi-agent system becomes

$$\mathbf{x}(t+1) = (I_N - \kappa L)\mathbf{x}(t) =: A\mathbf{x}(t), \tag{2}$$

where $L = [\ell_{ij}] := \operatorname{diag}\{\sum_{j\neq 1}[\mathcal{A}]_{1j}, \dots, \sum_{j\neq N}[\mathcal{A}]_{Nj}\} - \mathcal{A} \in \mathbb{R}^{N\times N}$ is the *Laplacian* [27] associated with the adjacency matrix $\mathcal{A}$. Note that, by the properties of the Laplacian, we have $A\mathbf{1}_N = \mathbf{1}_N$, and hence 1 is an eigenvalue of A. System (2) can be used to describe a wide variety of practical applications as it describes the situation when each agent/node shares information with its neighbors with the final goal of converging to a common constant value. If so, we refer to the multi-agent system as to a consensus network. More formally, system (2) is a *consensus network* if for every initial state $\mathbf{x}(0)$ there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} \mathbf{x}(t) = \alpha \mathbf{1}_N. \tag{3}$$

The constant α is called the *consensus value* [28] for system (2), corresponding to the given initial state. If the agents' communication graph is strongly connected, namely the Laplacian L is irreducible [28], and the coupling strength κ satisfies the following constraint:

$$0 < \kappa < \frac{1}{\max_{i=1,\dots,N} \ell_{ii}}, \tag{4}$$

system (2) is a consensus network (see Theorem 2 in [28]). Moreover, the consensus value is

$$\alpha = \frac{\mathbf{w}_A^\top \mathbf{x}(0)}{\mathbf{w}_A^\top \mathbf{1}_N}. \tag{5}$$

where $\mathbf{w}_A$ is a left eigenvector of A corresponding to the unitary eigenvalue. Note that the final value on which the agents agree is a linear function of the initial conditions $x_i(0), i \in \{1, 2, \ldots, N\}$, of the agents; this kind of agreement is known as *weighted-average consensus* [28]. In the following we assume that the above conditions are steadily satisfied.

Remark 1. *By assumption* (4) *on* κ, $A = I_N - \kappa L$ *is a positive matrix and, by the assumption that* L *is the Laplacian of a strongly connected graph, it is also irreducible. So, Perron-Frobenius theorem and condition* $A\mathbf{1}_N = \mathbf{1}_N$ *ensure [28] that* 1 *is a simple eigenvalue of* A *and that its modulus is greater than the modulus of any other eigenvalue of* A*.*

We now assume that, in the current set-up, one agent is a leader that can influence the group dynamics through a signal u (in addition to v_i) based on its own past dynamics. The purpose of the leader's action is to improve the performance of the group, e.g. to accelerate consensus, to achieve finite-time consensus, to ensure effective fault tolerance or formation control. Assuming without loss of generality (w.l.o.g.) that the leader is the first agent (if it is not, we can permute the state entries), the multi-agent system with consensus protocol (2) and one leader exerting the control action u becomes:

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + \mathbf{e}_1 u(t). \tag{6}$$

In the sequel, we introduce two different schemes to generate u, and investigate the features and performance of the multi-agent system under these two protocols. In particular, in this paper we explore consensus and finite-time consensus.

III. Proposed control protocols

First control protocol. We first consider the case when the leader elaborates its control input, $u(t)$, based on its own state evolution only. We assume that the leader has the possibility of storing the past values of its own state dynamics in a prescribed time window μ, and the control input $u(t)$ is generated by the moving average (MA) model:

$$u(t) = \sum_{i=0}^{\mu} a_i x_1(t-i), \tag{7}$$

where $\mu > 0$ and $a_0, a_1, \ldots, a_\mu$ are design parameters, with $a_\mu \neq 0$. The idea of resorting to additional control inputs, whose values are determined based on the past state evolutions of the agents, was recently proposed to accelerate convergence toward consensus, by assuming that each node adopts an update rule as in (7). By adopting this control strategy, good consensus speed can be ensured even when there is poor knowledge of the network [7]. Note that in [6], [7] only one memory sample is stored, namely $\mu = 1$. With respect to these references, we assume that only one agent has the possibility of storing its past state evolution and of elaborating an additional control input. If we introduce the augmented state vector

$$\boldsymbol{\chi}(t) := \begin{bmatrix} \mathbf{x}(t) \\ x_1(t-1) \\ \vdots \\ x_1(t-\mu) \end{bmatrix} \in \mathbb{R}^{N+\mu}, \tag{8}$$

then system (6) under the leader control action (7) can be compactly described by:

$$\boldsymbol{\chi}(t+1) = \begin{bmatrix} A + a_0 \mathbf{e}_1 \mathbf{e}_1^\top & \mathbf{e}_1 \mathbf{a}^\top \\ & \\ \mathbf{e}_1 \mathbf{e}_1^\top & F \end{bmatrix} \boldsymbol{\chi}(t) =: \mathcal{M}_1 \boldsymbol{\chi}(t), \tag{9}$$

where

$$\mathbf{a} := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_\mu \end{bmatrix} \in \mathbb{R}^\mu, \quad F := \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{\mu \times \mu}. \tag{10}$$

Second control protocol. An alternative control scheme is the one where the leader's additional control input $u(t)$ is elaborated based on the leader's state, $x_1(t)$, through a special ARMA model, implemented by a state-space model with the following characteristic features: the system input is $x_1(t)$, the system output is $\mathbf{u}(t)$, and the state-update depends on $x_1(t)$ as well as on a $\mathbf{u}(t)$, in a sort of static output feedback (see Fig. 1). The state vector of this μ-dimensional state-space model is $\boldsymbol{\varepsilon}(t) := \begin{bmatrix} \varepsilon_1(t) & \varepsilon_2(t) & \dots & \varepsilon_\mu(t) \end{bmatrix}^\top$, and updates according to the following discrete-time state-space model:

$$\begin{aligned} \varepsilon_1(t+1) &= a_1 x_1(t) - b_1 u(t), \\ \varepsilon_i(t+1) &= \varepsilon_{i-1}(t) + a_i x_1(t) - b_i u(t), \ i = 2, .., \mu, \end{aligned} \tag{11}$$

where b_i and $a_i \in \mathbb{R}$ are design parameters. The leader's input is expressed in terms of these auxiliary variables as follows

$$u(t) = \varepsilon_\mu(t) + a_0 x_1(t). \tag{12}$$

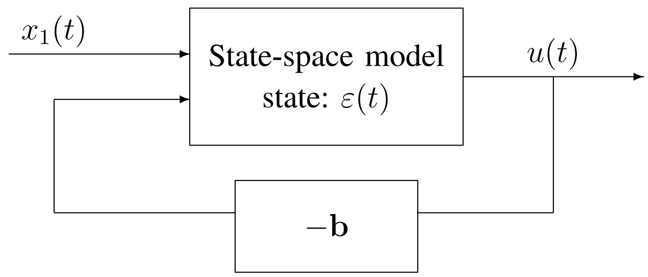

Fig. 1. Block scheme describing the second control protocol.

Also in this case, we can describe the overall muti-agent system in compact form. Upon introducing the augmented state vector

$$\boldsymbol{\chi}(t) := \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\varepsilon}(t) \end{bmatrix}, \tag{13}$$

we obtain

$$\boldsymbol{\chi}(t+1) = \begin{bmatrix} A + a_0 \mathbf{e}_1 \mathbf{e}_1^\top & \mathbf{e}_1 \mathbf{e}_\mu^\top \\ \\ (\mathbf{a} - \mathbf{b} a_0)\mathbf{e}_1^\top & F - \mathbf{b}\mathbf{e}_\mu^\top \end{bmatrix} \boldsymbol{\chi}(t) =: \mathcal{M}_2 \boldsymbol{\chi}(t), \tag{14}$$

where $a_0 \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^\mu$ and $F \in \mathbb{R}^{\mu \times \mu}$ have been defined in (10), and $\mathbf{b} := \begin{bmatrix} b_1 & \dots & b_\mu \end{bmatrix}^\top \in \mathbb{R}^\mu$. A comparison between (9) and (14) reveals that the first control protocol cannot be obtained as a special case of the second one, except when $\mu = 1$, $\mathbf{a} = \mathbf{e}_1$ and $\mathbf{b} = \mathbf{0}$, which is a trivial case.

IV. Characteristic polynomials of the leader-controlled systems

As a first step, in this section we investigate the structures of the characteristic polynomials of the closed-loop multi-agent systems obtained from (6), when the leader adopts either the control protocol (7) or (11)-(12). To this goal it is useful to recall the following well-known technical lemma pertaining the determinant of a block matrix.

Lemma 2. *[29] Given a matrix* $\mathcal{M} = \begin{bmatrix} R & S \\ P & Q \end{bmatrix}$, *with* $R \in \mathbb{R}^{N \times N}$, $S \in \mathbb{R}^{N \times \mu}$, $P \in \mathbb{R}^{\mu \times N}$, $Q \in \mathbb{R}^{\mu \times \mu}$ *and* Q *nonsingular, its determinant can be expressed as* $\det \mathcal{M} = \det Q \cdot \det (R - SQ^{-1}P)$.

Proposition 3. *Consider the multi-agent system (6) with one leader adopting the control protocol (7) or (11)-(12). The characteristic polynomial of the state matrix $\mathcal{M}_1$ of (9) is*

$$p_{\mathcal{M}_1}(z) = z^{\mu} p_A(z) - \sum_{i=0}^{\mu} a_i z^{\mu-i} p_1(z), \tag{15}$$

where $p_A(z) := \det(zI_N - A)$, $p_1(z) := \det\left(zI_{N-1} - A_{(1)}\right)$, and $A_{(1)}$ is the $(N-1)\times(N-1)$ submatrix of A obtained by deleting the first row and the first column. The characteristic polynomial of the state matrix $\mathcal{M}_2$ of system (14) is

$$p_{\mathcal{M}_2}(z) = p_A(z)q(z) - p_1(z)[a_0 z^{\mu} + r(z)], \tag{16}$$

where $q(z) := z^{\mu} + b_{\mu} z^{\mu-1} + \dots + b_2 z + b_1$ and $r(z) := a_{\mu} z^{\mu-1} + \dots + a_2 z + a_1$.

