PARTIAL INTERCONNECTION AND OBSERVER-BASED DEAD-BEAT CONTROL OF TWO-DIMENSIONAL BEHAVIORS

MAURO BISIACCO, MARIA ELENA VALCHER*

Abstract. In this paper the dead-beat control problem, by partial interconnection, of two-dimensional (2D) discrete behaviors, defined on the grid $\mathbb{Z}_+ \times \mathbb{Z}$ and having the time as (first) independent variable, is investigated. The possibility of driving to zero (in a finite number of "steps") either all or part of the system variables, by means of a partial interconnection controller, proves to be equivalent to the reconstructibility of the variables that are not accessible for control. On the other hand, if we search for "admissible" dead-beat controllers, the only ones providing meaningful results in practice, we have to introduce the zero-time-controllability assumption. These two properties are just the necessary and sufficient conditions for the existence of an observer-based (admissible) dead-beat controller, which consists of a dead-beat observer, to estimate the relevant variables from the measured ones, and of a full interconnection dead-beat controller, acting on both the measured and the estimated variables.

Key words. 2D behavior, nilpotent autonomous behavior, reconstructibility, zero-time-controllability, dead-beat observer/controller, full/partial interconnection, observer-based dead-beat controller.

Running title: Dead-beat control of two-dimensional behaviors by partial interconnection

1. Introduction

Most of the literature about two-dimensional (2D) and multi-dimensional (nD) systems has concentrated on mathematical models for which the two (n, in general) independent variables play the same role [6, 10, 15, 22]. In several engineering applications, however, one of the independent variables is time, and its role is distinguished from that of all the others. In fact, for that variable the concept of causality always makes sense. As a consequence, fundamental properties, like autonomy and controllability, need to be redefined in the light of this interpretation, in order to take into account the privileged role of the time variable.

The interest in the behavioral approach to multidimensional systems for which one of the independent variables represents time, started with the works of Sasane and co-authors. In [17, 18, 19] the properties of time-autonomy and time-controllability for systems described by partial differential equations were thoroughly investigated. Recent years have witnessed a renewed interest in this class of systems, starting with the papers [8, 14], where time-relevant autonomous 2D discrete behaviors and the related stability problems have been investigated. More recently, Oberst

*M.Bisiacco and M.E. Valcher are with the Department of Information Engineering, University of Padova, via Gradenigo 6/B, 35131 Padova, Italy. E-mails: {bisiacco,meme}@dei.unipd.it.

and Scheicher [12] have developed a general framework for the discrete nD case, and have provided characterizations of time-autonomy and time-controllability. Also, an extension of the stability analysis to general nD behaviors with a time coordinate has been presented in [11].

In [3, 4] time-controllability and zero-time-controllability of discrete 2D behaviors, defined on $\mathbb{Z}_+ \times \mathbb{Z}$ and having the time as an independent variable, have been characterized, and a complete solution of the full interconnection dead-beat control (DBC) problem has been provided. In these papers, the first variable, defined on $\mathbb{Z}_+$, has been regarded as the time variable, while the second coordinate, defined on the whole integer set, is a space variable. According to this perspective, *vertical strips* in the grid $\mathbb{Z}_+ \times \mathbb{Z}$, namely the sets $\{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z} : 0 \leq h \leq N-1\}$, and the *half-planes* $\{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z} : h \geq N\}$, for $N \in \mathbb{Z}_+$, have been given special interpretation. The former are the sets where "initial conditions" on the system variables are given, while the latter are the supports of "long term evolutions", where both concepts clearly refer to the time coordinate. The definitions of time-controllability and of zero-time-controllability, as well as the DBC problem, have been accordingly introduced and investigated.

In this paper, we extend the results obtained in [3, 4], by investigating the dead-beat control problem under different assumptions on both the control