Proof. By making use of Lemma 2, upon assuming $R = zI_N - (A + a_0 \mathbf{e}_1 \mathbf{e}_1^{\top})$, $S = -\mathbf{e}_1 \mathbf{a}^{\top}$, $Q = zI_{\mu} - F$ and $P = -\mathbf{e}_1 \mathbf{e}_1^{\top}$, we obtain $p_{\mathcal{M}_1}(z) = \det(zI_{\mu} - F) \det\left[zI_N - A - a_0 \mathbf{e}_1 \mathbf{e}_1^{\top} - \mathbf{e}_1 \mathbf{a}^{\top}(zI_{\mu} - F)^{-1} \mathbf{e}_1 \mathbf{e}_1^{\top}\right].$ Now, note that $\det(zI_{\mu} - F) = z^{\mu}$ and

$$\mathbf{a}^{\top}(zI_{\mu} - F)^{-1}\mathbf{e}_1 = \mathbf{a}^{\top} \begin{bmatrix} z^{-1} & 0 & \dots & 0 \\ z^{-2} & z^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z^{-\mu} & z^{-\mu+1} & \dots & z^{-1} \end{bmatrix} \mathbf{e}_1 = \sum_{i=1}^{\mu} a_i z^{-i}, \tag{17}$$

and hence

$$p_{\mathcal{M}_1}(z) = z^{\mu} \cdot \det\left[zI_N - A - \mathbf{e}_1 \left(\sum_{i=0}^{\mu} a_i z^{-i}\right) \mathbf{e}_1^{\top}\right].$$

By using the Laplace expansion rule along the first row [29], we obtain that $\det\left[zI - A - \mathbf{e}_1 (\sum_{i=0}^{\mu} a_i z^{-i}) \mathbf{e}_1^{\top}\right] = p_A(z) - (\sum_{i=0}^{\mu} a_i z^{-i}) p_1(z)$. So, we finally get

$$p_{\mathcal{M}_1}(z) = z^{\mu} p_A(z) - \left(\sum_{i=0}^{\mu} a_i z^{\mu-i}\right) p_1(z).$$

Analogously, in the case of $\mathcal{M}_2$ we have, by Lemma 2, $p_{\mathcal{M}_2}(z) = \det(zI_{\mu} - F + \mathbf{b}\mathbf{e}_{\mu}^{\top}) \det\left[zI_N - A - a_0 \mathbf{e}_1 \mathbf{e}_1^{\top} - \mathbf{e}\right.$ $\left.(\mathbf{a} - a_0 \mathbf{b})\mathbf{e}_1^{\top}\right]$. We note that

$$F - \mathbf{b}\mathbf{e}_{\mu}^{\top} = \begin{bmatrix} 0 & 0 & 0 & \dots & -b_1 \\ 1 & 0 & 0 & \dots & -b_2 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & & \ddots & 0 & -b_{\mu-1} \\ 0 & 0 & \dots & 1 & -b_{\mu} \end{bmatrix}$$

is the transposed of a companion matrix, and hence $\det(zI_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top) = z^\mu + \sum_{i=0}^{\mu-1} b_{i+1} z^i = q(z)$. On the other hand,

$$\mathbf{e}_\mu^\top (zI_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top)^{-1}(\mathbf{a} - a_0\mathbf{b}) = \frac{1}{z^\mu + \sum_{i=0}^{\mu-1} b_{i+1} z^i} \begin{bmatrix} 1 & z & \dots & z^{\mu-1} \end{bmatrix} (\mathbf{a} - a_0\mathbf{b})$$

$$= \frac{r(z) - a_0[q(z) - z^\mu]}{q(z)}.$$

Consequently, $p_{\mathcal{M}_2}(z) = q(z)\det\left[zI_N - A - \mathbf{e}_1\left(a_0 + \frac{r(z)-a_0[q(z)-z^\mu]}{q(z)}\right)\mathbf{e}_1^\top\right] = q(z)\det\left[zI_N - A - \mathbf{e}_1 \frac{r(z)+a_0 z^\mu}{q(z)}\mathbf{e}_1^\top\right]$, and by proceeding as before we get $p_{\mathcal{M}_2}(z) = q(z)\cdot\left[p_A(z) - p_1(z)\frac{r(z)+a_0 z^\mu}{q(z)}\right] = q(z)p_A(z) - p_1(z)[a_0 z^\mu + r(z)]$. $\square$

The expressions previously derived for the characteristic polynomials of the closed-loop multi-agent systems, obtained corresponding to the two proposed control protocols, allow to immediately solve the problem of attributing to the overall system a desired characteristic polynomial. Indeed, let $\Psi(z) \in \mathbb{R}[z]$ be a monic polynomial of degree $N + \mu$. In the case of the control law (7), the problem is to find a polynomial $y(z) := -\sum_{i=0}^{\mu} a_i z^{\mu-i}$, of degree at most μ and satisfying $y(0) \neq 0$ (due to the assumption $a_\mu \neq 0$), such that

$$p_A(z)z^\mu + p_1(z)y(z) = \Psi(z), \tag{18}$$

while for the control law (11)-(12), the problem is to determine polynomials $x(z) := z^\mu + \sum_{i=1}^{\mu} b_i z^{i-1}$, monic of degree μ, and $y(z) := -a_0 z^\mu - \sum_{i=1}^{\mu} a_i z^{i-1}$, of degree at most μ, so that

$$p_A(z)x(z) + p_1(z)y(z) = \Psi(z). \tag{19}$$

Note that $p_1(z) = \det(zI_{N-1} - A_{(1)})$ is a monic polynomial of degree $N - 1$. We have the following immediate result.

Proposition 4. *Consider the multi-agent system* (6) *with a single leader, and let* $\Psi(z) \in \mathbb{R}[z]$ *be a monic polynomial of degree* $N + \mu$.

i) There exist $a_0, a_1, \dots, a_\mu \in \mathbb{R}$, $a_\mu \neq 0$, *such that the control protocol* (7) *attributes to the overall multi-agent system* (9) *the characteristic polynomial* $\Psi(z)$ *if and only if (i)* $p_1(z) \mid \Psi(z) - z^\mu p_A(z)$ *and (ii) if* $\Psi(0) = 0$ *then the multiplicities of* 0 *as a zero of* $\Psi(z) - z^\mu p_A(z)$ *and of* $p_1(z)$ *coincide.*

ii) If $\mu \geq N - 1$ *and* $\mathrm{GCD}(p_A(z), p_1(z)) \mid \Psi(z)$, *then there exist* $a_0, a_1, \dots, a_\mu \in \mathbb{R}$ *and* $b_1, b_2 \dots, b_\mu \in \mathbb{R}$ *such that the control protocol* (11)-(12) *attributes to the overall multi-agent system* (14) *the characteristic polynomial* $\Psi(z)$.

Proof. i) Obvious from (18) and condition $y(0) \neq 0$.
ii) If $\mathrm{GCD}(p_A(z), p_1(z)) \mid \Psi(z)$ then the Diophantine equation (19) has necessarily a solution $(x(z), y(z)) \in \mathbb{R}[z] \times \mathbb{R}[z]$. We want to show that when $\mu \geq N-1$ then the solution of minimal degree w.r.t. $y(z)$ is such that $x(z)$ is monic of degree μ, and $y(z)$ has degree at most μ. Clearly, the solution of minimal degree w.r.t. $y(z)$ satisfies [24] $\partial y < \partial \frac{p_A}{\mathrm{GCD}(p_A(z),p_1(z))} \leq N \leq \mu + 1$, and hence $\partial(p_1 y) \leq (N-1) + (N-1) = 2N-2$. This implies that $\partial(p_1 y) < \partial \Psi = N + \mu$ and hence $\partial(p_A x) = \partial \Psi = N + \mu$. Consequently, $\partial x = \mu$ and $x(z)$ is monic because $p_A(z)$ and $\Psi(z)$ are. This ends the proof. $\square$

V. Leader-controlled distributed consensus

As discussed in [30], leaders in a multi-agent system can take different roles. "Power leaders" have dynamics which is independent of those of the other agents, and their role is that of imposing to the followers the direction to follow, the specific goal to achieve. Alternatively, "knowledge leaders" do not impose the final decision but rather the way and the speed at which to achieve it. In this paper we adhere to the second perspective and introduce a definition of consensus for system (14) that imposes the final value to be a linear function of the initial states of all agents, and hence to be a *distributed consensus* (see, e.g., [28]) rather than the prescribed value or signal chosen by the leader. We refer to this type of consensus as *leader-controlled distributed consensus*, in the sense that the consensus condition is achieved under the guidance of a leader, but it is *distributed* as each node contributes not only to the achievement but also to the value of the final consensus. The following definition is inspired by the one given in [22].

Definition 1 (Leader-controlled distributed consensus)**.** *Consider a multi-agent system with a single leader described as in* (6)*. We say that the control protocols* (7) *or* (11)-(12) *lead the system to* leader-controlled distributed consensus *(for short,* LCD consensus*), under the leader's action* $u(t)$*, if there is a vector* $\mathbf{c} \in \mathbb{R}^N$ *such that for every choice of the agents' initial state* $\mathbf{x}_0$ *(and assuming zero initial conditions for the auxiliary variables, i.e.* $\boldsymbol{\chi}(0) = [\mathbf{x}_0^\top \ 0^\top]^\top$*) one has*

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{1}_N (\mathbf{c}^\top \mathbf{x}_0). \tag{20}$$

In this section we investigate under what conditions the closed-loop systems obtained corresponding to the two control protocols achieve leader-controlled distributed consensus. To this goal, to highlight the fact that the leader is the first agent, we assume w.l.o.g. that $A \in \mathbb{R}^{N\times N}$ is block partitioned as:

$$A = I_N - \kappa L = \begin{bmatrix} a_{11} & \mathbf{h}^\top \\ \mathbf{g} & A_{(1)} \end{bmatrix}, \tag{21}$$

where $a_{11} \in \mathbb{R}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{N-1}$ and $A_{(1)} \in \mathbb{R}^{(N-1)\times(N-1)}$ is the principal submatrix of A obtained by deleting its first row and its first column. Also, in the following we will assume that *the triple* $(A_{(1)}, \mathbf{g}, \mathbf{h}^\top)$ *is a minimal realization, namely that the pair* $(A_{(1)}, \mathbf{g})$ *is reachable and the pair* $(A_{(1)}, \mathbf{h}^\top)$ *is observable*. It is worthwhile to comment on the meaning of this assumption in the current set-up. From the perspective of a group of N agents consisting of a leader and $N-1$ followers, the reachability of the pair $(A_{(1)}, \mathbf{g})$ amounts to saying that the leader has the possibility of arbitrarily shaping the dynamics of the followers, while the observability of the pair $(A_{(1)}, \mathbf{h}^\top)$ means that the actions of the followers are completely observable by the leader. These seem to be rather natural assumptions if we want the leader to be able to influence the overall system dynamics to achieve LCD consensus and/or different targets. Note that controllability/observability of leader-follower consensus networks and their relationships with the graph structure have been the subjects of a good number of studies, e.g., [31], [32]. Necessary and sufficient conditions for controllability/observability in terms of graph features are available only for special graph structures (such as path graphs, cycle graphs and some others), but these properties are very easy conditions to check. As it has been shown in the Appendix (see Lemma 20), the assumption that $(A_{(1)}, \mathbf{g}, \mathbf{h}^\top)$ is a minimal realization is equivalent to assuming that the characteristic polynomials of A and $A_{(1)}$, namely $p_A(z)$ and $p_1(z)$, are coprime.

Remark 5. *Note that since A is positive and Schur, then $A_{(1)}$, being a principal submatrix of A, is Schur in turn. So, even if the matrices would have some common eigenvalues, this would result in the fact that $(A_{(1)}, \mathbf{g})$ is nonetheless always stabilizable and $(A_{(1)}, \mathbf{h}^\top)$ is always detectable. So, the polynomials $p_A(z)$ and $p_1(z)$ would have some common zero of modulus strictly smaller than 1. These common eigenvalues are inherited by the matrices $\mathcal{M}_i, i = 1, 2$, and represent a lower bound on the modulus of the second largest eigenvalue of the matrices $\mathcal{M}_i$ and hence a limit to the best achievable convergence speed. In particular, if such common zeros are not 0 then finite-time LCD consensus cannot be achieved. We omit this analysis due to space constraints.*

Under the minimality assumption on the triple $(A_{(1)}, \mathbf{g}, \mathbf{h}^\top)$, the multi-agent system (9), obtained corresponding to the control protocol (7), achieves LCD consensus if and only if the spectrum of the matrix $\mathcal{M}_1$ satisfies certain specific conditions, as shown in Theorem 6, below.

Theorem 6. *Consider the multi-agent system (6) with one leader adopting the control protocol (7). The resulting multi-agent system reaches leader-controlled distributed consensus (when all the auxiliary variables are zero at $t=0$) if and only if:*

- **(C-1)** *The state matrix $\mathcal{M}_1$ has an eigenvalue in 1 (and hence, by Lemma 21, part i), the corresponding right eigenvector is $\begin{bmatrix}\mathbf{1}_N^\top & \mathbf{1}_\mu^\top\end{bmatrix}^\top$).*
- **(C-2)** *1 is a simple eigenvalue of $\mathcal{M}_1$ and it is strictly dominant, namely all the other eigenvalues have moduli smaller than 1.*

*If conditions **(C-1)** and **(C-2)** hold, then*

(a1) The parameters $a_0 \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^\mu$ satisfy

$$a_0 + \mathbf{a}^\top \mathbf{1}_\mu = 0. \tag{22}$$

(a2) The definition of LCD consensus holds for (see (20))

$$\mathbf{c} = \frac{\mathbf{w}}{\mathbf{w}^\top \mathbf{1}_N + \mathbf{w}_{ext}^\top \mathbf{1}_\mu}, \tag{23}$$

where $\begin{bmatrix}\mathbf{w}^\top & \mathbf{w}_{ext}^\top\end{bmatrix}^\top$, $\mathbf{w} \in \mathbb{R}^N$ and $\mathbf{w}_{ext} \in \mathbb{R}^\mu$, is a left eigenvector of $\mathcal{M}_1$ corresponding to the eigenvalue 1.

(a3) The vector $\mathbf{w} \in \mathbb{R}^N$ in (a2) is an eigenvector of A corresponding to 1, and if we assume w.l.o.g. that $\mathbf{w}^\top \mathbf{e}_1 = 1$[1], then

(a4) the vector $\mathbf{c}$ in (23) becomes

$$\mathbf{c} = \frac{\mathbf{w}}{\mathbf{w}^\top \mathbf{1}_N + \mathbf{a}^\top (I_\mu - F)^{-1} \mathbf{1}_\mu}.$$

Proof. The overall controlled system, corresponding to the control protocol (7), is described as in (9), and we assume the variable $\mathbf{x}(t)$ as the output of the overall state-space model:

$$\mathbf{x}(t) = \begin{bmatrix} I_N & 0 \end{bmatrix} \boldsymbol{\chi}(t) =: H\boldsymbol{\chi}(t). \tag{24}$$

[Sufficiency + (a1)-(a2)] First of all note that if **(C-1)** holds, then by Lemma 21, part i), condition (a1) holds. Assume **(C-1)** and **(C-2)** hold, and consider a basis of $\mathbb{R}^{N+\mu}$ consisting

[1]Being A an irreducible Metzler matrix, with strictly dominant eigenvalue 1, it admits a strictly positive left eigenvector corresponding to 1, and hence all the other eigenvectors, being multiple of that eigenvector, have nonzero entries. So, in particular, the first entry is nonzero.

of (generalised) eigenvectors of $\mathcal{M}_1$, having $\mathbf{v} := \begin{bmatrix} \mathbf{1}_N^\top & \mathbf{1}_\mu^\top \end{bmatrix}^\top$ as its first element. Every $\boldsymbol{\chi}(0) = \begin{bmatrix} \mathbf{x}_0^\top & 0 \end{bmatrix}^\top$ can be expressed as a linear combination of such eigenvectors (see e.g. [33]). We let α be the coefficient weighting the eigenvector $\mathbf{v}$ in the expression of $\boldsymbol{\chi}(0)$. The state evolution of system (9) asymptotically converges to $\alpha \cdot \begin{bmatrix} \mathbf{1}_N^\top & \mathbf{1}_\mu^\top \end{bmatrix}^\top$, and hence $\mathbf{x}(t) \to \alpha \mathbf{1}_N$ as t goes to $+\infty$. This ensures leader-controlled distributed consensus (and hence completes the sufficiency part). To determine the value of α, note that for every $t \geq 0$ one has $\begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix}^\top \boldsymbol{\chi}(t) = \begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix}^\top \mathcal{M}_1^t \boldsymbol{\chi}(0) = \begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix}^\top \boldsymbol{\chi}(0) = \mathbf{w}^\top \mathbf{x}_0$. On the other hand, $\lim_{t\to+\infty} \begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix}^\top \boldsymbol{\chi}(t) = \alpha \begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix}^\top \begin{bmatrix} \mathbf{1}_N \\ \mathbf{1}_\mu \end{bmatrix} = \alpha(\mathbf{w}^\top \mathbf{1}_N + \mathbf{w}_{ext}^\top \mathbf{1}_\mu)$. This allows to say that $\alpha = \dfrac{\mathbf{w}^\top \mathbf{x}_0}{\mathbf{w}^\top \mathbf{1}_N + \mathbf{w}_{ext}^\top \mathbf{1}_\mu}$, and hence $\mathbf{c}$ is expressed as in (23), thus proving (a2).

[Necessity] By definition of leader-controlled distributed consensus, for every initial condition $\boldsymbol{\chi}(0) = \begin{bmatrix} \mathbf{x}_0^\top & 0 \end{bmatrix}^\top$ the trajectory $\mathbf{x}(t)$ converges to $(\mathbf{c}^\top \mathbf{x}_0)\mathbf{1}_N$. This implies that 1 is an eigenvalue of $\mathcal{M}_1$ and hence **(C-1)** holds. Also, by Lemma 21 part i), we can claim that $\mathbf{v} := \begin{bmatrix} \mathbf{1}_N^\top & \mathbf{1}_\mu^\top \end{bmatrix}^\top$ is an eigenvector of $\mathcal{M}_1$ corresponding to 1, and there are no eigenvectors of $\mathcal{M}_1$ corresponding to 1 that are linearly independent of $\mathbf{v}$. So, the geometric multiplicity of 1 as an eigenvalue of $\mathcal{M}_1$ is unitary. To complete the proof of **(C-2)**, we preliminarily show that the pair $(\mathcal{M}_1, H) = \left(\mathcal{M}_1, \begin{bmatrix} I_N & 0 \end{bmatrix}\right)$ is observable. To this end it is sufficient to note that if this were not the case then $\lambda \in \sigma(\mathcal{M}_1)$ and $\begin{bmatrix} \mathbf{z}^\top & \mathbf{z}_{ext}^\top \end{bmatrix}^\top \neq 0$ could be found (see PBH test [34]) such that

$$\begin{bmatrix} \lambda I_{N+\mu} - \mathcal{M}_1 \\ H \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_{ext} \end{bmatrix} = 0,$$

but this amounts to saying that $\lambda \in \sigma(\mathcal{M}_1)$ and $\mathbf{z}_{ext} \neq 0$ could be find such that $\begin{bmatrix} -\mathbf{e}_1 \mathbf{a}^\top \\ \lambda I_\mu - F \end{bmatrix} \mathbf{z}_{ext} = 0$. By the assumption $a_\mu \neq 0$ and the structure of F such a vector $\mathbf{z}_{ext} \neq 0$ does not exist, and hence $(\mathcal{M}_1, H)$ is observable.

We now want to show that if one of the following situations arises: (a) there exists $\lambda \in \sigma(\mathcal{M}_1)$, $|\lambda| \geq 1$ and $\lambda \neq 1$, or (b) $\lambda = 1$ is an eigenvalue of $\mathcal{M}_1$ with multiplicity in the minimal annihilating polynomial [33] of $\mathcal{M}_1$ greater than 1, then there exists $\mathbf{x}_0 \neq 0$ such that LCD consensus among the agents is not reached. Assume, w.l.o.g., that λ is an eigenvalue of $\mathcal{M}_1$ with largest modulus. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{N+\mu}\}$ be a basis of (right) generalised eigenvectors of $\mathcal{M}_1$, and assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are generalised eigenvectors corresponding to λ, while all the remaining ones are generalised eigenvectors corresponding to eigenvalues of $\mathcal{M}_1$ different from λ. Let $\tilde{\mathbf{w}} \in \mathbb{R}^\mu$ be a left eigenvector of $\mathcal{M}_1$ corresponding to λ. By Lemma 21, part ii), we know that the first N entries of $\tilde{\mathbf{w}}$ cannot be zero, and hence $\mathbf{x}_0 \in \mathbb{R}^N$ can be found such that $\tilde{\mathbf{w}}^\top \begin{bmatrix} \mathbf{x}_0 \\ 0 \end{bmatrix} \neq 0$. On the other hand, being $\mathcal{B}$ a basis of $\mathbb{R}^{N+\mu}$, complex coefficients α_i can be found such that $\begin{bmatrix} \mathbf{x}_0 \\ 0 \end{bmatrix} = \sum_{i=1}^{N+\mu} \alpha_i \mathbf{v}_i$. Since $\tilde{\mathbf{w}}^\top \mathbf{v}_j = 0, j = r+1, \dots, N+\mu$, this means that $0 \neq \tilde{\mathbf{w}}^\top \begin{bmatrix} \mathbf{x}_0 \\ 0 \end{bmatrix} = \sum_{i=1}^{r} \alpha_i \tilde{\mathbf{w}}^\top \mathbf{v}_i$. And by the observability of the pair $(\mathcal{M}_1, \begin{bmatrix} I_N & 0 \end{bmatrix})$ it follows that as t goes to $+\infty$, $\mathbf{x}(t)$ tends to align to some vector that has a nonzero projection on at least one of the eigenvectors in the set $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$. In case (a) it is obvious that the state evolution has a dominant mode which is not constant, and hence the state $\boldsymbol{\chi}(t)$ cannot converge to a constant vector with all identical entries. In case (b) we may assume w.l.o.g. that $\mathbf{v}_i, i = 1, 2, \dots, r$, is a generalized eigenvector of order i of $\mathcal{M}_1$ corresponding to 1 and $\mathbf{v}_{i-1} = (\mathcal{M}_1 - I_{N+\mu})\mathbf{v}_i$. Since $\tilde{\mathbf{w}}^\top(\mathcal{M}_1 - I_{N+\mu}) = \mathbf{0}^\top$, it follows that $\tilde{\mathbf{w}}^\top \mathbf{v}_i = \mathbf{0}^\top$ for $i = 1, 2, \dots, r-1$, and hence it must be $\tilde{\mathbf{w}}^\top \mathbf{v}_r \neq \mathbf{0}^\top$. Consequently, also in this case we know that the state evolution has a dominant mode which is not constant, and LCD consensus is not achieved. Therefore condition **(C-2)** holds.

To conclude the proof, we only need to show that if **(C-1)** and **(C-2)** hold, then (a3) and (a4) hold. To prove (a3), we have to show that $\mathbf{w}$ is a left eigenvector of A corresponding to the unitary eigenvalue. The vector $\begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix}$ is a left eigenvalue of $\mathcal{M}_1$ corresponding to 1 if and only if

$$\begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix} \begin{bmatrix} A + a_0 \mathbf{e}_1 \mathbf{e}_1^\top & \mathbf{e}_1 \mathbf{a}^\top \\ \\ \mathbf{e}_1 \mathbf{e}_1^\top & F \end{bmatrix} = \begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix},$$

which is equivalent, after a few manipulations (see (17)), to

$$\mathbf{w}_{ext}^\top = \mathbf{w}^\top \mathbf{e}_1 \mathbf{a}^\top (I_\mu - F)^{-1}, \tag{25}$$

$$0 = \mathbf{w}^\top [I_\mu - A - (a_0 + \mathbf{a}^\top \mathbf{1}_\mu) \mathbf{e}_1 \mathbf{e}_1^\top]. \tag{26}$$

By (a1), the second identity becomes $0 = \mathbf{w}^\top [I_\mu - A]$, thus proving that $\mathbf{w}$ is an eigenvector of A corresponding to 1, namely $\mathbf{w} = \mathbf{w}_A$ (see (5)). Based on the expression of $\mathbf{c}$ given in (23), in order to prove (a4) we simply need to show that if $\mathbf{w}^\top \mathbf{e}_1 = 1$ then $\mathbf{w}_{ext}^\top = \mathbf{a}^\top (I_\mu - F)^{-1}$, but this follows immediately from (25). □

We now consider the LCD consensus problem in case of the second protocol and derive the same spectral conditions on the matrix $\mathcal{M}_2$.

Theorem 7. *Consider the multi-agent system (6) with one leader adopting the control protocol (11)-(12), and assume that the two polynomials $q(z)$ and $a_0 z^\mu + r(z)$, where $q(z) = z^\mu + b_\mu z^{\mu-1} + \cdots + b_2 z + b_1$ and $r(z) = a_\mu z^{\mu-1} + \cdots + a_2 z + a_1$, have no common zero of modulus greater than or equal to 1, and that $q(1) = 1 + \mathbf{b}^\top \mathbf{1}_\mu \neq 0$. The resulting multi-agent system reaches leader-controlled distributed consensus (when all the auxiliary variables are zero at $t = 0$) if and only if:*

- **(C-1)** *The state matrix $\mathcal{M}_2$ has an eigenvalue in 1 (and hence, by Lemma 22, part i) the corresponding right eigenvector is $\begin{bmatrix} \mathbf{1}_N^\top & ((I_\mu - F)^{-1}\mathbf{a})^\top \end{bmatrix}^\top$).*
- **(C-2)** *1 is a simple eigenvalue of $\mathcal{M}_2$ and it is strictly dominant.*

*If conditions **(C-1)** and **(C-2)** hold, then*

(b1) The parameters $a_0 \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^\mu$ satisfy (22).

(b2) The definition of LCD consensus holds for

$$\mathbf{c} = \frac{\mathbf{w}}{\mathbf{w}^\top \mathbf{1}_N + \mathbf{w}_{ext}^\top (I_\mu - F)^{-1}\mathbf{a}}, \tag{27}$$

where $\begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix}^\top$, with $\mathbf{w} \in \mathbb{R}^N$ and $\mathbf{w}_{ext} \in \mathbb{R}^\mu$, is a left eigenvector of $\mathcal{M}_2$ corresponding to the eigenvalue 1.

(b3)] The vector $\mathbf{w} \in \mathbb{R}^N$ in (b2) is an eigenvector of A corresponding to 1, and if we assume w.l.o.g. that $\mathbf{w}^\top \mathbf{e}_1 = 1$, then

(b4) the vector $\mathbf{c}$ in (27) becomes

$$\mathbf{c} = \frac{\mathbf{w}}{\mathbf{w}^\top \mathbf{1}_N + \dfrac{1}{1 + \mathbf{1}^\top \mathbf{b}} \mathbf{1}_\mu^\top (I_\mu - F)^{-1}\mathbf{a}}.$$

Proof. Also in this case we assume that the system output is described as in (24). The [Sufficiency] is identical to the one for the first protocol, and it also leads, through Lemma 22 part i), to conditions (b1) and (b2). The [Necessity] is also similar to the one for the first protocol and it also relies on Lemma 22. So, for the sake of brevity we omit them.

We now show that if **(C-1)** and **(C-2)** hold, then (b3) and (b4) hold. To prove (b3), let $\begin{bmatrix}\mathbf{w}^\top & \mathbf{w}_{ext}^\top\end{bmatrix}$ be a left eigenvector of $\mathcal{M}_2$ corresponding to 1. This amounts to saying that

$$\begin{bmatrix}\mathbf{w}^\top & \mathbf{w}_{ext}^\top\end{bmatrix}\begin{bmatrix} A + a_0\mathbf{e}_1\mathbf{e}_1^\top & \mathbf{e}_1\mathbf{e}_\mu^\top \\ \\ (\mathbf{a}-a_0\mathbf{b})\mathbf{e}_1^\top & F - \mathbf{b}\mathbf{e}_\mu^\top \end{bmatrix} = \begin{bmatrix}\mathbf{w}^\top & \mathbf{w}_{ext}^\top\end{bmatrix},$$

which is equivalent to

$$\begin{aligned} \mathbf{w}^\top[A + a_0\mathbf{e}_1\mathbf{e}_1^\top - I_\mu] + \mathbf{w}_{ext}^\top(\mathbf{a}-a_0\mathbf{b})\mathbf{e}_1^\top &= 0, \\ \mathbf{w}_{ext}^\top(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top) &= \mathbf{w}^\top\mathbf{e}_1\mathbf{e}_\mu^\top. \end{aligned}$$

By assumption, $q(1) = 1 + \mathbf{b}^\top\mathbf{1}_\mu \neq 0$, and since $\det(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top) = 1 + \mathbf{b}^\top\mathbf{1}_\mu$, it follows that $I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top$ is nonsingular. Therefore we obtain

$$\mathbf{w}_{ext}^\top = (\mathbf{w}^\top\mathbf{e}_1)\mathbf{e}_\mu^\top(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top)^{-1}, \tag{28}$$

$$0 = \mathbf{w}^\top[A + a_0\mathbf{e}_1\mathbf{e}_1^\top - I_\mu] + (\mathbf{w}^\top\mathbf{e}_1)\mathbf{e}_\mu^\top(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top)^{-1}(\mathbf{a}-a_0\mathbf{b})\mathbf{e}_1^\top. \tag{29}$$

We observe that

$$\mathbf{e}_\mu^\top(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top)^{-1}(\mathbf{a}-a_0\mathbf{b}) = \frac{1}{1+\mathbf{b}^\top\mathbf{1}_\mu}\mathbf{1}_\mu^\top(\mathbf{a}-a_0\mathbf{b}) \tag{30}$$

$$= -\frac{a_0}{1+\mathbf{b}^\top\mathbf{1}_\mu}(1 + \mathbf{1}_\mu^\top\mathbf{b}) = -a_0, \tag{31}$$

where we made use of (b1). Therefore condition (29) becomes

$$0 = \mathbf{w}^\top[A + a_0\mathbf{e}_1\mathbf{e}_1^\top - I_\mu] - a_0(\mathbf{w}^\top\mathbf{e}_1)\mathbf{e}_1^\top = \mathbf{w}^\top[A - I_\mu]$$

which proves that $\mathbf{w}$ is a left eigenvector of A corresponding to 1. Finally, based on the expression of $\mathbf{c}$ given in (27), in order to prove (b4) we simply need to show that if $\mathbf{w}^\top\mathbf{e}_1 = 1$ then $\mathbf{w}_{ext}^\top = \frac{1}{1+\mathbf{b}^\top\mathbf{1}_\mu}\mathbf{1}_\mu^\top$. Indeed, from (28) and (30), we get $\mathbf{w}_{ext}^\top = [\mathbf{w}^\top\mathbf{e}_1]\mathbf{e}_\mu^\top[I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top]^{-1} = \frac{1}{1+\mathbf{b}^\top\mathbf{1}_\mu}\mathbf{1}_\mu^\top$. □

Remark 8. *It is worthwhile underlying that the assumption that the polynomials $q(z)$ and $a_0z^\mu + r(z)$ have no common zero of modulus greater than or equal to 1 plays a fundamental role in the necessity part of Theorem 7. Indeed, if these polynomials have a common zero λ then, by the expression of $p_{\mathcal{M}_2}(z)$ derived in (16), it follows that $p_{\mathcal{M}_2}(\lambda) = 0$, namely λ is an eigenvalue of $\mathcal{M}_2$. On the other hand, it is also clear (see the proof of Lemma 22) that the left eigenvectors of $\mathcal{M}_2$ corresponding to λ take the form $\tilde{\mathbf{w}} = \begin{bmatrix}\mathbf{0}_N^\top & \mathbf{w}_2^\top\end{bmatrix}^\top$, $\mathbf{w}_2 \in \mathbb{R}^\mu, \mathbf{w}_2 \neq 0$.*

Consequently, for every $\mathbf{x}_0 \in \mathbb{R}^N$, $\tilde{\mathbf{w}}^\top \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0}_\mu \end{bmatrix} = 0$, *and hence even if* $|\lambda| \geq 1$ *LCD consensus is still possible. This is strictly related to the fact that even if the pair* $\left(\mathcal{M}_2, \begin{bmatrix} I_N & 0 \end{bmatrix}\right)$ *is observable, we are restricting the set of admissible initial conditions to those whose last* μ *entries are zero. It is worthwhile noticing, however, that this is by no means a problem, since the coefficients of the polynomials* $q(z)$ *and* $a_0 z^\mu + r(z)$ *are just design parameters, and as it will be shown in the following section we can always avoid selecting such coefficients in a way that the polynomials have a common zero whose modulus is not smaller than* 1. *In the next section we will also provide an example illustrating this situation in the specific case of finite-time LCD consensus achieved by means of the second control protocol.*

Remark 9. *Theorems 6 and 7 provide necessary and sufficient conditions for the two closed-loop systems, obtained corresponding to the two control protocols, to achieve LCD consensus, according to Definition 1. It is worth noting that, in general, the leader-controlled distributed consensus value may be different from the one the agents would converge to without the additional control protocol (i.e., (5)), but the relative weights of the agents in contributing to the final consensus value are unaltered. Indeed, the vector* $\mathbf{c}$ *does not necessarily coincide with* $\mathbf{w}_A$, *but it is always a scalar multiple of it (see the expressions of the vectors* $\mathbf{c}$ *and points (a3) and (b3) in the previous theorems). Therefore the final consensus value achieved with either one of the two protocols is always related to the original consensus value (5) through some scalar coefficient that depends on the protocol but not on the initial condition (and hence could be easily adjusted if one wants to maintain also the consensus value unaltered).*

Theorems 6 and 7 state that, under certain hypotheses involving the triple $(A_{(1)}, \mathbf{g}, \mathbf{h}^\top)$ (and the polynomials $q(z)$ and $a_0 z^\mu + r(z)$ in the case of the second protocol), the overall multi-agent system achieves LCD consensus if and only if the resulting matrix $\mathcal{M}_i$ has a simple eigenvalue in 1 which is strictly dominant. These are not trivial results since the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ have a peculiar structure, and they exhibit none of the standard properties of the matrices involved in consensus algorithms, e.g. positivity, symmetry or irreducibility. Even more, we had to translate into spectral properties asymptotic performances that need to be achieved only corresponding to certain initial conditions. The outcome of Theorems 6 and 7 is that LCD consensus is achieved if and only if the characteristic polynomial of $\mathcal{M}_i$ satisfies certain conditions. This result, formalized in Proposition 10 below, will allow us in the next section to make use of the expressions derived in Section IV for $p_{\mathcal{M}_i}(z)$ and hence to determine under what conditions a control protocol (7) or (11)-(12) of suitable size μ can be designed to achieve LCD consensus.

Proposition 10. *Consider the multi-agent system* (6) *with one leader and control protocol* (7) *or* (11)-(12)*. Also, in the case of the protocol* (11)-(12)*, assume that the two polynomials $q(z)$ and $a_0 z^\mu + r(z)$, where $q(z) = z^\mu + b_\mu z^{\mu-1} + \cdots + b_2 z + b_1$ and $r(z) = a_\mu z^{\mu-1} + \cdots + a_2 z + a_1$, have no common zero of modulus greater than or equal to 1, and that $q(1) \neq 0$. The resulting multi-agent system reaches leader-controlled distributed consensus if and only if there exists a monic Schur polynomial $\psi(z) \in \mathbb{R}[z]$ of degree $N + \mu - 1$ such that $p_{\mathcal{M}_i}(z) = \psi(z)(z-1)$.*

VI. Conditions for the existence of protocols that lead to LCD consensus with desired speed

In the previous section we have assumed that the parameters of the two proposed control protocols have been assigned, and we have investigated under what conditions the resulting closed-loop multi-agent systems achieve LCD consensus. In this section, we assume to simply start with the original multi-agent system (6) and we investigate under what conditions, for each specific control protocol, parameters can be found that ensure the closed-loop system to achieve LCD consensus. Moreover, the issue of which convergence speed can be obtained is investigated.

As far as the first protocol is concerned, by Propositions 3 and 10, the overall system obtained corresponding to (7) achieves LCD consensus if and only if there exists a monic Schur polynomial $\psi(z) \in \mathbb{R}[z]$ of degree $N + \mu - 1$ such that

$$p_A(z) z^\mu + p_1(z) y(z) = \psi(z)(z-1), \tag{32}$$

where $y(z) := -\sum_{i=0}^{\mu} a_i z^{\mu-i} \in \mathbb{R}[z]$ is a polynomial of degree at most μ with $y(0) \neq 0$. In order to address the problem of determining when $\mu \in \mathbb{Z}_+, \mu > 0$, and a monic Schur polynomial $\psi(z) \in \mathbb{R}[z]$ of degree $N + \mu - 1$ can be found such that the previous Diophantine equation admits a solution of appropriate degree, we recall (see Remark 1) that, since $A = I_N - \kappa L$, assumption (4) on κ and the irreducibility assumption on L ensure that $p_A(z) = (z-1)\hat{p}_A(z)$, for some monic Schur polynomial $\hat{p}_A(z) \in \mathbb{R}[z]$, with $\partial \hat{p}_A = N - 1$.

Proposition 11. *Consider the multi-agent system* (6) *with a single leader and control protocol* (7)*. For every choice of μ there exist $a_0, a_1, \ldots, a_\mu \in \mathbb{R}$ such that LCD consensus is achieved.*

Proof. As $p_A(z) = (z-1)\hat{p}_A(z)$, where $\hat{p}_A(z)$ is a monic Schur polynomial of degree $N-1$, the LCD consensus problem is solved if and only if there exist $\mu \in \mathbb{Z}_+, \mu > 0$, and a Schur polynomial $\psi(z) \in \mathbb{R}[z]$, with $\partial\psi = N + \mu - 1$, such that the polynomial equation $\hat{p}_A(z)z^\mu(z-1) + p_1(z)y(z) = \psi(z)(z-1)$, has a solution $y(z) \in \mathbb{R}[z]$ of degree μ and $y(0) \neq 0$. Set $y(z) = (z-1)\hat{y}(z)$, with $\hat{y}(z) \in \mathbb{R}[z]$ of degree $\mu - 1$ (note that this amounts to assuming $a_0 + \mathbf{a}^\top \mathbf{1}_\mu = \sum_{i=0}^{\mu} a_i = 0$) and $\hat{y}(0) \neq 0$. Then the previous equation becomes $\hat{p}_A(z)z^\mu + p_1(z)\hat{y}(z) = \psi(z)$. Let $\tilde{y}(z) \in \mathbb{R}[z]$ be an arbitrary polynomial of degree $\mu - 1$ (in particular one with $\tilde{y}(0) \neq 0$). As the zeros of a polynomial are a continuous function of its coefficients, for every value of K in a sufficiently small neighbourhood of zero, all the zeros of $\hat{p}_A(z)z^\mu + K[p_1(z)\tilde{y}(z)]$ remain inside the unitary circle and hence it is a Schur polynomial. Therefore, for every such K, the identity $K(z-1)\tilde{y}(z) = y(z) = -\sum_{i=0}^{\mu} a_i z^{\mu-i}$ provides the desired parameters a_0 and $\mathbf{a}$. □

Once LCD consensus is achieved, the speed of convergence is determined by the second largest eigenvalue of $\mathcal{M}_1$, namely by the zero of largest modulus of $\psi(z)$. So, the natural question arises: What are the convergence speeds that can be achieved? Equivalently: For which Schur polynomials $\psi(z)$ equation (32) has a solution? Proposition 4 has provided algebraic conditions for the solvability of (32), however no bound on the best possible speed of convergence attainable can be easily derived from those conditions. Even if the best possible performance of the first protocol are mostly unexplored, there are cases when even for $\mu = 1$, a suitable choice of the parameters a_0 and a_1 can always ensure an improvement of the convergence speed to LCD consensus, as discussed in the following remark.

Remark 12. *If $\mu = 1$, the characteristic polynomial of $\mathcal{M}_1$ becomes $p_{\mathcal{M}_1}(z) = z \cdot p_A(z) + y(z)p_1(z)$, where $y(z) = -a_0 z - a_1$. In order to achieve LCD consensus we have to impose that $y(1) = 0$ (and hence $a_1 = -a_0$) and that $p_{\mathcal{M}_1}(z)/(z-1)$ is a Schur polynomial. This leads us to the polynomial equation $z \cdot \hat{p}_A(z) - a_0 p_1(z) = \psi(z)$, where $\hat{p}_A(z) = \prod_{i=2}^{N}(z - \lambda_i)$, $p_1(z) = \prod_{i=1}^{N-1}(z - \nu_i)$, and $\psi(z)$ must be Schur. The expression $z \cdot \hat{p}_A(z) - a_0 p_1(z)$ can be regarded as a root locus (positive for $a_0 < 0$ and negative for $a_0 > 0$). The second largest eigenvalue of $\mathcal{M}_1$, say ρ, coincides with the zero of maximum modulus of $\psi(z)$, and we want to prove that there are situations when a suitable choice of a_0 always makes ρ smaller than the second largest eigenvalue of A.*

Case 1: If the communication graph is undirected and connected, and the coupling strength

κ satisfies (4). Then, $A = I_N - kL$ is a symmetric, Schur, positive matrix, whose eigenvalues satisfy [28] $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N > -1$. On the other hand, by [35], the eigenvalues $\nu_i, i \in \{1, \ldots, N-1\}$, of the symmetric and positive $(N-1) \times (N-1)$ submatrix of A, $A_{(1)}$, say $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{N-1}$, satisfy the interlacing property *and hence $1 = \lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \cdots \geq \nu_{N-1} \geq \lambda_N > -1$. The coprimality of $p_A(z)$ and $p_1(z)$ allows to further refine the previous identity as*

$$1 = \lambda_1 > \nu_1 > \lambda_2 > \nu_2 \cdots > \nu_{N-1} > \lambda_N > -1. \tag{33}$$

Simple arguments based on the root locus show that, except in the case when $N = 2$ and $p_A(z) = (z-1)z$ (the case when the original multi-agent system achieves finite-time LCD consensus and hence one cannot improve the convergence speed), in all the other cases small positive values of a_0 surely allow to make ρ smaller than $|\lambda_2|$.

Case 2: Similarly, elementary reasonings based on the root locus allow to say that when the second largest eigenvalue of A, say λ_2, namely the zero of maximum modulus of $\hat{p}_A(z)$, is real, nonzero and simple, then by suitably choosing either $a_0 > 0$ or $a_0 < 0$ we improve the convergence speed. Indeed, independently of the location of all the other "poles" ($\lambda_i, i \in \{2, \ldots, N\}$, plus the "pole" in 0 due to the factor z) and "zeros" ν_i, both the positive and the negative root loci contain an interval of the real axis having λ_2 at one of its extreme, and we can always choose a_0 to have the modulus decreasing along that interval. By continuity, for small values of $|a_0|$ the zero moving on that interval will necessarily be the one of maximum modulus.

In the case of the closed-loop system (14), derived when a single leader uses the control law (11)-(12), we can reach LCD consensus if we can find $\mu > 0, \mu \in \mathbb{Z}$, and a monic Schur polynomial $\psi(z) \in \mathbb{R}[z]$ of degree $N + \mu - 1$ such that

$$p_A(z)x(z) + p_1(z)y(z) = \psi(z)(z-1), \tag{34}$$

where $x(z) := z^\mu + \sum_{i=1}^{\mu} b_i z^{i-1}$ is monic of degree μ with $x(1) \neq 0$, while $y(z) := -a_0 z^\mu - \sum_{i=1}^{\mu} a_i z^{i-1}$ has degree at most μ, and these two polynomials have no common zero of modulus greater than or equal to 1. Now, it is clear that if (34) is solvable for some Schur polynomial $\psi(z)$, then $x(z)$ and $y(z)$ cannot have a common zero λ of modulus $|\lambda| \geq 1$ except, possibly, for $\lambda = 1$. On the other hand, if we ensure that $x(1) \neq 0$, then necessarily $x(z)$ and $y(z)$ cannot have a common zero in 1. So, we need to show that $\mu > 0, \mu \in \mathbb{Z}$, and a monic Schur polynomial

$\psi(z) \in \mathbb{R}[z]$ of degree $N+\mu-1$ can be found such that (34) is solvable, the two polynomials $x(z)$ and $y(z)$ have suitable degrees and $x(1) \neq 0$.

Proposition 13. *Consider the multi-agent system* (6) *with a single leader and control protocol* (11)-(12)*. For every $\mu \in \mathbb{Z}_+$ there exist parameters $a_0, a_1, \dots, a_\mu, b_1, b_2, \dots, b_\mu \in \mathbb{R}$ such that LCD consensus is achieved. Moreover, if $\mu \geq N-1$, the monic Schur polynomial $\psi(z)$ in* (34) *can be arbitrarily chosen.*

Proof. By following the same reasoning as in the previous proof, and hence assuming, in particular, that $p_A(z) = (z-1)\hat{p}_A(z)$, where $\hat{p}_A(z)$ is a monic Schur polynomial of degree $N-1$, and that $y(z) = (z-1)\hat{y}(z)$, where $\hat{y}(z) \in \mathbb{R}[z]$ has degree $\mu-1$, we have reduced ourselves to show that there exists a monic Schur polynomial $\psi(z) \in \mathbb{R}[z]$ of degree $N+\mu-1$ such that the Diophantine equation

$$\hat{p}_A(z)x(z) + p_1(z)\hat{y}(z) = \psi(z) \tag{35}$$

has a solution $(x(z), \hat{y}(z))$ with $x(z)$ monic of degree μ, $x(1) \neq 0$, and $\hat{y}(z)$ of degree at most $\mu-1$. Since $\hat{p}_A(z)$ is Schur and monic, for every choice of a monic Schur polynomial $x(z)$ of degree μ (and hence satisfying $x(1) \neq 0$), for every choice of a polynomial $\tilde{y}(z) \in \mathbb{R}[z]$ of degree $\mu-1$, and for every value of K in a sufficiently small neighbourhood of zero, all the zeros of $\hat{p}_A(z)x(z) + K[p_1(z)\tilde{y}(z)]$ remain inside the unitary circle and hence this is a Schur polynomial. So, from the coefficients of $x(z)$ and $y(z)$ one uniquely determines the coefficients of the control protocol (11)-(12). If $\mu \geq N-1$, by making use of the assumption $\mathrm{GCD}(p_A(z), p_1(z)) = 1$, we can follow the same reasoning as in Proposition 4 and show that equation (35) admits solution with the desired properties for every monic Schur polynomial $\psi(z)$. The coefficients of the polynomials $x(z)$ and $y(z)$ immediately allow to determine the parameters $a_0, \mathbf{a}$ and $\mathbf{b}$. □

Remark 14. *The main contribution of Proposition 13 is to have shown that by resorting to the second protocol one can ensure not only LCD consensus for the resulting closed-loop system, but also arbitrary convergence speed, provided that the leader can make use of a control protocol of complexity $\mu = N-1$. This result shows that as far as it is possible and economically convenient to concentrate all the resources in a single node, then the second protocol we propose ensures arbitrary convergence speed to the overall multi-agent system at a slightly lower complexity with respect to the solutions proposed in [7] and [8], where each agent was injecting a control* *input based on its current and past state values. Indeed, the schemes in [7] and [8] can be equivalently described by a state-space model $\chi(t+1) = \mathcal{M}\chi(t)$ for some matrix $\mathcal{M}$ of size $2N$, and the optimal value of the convergence speed that can be achieved is bounded by certain parameters that depend on the values of the eigenvalues of the matrix A. In our case, we can ensure that with a closed-loop system of size $N + \mu \leq 2N - 1$ we can achieve arbitrary speed of convergence.*

Remark 15. *Polynomial equations like (32) and (34), often arise in fundamental problems of control, signal processing and applied mathematics, and their solvability has been the subject of extended research efforts for some decades. A valuable study on different algorithms to cope with linear polynomial Diophantine equations is the Ph.D. dissertation of D. Henrion [36]. In the thesis efficient and numerically reliable algorithms for polynomial matrices are sought and several different numerical techniques are compared. Among them, the most efficient approach is the one based on the Sylvester Matrix, whose inverse can be efficiently computed by taking advantage of its structure [37]. Using this method, the accuracy of the solution is related to the structure of the matrix, in terms of closeness between the zeros of $p_A(z)$ and $p_1(z)$ [38]. Based on these theoretical results several software tools have been developed. One of the first and most renowned is the Matlab toolbox Polyx [39], which is a Polynomial Toolbox for polynomials, polynomial matrices and their applications in systems, signals and control. Polyx is able to efficiently solve scalar polynomial matrix equations, with the desired precision. Of course, the case may occur that the communication graph, and hence the coefficients of the Laplacian L, are known only in an approximate way. However, as far as A takes the form $A = I - \kappa L$ and L is the Laplacian of an irreducible graph, its characteristic polynomial necessarily takes the form $p_A(z) = (z-1)\tilde{p}_A(z)$ for some Schur monic polynomial $\tilde{p}_A(z)$ that may be slightly different from the nominal one $\hat{p}_A(z)$. Accordingly, the obtained solution of the Diophantine equation will result in some polynomial $(z-1)\tilde{\psi}(z)$ where $\tilde{\psi}(z)$ is different from the polynomial $\psi(z)$ originally designed. However, the zeros of a polynomial are continuous functions of its coefficients. So, as far as the errors in the coefficients of $\hat{p}_A(z)$ are small, also the zeros of the obtained polynomial $\tilde{\psi}(z)$ will not be very far from the designed ones, thus guaranteeing that the achieved speed of convergence is not too far from the desired one. Note that within Polyx several tools for robust stabilization are available that can be used to deal with the problem of imprecise knowledge of $\hat{p}_A(z)$. Of course, if knowledge of the communication graph is very poor, the methodology* *proposed in this paper is not the most appropriate one to use.*

VII. Finite-time LCD consensus

We now apply the results about the allocation of the characteristic polynomials of the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ presented in Section IV and the results about LCD consensus of the previous section to design decentralized policies that make the agents reach leader-controlled distributed consensus in finite-time. Finite-time consensus has been the object of a considerable interest in the last years (e.g. [40], [41], [42]). Not only it leads the agents to agree on the same decision within a finite number of steps, but also shows better disturbance rejection properties and robustness with respect to uncertainties compared to most consensus schemes [43].

Definition 2 (Leader-controlled distributed finite-time consensus.)**.** *Consider a multi-agent system with a leader, described as in* (6)*. We say that the control protocols* (7) *or* (11)-(12) *lead the system to* leader-controlled distributed finite-time consensus*, under the leader's action $u(t)$, if there exist a vector $\mathbf{c} \in \mathbb{R}^N$ and $T \in \mathbb{Z}_+$ such that for every choice of the agents' initial state $\mathbf{x}_0$ (and assuming zero initial conditions for the auxiliary variables) one has*

$$\mathbf{x}(t) = \mathbf{1}_N(\mathbf{c}^\top \mathbf{x}_0), \qquad \forall\, t \in \mathbb{Z}_+, t \geq T. \tag{36}$$

When (36) *holds, we refer to T as to the* consensus time.

Proposition 16. *Consider the closed-loop multi-agent system* (9)*, derived in the case of a single leader using the control law* (7)*. The overall system achieves finite-time LCD consensus if and only if the characteristic polynomial of the matrix $\mathcal{M}_1$ satisfies*

$$p_{\mathcal{M}_1}(z) = z^{N+\mu-1}(z-1). \tag{37}$$

Similarly, under the assumption that the two polynomials $q(z)$ and $a_0 z^\mu + r(z)$, where $q(z) = z^\mu + b_\mu z^{\mu-1} + \dots + b_2 z + b_1$ and $r(z) = a_\mu z^{\mu-1} + \dots + a_2 z + a_1$, have no common zero, except possibly in 0, the closed-loop multi-agent system (14)*, derived in the case of a single leader using the control law* (11)-(12)*, achieves finite-time LCD consensus if and only if the characteristic polynomial of the matrix $\mathcal{M}_2$ satisfies*

$$p_{\mathcal{M}_2}(z) = z^{N+\mu-1}(z-1). \tag{38}$$

In both cases the LCD consensus time T is not greater than $N + \mu - 1$.

Proof. Consider, first, the case of a single leader using the control law (7). The corresponding closed-loop multi-agent system is hence described as in (9).

[Sufficiency] Suppose that the characteristic polynomial of the matrix $\mathcal{M}_1$ is described as in (37). By the Cayley-Hamilton theorem, $p_{\mathcal{M}_1}(z)$ is an annihilating polynomial of $\mathcal{M}_1$ [33], which means that $\mathcal{M}_1^{N+\mu} - \mathcal{M}_1^{N+\mu-1} = 0$. This implies that

$$\mathcal{M}_1^t = \mathcal{M}_1^{N+\mu-1} \qquad \forall\ t \in \mathbb{Z}_+, t \geq N+\mu-1. \tag{39}$$

On the other hand, by Proposition 10, if the characteristic polynomial $p_{\mathcal{M}_1}(z)$ is described as in (37), the closed-loop multi-agent system (9) achieves LCD consensus, i.e.

$$\lim_{t\to+\infty} \begin{bmatrix} I & 0 \end{bmatrix} \boldsymbol{\chi}(t) = \lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{1}_N(\mathbf{c}^\top \mathbf{x}_0).$$

Identity (39) ensures that $\boldsymbol{\chi}(t) = M_1^t \boldsymbol{\chi}(0) = M_1^{N+\mu-1}\boldsymbol{\chi}(0)$ for every $t \geq N+\mu-1$, and therefore (36) is satisfied (at least) for $T = N+\mu-1$, thus proving that the system reaches finite-time LCD consensus.

[Necessity] If the multi-agent system (6) with one leader and control protocol (7), achieves finite-time LCD consensus then it achieves standard LCD consensus, and hence by Proposition 10 there exists a monic Schur polynomial $\psi(z) \in \mathbb{R}[z]$ of degree $N+\mu-1$ such that $p_{\mathcal{M}_1}(z) = \psi(z)(z-1)$. We want to prove that $\psi(z) = z^{N+\mu-1}$. If this were not the case, there would be $\lambda \in \mathbb{C}$, with $0 < |\lambda| < 1$, such that $p_{\mathcal{M}_1}(\lambda) = 0$ and hence $\lambda \in \sigma(\mathcal{M}_1)$. Denote by $\tilde{\mathbf{w}}$ a left eigenvector of $\mathcal{M}_1$, respectively, corresponding to λ. By Lemma 21, part ii), we know that the first N entries of $\tilde{\mathbf{w}}$ cannot be zero and hence there exists $\mathbf{x}_0 \in \mathbb{R}^N$ such that $\tilde{\mathbf{w}}^\top \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0}_\mu \end{bmatrix} \neq 0$. By the same reasoning adopted in the proof of Theorem 6, we can claim that the state evolution $\boldsymbol{\chi}(t)$ has a nonzero projection over the set of right generalised eigenvectors corresponding to λ and hence at least one of the entries of $\boldsymbol{\chi}(t)$, say the i-th, is a linear combination of elementary modes where λ^t is weighted by a nonzero coefficient. But this makes it impossible for $\chi_i(t)$ to become constant in a finite number of steps.

Consider, now, the multi-agent system (6) and assume that there is one leader adopting the control protocol (11)-(12). The proof of the [Sufficiency] is identical to the previous one. The proof of [Necessity] is similar and we only give a sketch of it. By proceeding as in the previous case, we can claim that if finite-time LCD consensus is achieved, then LCD consensus is achieved and hence there exists a monic Schur polynomial $\psi(z) \in \mathbb{R}[z]$ of degree $N+\mu-1$ such that $p_{\mathcal{M}_2}(z) = \psi(z)(z-1)$. As the two polynomials $q(z)$ and $a_0 z^\mu + r(z)$, where $q(z) =$

$z^\mu + b_\mu z^{\mu-1} + \cdots + b_2 z + b_1$ and $r(z) = a_\mu z^{\mu-1} + \cdots + a_2 z + a_1$ have no common zero (except possibly in zero), then we can apply Lemma 22, part ii), and show, as in the previous case that if $\psi(z)$ has a zero $\lambda \neq 0$, then initial conditions can be found corresponding to which the LCD consensus is not achieved in a finite-time. $\square$

Remark 17. *As remarked immediately after Theorem 7, the assumption that the polynomials $q(z)$ and $a_0 z^\mu + r(z)$ have no common zero (except possibly in 0) plays a fundamental role also in the necessity part of Proposition 16, when dealing with the second protocol. Nevertheless, if these polynomials have a common zero $\lambda \neq 0$ finite-time LCD consensus is still possible, as illustrated in the following example.*

Example 1. *Assume $N = 3$, and consider the strongly connected graph with Laplacian*

$$L = \begin{bmatrix} 2.25 & -2.25 & 0 \\ -1 & 3 & -2 \\ -3 & -1 & 4 \end{bmatrix}.$$

If $\kappa = 1/5$, then

$$A = I_3 - \kappa L = \begin{bmatrix} 0.55 & 0.45 & 0 \\ 0.2 & 0.4 & 0.4 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} \qquad A_{(1)} = \begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}.$$

A direct calculation leads to $p_A(z) = (z-1)(z^2 - 0.15z + 0.09)$ and $p_1(z) = z(z-0.6)$, and hence the two polynomials are coprime. Now assume $\mu = 3$, $x(z) = z^3 - 0.5z^2$ and $y(z) = 0.15z^3 - 0.225z + 0.075$. These two polynomials have a common zero in 0.5. It is a matter of simple calculations to verify that $p_A(z)x(z) + p_1(z)y(z) = (z-1)z^4(z-0.5)$. The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to the above polynomials $x(z)$ and $y(z)$ can be easily found through the identities

$$\begin{aligned} x(z) &= z^3 - 0.5z^2 = z^3 + b_3 z^2 + b_2 z + b_1 \\ y(z) &= 0.15z^3 - 0.225z^2 + 0.075z = -a_0 z^3 - (a_3 z^2 + a_2 z + a_1), \end{aligned}$$

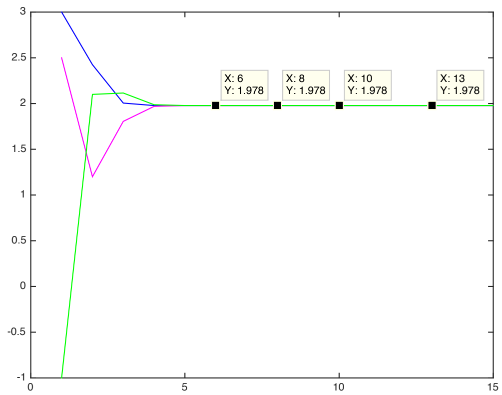

Fig. 2. Finite-time LCD consensus achieved by the multi-agent system of Example 1.

obtaining $a_0 = -0.15, \mathbf{a} = \begin{bmatrix} 0 & -0.075 & 0.225 \end{bmatrix}^\top, \mathbf{b} = \begin{bmatrix} 0 & 0 & -0.5 \end{bmatrix}^\top$, *and*

$$\mathcal{M}_2 = \begin{bmatrix} 0.4 & 0.45 & 0 & 0 & 0 & 1 \\ 0.2 & 0.4 & 0.4 & 0 & 0 & 0 \\ 0.6 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.075 & 0 & 0 & 1 & 0 & 0 \\ 0.15 & 0 & 0 & 0 & 1 & 0.5 \end{bmatrix}.$$

It is easy to verify that $\tilde{\mathbf{w}}^\top = \begin{bmatrix} 0 & 0 & 0 & 4 & 2 & 1 \end{bmatrix}^\top$ *is a left eigenvector of* $\mathcal{M}_2$ *corresponding to* $\lambda = 0.5$*. On the other hand, for every* $\mathbf{x}_0 \in \mathbb{R}^3$*, one has* $\tilde{\mathbf{w}}^\top \mathbf{x}_0 = 0$ *and, in fact, the overall multi-agent system always reaches finite-time LCD consensus, as illustrated in Fig. 2.*

By the previous result, we can claim that the overall system obtained corresponding to (7) achieves finite-time LCD consensus if and only if

$$p_A(z)z^\mu + p_1(z)y(z) = z^{N+\mu-1}(z-1), \tag{40}$$

where $y(z) := -\sum_{i=0}^{\mu} a_i z^{\mu-i}$ is a polynomial of degree at most μ (with $y(0) \neq 0$). Similarly, in the case of the closed-loop multi-agent system (14), derived in the case of a single leader using the control law (11)-(12), the finite-time LCD consensus condition is equivalent to

$$p_A(z)x(z) + p_1(z)y(z) = z^{N+\mu-1}(z-1), \tag{41}$$

where $x(z) := z^\mu + \sum_{i=1}^{\mu} b_i z^{i-1}$, is monic of degree μ, with $x(1) \neq 0$, $y(z) := -a_0 z^\mu - \sum_{i=1}^{\mu} a_i z^{i-1}$, has degree at most μ, and they are coprime polynomials. We are now in a position to provide conditions for the existence of coefficients μ, a_0 and $\mathbf{a}$ such that the corresponding control scheme (7) achieves finite-time LCD consensus.

Proposition 18. *Consider the multi-agent system* (6) *with a single leader and control protocol* (7)*. There exist coefficients $a_0, a_1, \dots, a_\mu \in \mathbb{R}, a_\mu \neq 0$, such that finite-time LCD consensus is achieved if and only if*

$$p_1(z) \mid z^\mu [z^{N-1} - \hat{p}_A(z)] \tag{42}$$

and $p_1(z)$ has a zero in 0 of multiplicity μ.

Proof. As $p_A(z) = (z-1)\hat{p}_A(z)$, where $\hat{p}_A(z)$ is a monic Schur polynomial of degree $N-1$, the Diophantine equation (40) can be rewritten as

$$\hat{p}_A(z) z^\mu (z-1) + p_1(z) y(z) = z^{N+\mu-1}(z-1), \tag{43}$$

and hence as $z^\mu (z-1)[z^{N-1} - \hat{p}_A(z)] = p_1(z)y(z)$. Note that the Schur property of $\hat{p}_A(z)$ ensures that $z^{N-1} - \hat{p}_A(z)$ has no zero in 0. If coefficients $a_0, a_1, \dots, a_\mu \in \mathbb{R}, a_\mu \neq 0$, exist such that finite-time LCD consensus is achieved, (43) holds for some $y(z)$ of degree at most μ and therefore $p_1(z) \mid z^\mu (z-1)[z^{N-1} - \hat{p}_A(z)]$. This ensures that the multiplicity of 0 as a zero of $p_1(z)$ is not greater than μ. On the other hand, if this multiplicity would be lower than μ then $y(0) = 0$ and hence $a_\mu = 0$, a contradiction. So, 0 must be a zero of $p_1(z)$ of mulitplicity μ.

Conversely, if $p_1(z) \mid z^\mu (z-1)[z^{N-1} - \hat{p}_A(z)]$ and $p_1(z)$ has a zero in 0 of multiplicity μ, then there is a monic polynomial $y(z)$, with $\partial y \leq \mu$ and $y(0) \neq 0$ such that (40) holds. Consequently, coefficients $a_0, a_1, \dots, a_\mu \in \mathbb{R}, a_\mu \neq 0$, can be found such that finite-time LCD consensus is achieved. $\square$

As an example, consider again the multi-agent system of Example 1. Polynomials $\hat{p}_A(z) = z^2 - 0.15z + 0.09$ and $p_1(z) = z(z - 0.6)$ satisfy the above divisibility condition (42) for $\mu = 1$ and the polynomial $y(z)$ solution of (43) is equal to $0.15z - 0.15$. This, in turn, means that the (first) protocol $u_1(t) = 0.15x_1(t) - 0.15x_1(t-1)$ makes the overall system reach LCD consensus in finite time. We now consider the finite-time LCD consensus problem for the case of one leader and control law (11)-(12).

Proposition 19. *Consider the multi-agent system* (6) *with a single leader and control protocol* (11)-(12)*. There always exist* $\mu \in \mathbb{Z}_+$ *and parameters* $a_0, a_1, \dots, a_\mu, b_1, b_2, \dots, b_\mu \in \mathbb{R}$ *such that finite-time LCD consensus is achieved.*

Proof. Follows immediately from the second part of Proposition 13. □

Proposition 19 supports our previous claim in Remark 14. The control protocol (11)-(12) allows to achieve LCD consensus with arbitrary convergence speed and even finite-time LCD consensus, at the cost of the increased size of the overall system. Indeed, the leader needs to resort to an additional feedback control algorithm whose memory size is at most $N-1$. So, the solution of attributing all the memory and the computational capabilities to a single-leader proves to be more effective and efficient than distributing them uniformly among the N agents [7], [8]. As far as the first control protocol is concerned, the conditions for finite-time LCD consensus are admittedly very restrictive, since a necessary condition for it to happen is that $p_1(z) = \det(zI_{N-1} - A_{(1)})$ has a zero in 0, as well as it satisfies the divisibility condition (42). Remark 12 shows there are cases when the protocol always leads to improved performance, but it is difficult to provide a complete analysis of the potentialities of this control law, due to the very special structure of the associated Diophantine equation.

VIII. Conclusions

In this paper the LCD consensus problem for a multi-agent system has been investigated by assuming that one of the nodes acts as a leader and it is allowed to inject in the multi-agent system a control signal $u(t)$ to improve some group dynamics performance. Two control protocols to generate the signal u have been proposed. First the characteristic polynomials of the two controlled systems have been derived, and conditions for the free allocation of all the multi-agent system eigenvalues has been investigated. A complete analysis of the LCD consensus problem is carried on in Sections V and VI. Finally, necessary and sufficient conditions for finite-time LCD consensus have been provided.

Future research will focus on determining conditions that ensure that the first protocol always brings an improvement in terms of convergence speed and on providing bounds on the second largest eigenvalue of the resulting closed-loop matrix $\mathcal{M}_1$. Challenging open problems are: How can one select the leader in such a way to maximize the convergence speed? Is it convenient to assume that there are more leaders and if so, what is the best possible choice for them?

APPENDIX: TECHNICAL RESULTS

We first provide a necessary and sufficient condition for the two polynomials $p_A(z) = \det(zI_n - A)$ and $p_1(z) = \det(zI_\mu - A_{(1)})$ to be coprime.

Lemma 20. *Let A be an $N \times N$ real matrix and suppose that it is block partitioned as in (21), where $a_{11} \in \mathbb{R}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{N-1}$ and $A_{(1)} \in \mathbb{R}^{(N-1)\times(N-1)}$ is the principal submatrix of A obtained by deleting its first row and its first column. The following facts are equivalent:*

i) $(A_{(1)}, \mathbf{g}, \mathbf{h}^\top)$ is a minimal realization, i.e., the pair $(A_{(1)}, \mathbf{g})$ is reachable and the pair $(A_{(1)}, \mathbf{h}^\top)$ is observable;

ii) $\sigma(A) \cap \sigma(A_{(1)}) = \emptyset$, i.e., $p_A(z)$ and $p_1(z)$ are coprime polynomials.

Proof. i) $\Rightarrow$ ii) Suppose by contradiction that there exists $\lambda \in \sigma(A) \cap \sigma(A_{(1)})$. By Lemma 2, $\det(zI_N - A) = \det(zI_{N-1} - A_{(1)})[z - a_{11} - \mathbf{h}^\top(zI_{N-1} - A_{(1)})^{-1}\mathbf{g}] = \det(zI_{N-1} - A_{(1)})(z - a_{11}) - \mathbf{h}^\top \mathrm{adj}(zI_{N-1} - A_{(1)})\mathbf{g}$.
So, condition $0 = \det(\lambda I_N - A) = \det(\lambda I_{N-1} - A_{(1)})$ implies $\mathbf{h}^\top \mathrm{adj}(\lambda I_{N-1} - A_{(1)})\mathbf{g} = 0$. But this means that the two polynomials $\det(zI_{N-1} - A_{(1)})$ and $\mathbf{h}^\top \mathrm{adj}(zI_{N-1} - A_{(1)})\mathbf{g}$ have a common zero, and hence $(A_{(1)}, \mathbf{g}, \mathbf{h}^\top)$ is not a minimal realization of its transfer function. ii) $\Rightarrow$ i) It is easily proved by reversing the previous arguments. $\square$

The following two technical lemmas provide technical details about the eigenvectors of the closed-loop matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ that describe the multi-agent systems obtained when the leader adopts either the control protocol (7) or (11)-(12).

Lemma 21. *Let $A = I_N - \kappa L$, where L is an $N \times N$ irreducible Laplacian matrix and $\kappa \in \mathbb{R}$ satisfies (4), and suppose that A is block partitioned as in (21), where $a_{11} \in \mathbb{R}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{N-1}$ and $A_{(1)} \in \mathbb{R}^{(N-1)\times(N-1)}$. Also, assume that the triple $(A_{(1)}, \mathbf{g}, \mathbf{h}^\top)$ is a minimal realization. Consider the matrix*

$$\mathcal{M}_1 = \begin{bmatrix} A + a_0 \mathbf{e}_1 \mathbf{e}_1^\top & \mathbf{e}_1 \mathbf{a}^\top \\ \\ \mathbf{e}_1 \mathbf{e}_1^\top & F \end{bmatrix} \in \mathbb{R}^{(N+\mu)\times(N+\mu)},$$

obtained corresponding to the control protocol (7), where $a_0 \in \mathbb{R}$, while $\mathbf{a} \in \mathbb{R}^\mu$ and $F \in \mathbb{R}^{\mu\times\mu}$ are defined in (10). Then *i) If $1 \in \sigma(\mathcal{M}_1)$, then $a_0 + \mathbf{a}^\top \mathbf{1}_\mu = 0$, and every right eigenvector of $\mathcal{M}_1$ corresponding to the eigenvalue 1, say $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1^\top & \mathbf{v}_2^\top \end{bmatrix}^\top$, with $\mathbf{v}_1 \in \mathbb{R}^N$ and $\mathbf{v}_2 \in \mathbb{R}^\mu$, takes the form $\mathbf{v} = \alpha \mathbf{1}_{N+\mu}$, for some $\alpha \in \mathbb{R}, \alpha \neq 0$.*

ii) If λ is an eigenvalue of $\sigma(\mathcal{M}_1)$, and $\tilde{\mathbf{w}} = \begin{bmatrix} \mathbf{w}_1^\top & \mathbf{w}_2^\top \end{bmatrix}, \mathbf{w}_1 \in \mathbb{R}^N, \mathbf{w}_2 \in \mathbb{R}^\mu$, is a left eigenvector of $\mathcal{M}_1$ corresponding to λ, then $\mathbf{w}_1 \neq 0$.

Proof. i) Assume that $1 \in \sigma(\mathcal{M}_1)$. If $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1^\top & \mathbf{v}_2^\top \end{bmatrix}^\top \neq 0$ is a right eigenvector of $\mathcal{M}_1$ corresponding to 1, then

$$\begin{bmatrix} A + a_0 \mathbf{e}_1 \mathbf{e}_1^\top & \mathbf{e}_1 \mathbf{a}^\top \\ \\ \mathbf{e}_1 \mathbf{e}_1^\top & F \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \\ \mathbf{v}_2 \end{bmatrix},$$

which leads, after a few calculations and by the identity $(I_\mu - F)^{-1}\mathbf{e}_1 = \mathbf{1}_\mu$, to

$$\begin{aligned} \mathbf{v}_2 &= (I_\mu - F)^{-1}\mathbf{e}_1(\mathbf{e}_1^\top \mathbf{v}_1) = \mathbf{1}_\mu(\mathbf{e}_1^\top \mathbf{v}_1), \\ 0 &= [I_N - A - (a_0 + \mathbf{a}^\top \mathbf{1}_\mu)\mathbf{e}_1\mathbf{e}_1^\top]\mathbf{v}_1. \end{aligned} \tag{44}$$

If $I_N - A - (a_0 + \mathbf{a}^\top \mathbf{1}_\mu)\mathbf{e}_1\mathbf{e}_1^\top$ is nonsingular then $\mathbf{v}_1 = \mathbf{0}_N$ and $\mathbf{v}_2 = \mathbf{0}_\mu$, a contradiction. If $I_N - A - (a_0 + \mathbf{a}^\top \mathbf{1}_\mu)\mathbf{e}_1\mathbf{e}_1^\top$ is singular then $0 = \det[I_N - A - (a_0 + \mathbf{a}^\top \mathbf{1}_\mu)\mathbf{e}_1\mathbf{e}_1^\top] = \det(I_N - A) - (a_0 + \mathbf{a}^\top \mathbf{1}_\mu)\mathbf{e}_1^\top \mathrm{adj}(I_N - A)\mathbf{e}_1 = p_A(1) - (a_0 + \mathbf{a}^\top \mathbf{1}_\mu)p_1(1) = -(a_0 + \mathbf{a}^\top \mathbf{1}_\mu)p_1(1)$.
Since $p_1(1) \neq 0$, by the coprimality of $p_A(z)$ and $p_1(z)$ (see Lemma 20), it follows that $a_0 + \mathbf{a}^\top \mathbf{1}_\mu = 0$. But this means that (44) becomes $[I_N - A]\mathbf{v}_1 = 0$, and since A is an irreducible Metzler matrix and 1 is its strictly dominant eigenvalue then $\mathbf{v}_1 = \alpha \mathbf{1}_N$ and it is immediate to see that $\mathbf{v}_2 = \alpha \mathbf{1}_\mu$.
ii) $\mathcal{M}_1$ admits an eigenvector taking the form $\begin{bmatrix} \mathbf{0}_N^\top & \mathbf{w}_2^\top \end{bmatrix}, \mathbf{w}_2 \in \mathbb{R}^\mu, \mathbf{w}_2 \neq 0$, corresponding to some $\lambda \in \sigma(\mathcal{M}_1)$, if and only if there exists $\mathbf{w}_2 \in \mathbb{R}^\mu, \mathbf{w}_2 \neq 0$, and $\lambda \in \sigma(\mathcal{M}_1)$ such that $\mathbf{w}_2^\top \mathbf{e}_1 = 0, \ \mathbf{w}_2^\top F = \lambda \mathbf{w}_2^\top$. However, for every $\lambda \in \mathbb{C}$, no vector $\mathbf{w}_2 \neq \mathbf{0}_\mu$ can be found such that both previous identities hold. Therefore, for every $\lambda \in \sigma(\mathcal{M}_1)$, in the corresponding left eigenvectors the block $\mathbf{w}_1$ is nonzero. □

Lemma 22. *Let $A = I_N - \kappa L$, where L is an $N \times N$ irreducible Laplacian matrix and $\kappa \in \mathbb{R}$ satisfies (4), and suppose that A is block partitioned as in (21), where $a_{11} \in \mathbb{R}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{N-1}$*

and $A_{(1)} \in \mathbb{R}^{(N-1)\times(N-1)}$. *Also, assume that the triple* $(A_{(1)}, \mathbf{g}, \mathbf{h}^\top)$ *is a minimal realization. Consider the matrix*

$$\mathcal{M}_2 = \begin{bmatrix} A + a_0\mathbf{e}_1\mathbf{e}_1^\top & \mathbf{e}_1\mathbf{e}_\mu^\top \\ \\ (\mathbf{a} - \mathbf{b}a_0)\mathbf{e}_1^\top & F - \mathbf{b}\mathbf{e}_\mu^\top \end{bmatrix} \in \mathbb{R}^{(N+\mu)\times(N+\mu)},$$

obtained corresponding to the control protocol (11)-(12)*, where* $a_0 \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^\mu$ *and* $\mathbf{b} \in \mathbb{R}^\mu$ *are real parameters, and* F *has been defined in* (10)*, and assume that* $1 + \mathbf{b}^\top \mathbf{1}_\mu \neq 0$. *Then*

i) If $1 \in \sigma(\mathcal{M}_2)$*, then* $a_0 + \mathbf{a}^\top \mathbf{1}_\mu = 0$*, and every right eigenvector of* $\mathcal{M}_2$ *corresponding to the eigenvalue* 1*, say* $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1^\top & \mathbf{v}_2^\top \end{bmatrix}^\top$*, with* $\mathbf{v}_1 \in \mathbb{R}^N$ *and* $\mathbf{v}_2 \in \mathbb{R}^\mu$*, takes the form* $\mathbf{v} = \alpha \begin{bmatrix} \mathbf{1}_N^\top & ((I_\mu - F)^{-1}\mathbf{a})^\top \end{bmatrix}^\top$*, for some* $\alpha \in \mathbb{R}, \alpha \neq 0$.

ii) If λ *is an eigenvalue of* $\sigma(\mathcal{M}_2)$*, and* λ *is not a common zero of the two polynomials* $q(z)$ *and* $a_0 z^\mu + r(z)$*, where* $q(z) = z^\mu + b_\mu z^{\mu-1} + \cdots + b_2 z + b_1$ *and* $r(z) = a_\mu z^{\mu-1} + \cdots + a_2 z + a_1$*, then in every eigenvector of* $\mathcal{M}_2$*,* $\tilde{\mathbf{w}} = \begin{bmatrix} \mathbf{w}_1^\top & \mathbf{w}_2^\top \end{bmatrix}, \mathbf{w}_1 \in \mathbb{R}^N, \mathbf{w}_2 \in \mathbb{R}^\mu$*, corresponding to* λ *the vector* $\mathbf{w}_1$ *is not zero.*

Proof. i) Assume that $1 \in \sigma(\mathcal{M}_2)$. If $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1^\top & \mathbf{v}_2^\top \end{bmatrix}^\top \neq 0$ is a right eigenvector of $\mathcal{M}_2$ corresponding to 1, then

$$\begin{bmatrix} A + a_0\mathbf{e}_1\mathbf{e}_1^\top & \mathbf{e}_1\mathbf{e}_\mu^\top \\ \\ (\mathbf{a} - \mathbf{b}a_0)\mathbf{e}_1^\top & F - \mathbf{b}\mathbf{e}_\mu^\top \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \\ \mathbf{v}_2 \end{bmatrix},$$

which leads to

$$[I_N - A - a_0\mathbf{e}_1\mathbf{e}_1^\top]\mathbf{v}_1 = \mathbf{e}_1\mathbf{e}_\mu^\top\mathbf{v}_2, \tag{45}$$

$$(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top)\mathbf{v}_2 = (\mathbf{a} - \mathbf{b}a_0)\mathbf{e}_1^\top\mathbf{v}_1. \tag{46}$$

The matrix $I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top$ is nonsingular. If this was not the case, it would be $0 = \det(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top) = 1 + \mathbf{b}^\top\mathbf{1}_\mu$, a contradiction. Therefore, equation (46) leads to $\mathbf{v}_2 = (I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top)^{-1}(\mathbf{a} - \mathbf{b}a_0)\mathbf{e}_1^\top\mathbf{v}_1$, that substituted in (45) leads to

$$[I_N - A - a_0\mathbf{e}_1\mathbf{e}_1^\top - \mathbf{e}_\mu^\top(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top)^{-1}(\mathbf{a} - a_0\mathbf{b})\mathbf{e}_1\mathbf{e}_1^\top]\mathbf{v}_1 = 0. \tag{47}$$

Since (see, also, (30)) $\mathbf{e}_\mu^\top(I_\mu - F + \mathbf{b}\mathbf{e}_\mu^\top)^{-1}(\mathbf{a} - a_0\mathbf{b}) = \frac{1}{1+\mathbf{b}^\top\mathbf{1}_\mu}\mathbf{1}_\mu^\top(\mathbf{a} - a_0\mathbf{b})$, identity (47) becomes

$$0 = \left[I_N - A - a_0\mathbf{e}_1\mathbf{e}_1^\top - \frac{1}{1 + \mathbf{b}^\top\mathbf{1}_\mu}\mathbf{1}_\mu^\top(\mathbf{a} - a_0\mathbf{b})\mathbf{e}_1\mathbf{e}_1^\top\right]\mathbf{v}_1 = \left[I_N - A - \frac{a_0 + \mathbf{a}^\top\mathbf{1}_\mu}{1 + \mathbf{b}^\top\mathbf{1}_\mu}\mathbf{e}_1\mathbf{e}_1^\top\right]\mathbf{v}_1.$$

So, by reasoning as in the proof of part i) of Lemma 21, we can claim that $\frac{a_0+\mathbf{a}^\top \mathbf{1}_\mu}{1+\mathbf{b}^\top \mathbf{1}_\mu} = 0$ and hence $a_0 + \mathbf{a}^\top \mathbf{1}_\mu = 0$, $\mathbf{v}_1 = \alpha \mathbf{1}_N$ and it is easy to verify that $\mathbf{v}_2 = \alpha(I_\mu - F)^{-1}\mathbf{a}$.
ii) $\mathcal{M}_2$ admits an eigenvector taking the form $\begin{bmatrix}\mathbf{0}_N^\top & \mathbf{w}_2^\top\end{bmatrix}, \mathbf{w}_2 \in \mathbb{R}^\mu, \mathbf{w}_2 \neq 0$, corresponding to some $\lambda \in \sigma(\mathcal{M}_2)$, if and only if there exists $\mathbf{w}_2 \in \mathbb{R}^\mu, \mathbf{w}_2 \neq 0$, and $\lambda \in \mathbb{C}$ such that $\mathbf{w}_2^\top(\mathbf{a} - a_0\mathbf{b}) = 0, \mathbf{w}_2^\top[F - \mathbf{b}\mathbf{e}_\mu^\top] = \lambda \mathbf{w}_2^\top$, namely

$$\mathbf{w}_2^\top \left[\begin{array}{cccc|c} \lambda & 0 & \dots & b_1 & a_1 - a_0 b_1 \\ -1 & \lambda & & b_2 & a_2 - a_0 b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & -1 & \lambda + b_\mu & a_\mu - a_0 b_\mu \end{array}\right] = \begin{bmatrix}\mathbf{0}_\mu^\top & 0\end{bmatrix}.$$

This is equivalent to saying that $\lambda \in \mathbb{C}$ exists such that $\mathbf{w}_2^\top = \begin{bmatrix}1 & \lambda & \lambda^2 & \dots & \lambda^{\mu-1}\end{bmatrix}$ and $0 = \lambda^\mu + b_\mu \lambda^{\mu-1} + \dots + b_2\lambda + b_1, \quad (a_\mu \lambda^{\mu-1} + \dots + a_2\lambda + a_1) - a_0(b_\mu \lambda^{\mu-1} + \dots + b_2\lambda + b_1) = 0$. This happens if and only if there exists $\lambda \in \mathbb{C}$ such that $0 = q(\lambda)$ and $0 = (a_\mu \lambda^{\mu-1} + \dots + a_2\lambda + a_1) - a_0(-\lambda^\mu) = a_0\lambda^\mu + r(\lambda)$. So, for every λ which is not a common zero of $q(z)$ and $a_0 z^\mu + r(z)$, the block $\mathbf{w}_1$ in the eigenvector cannot be zero. □

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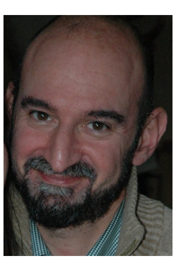

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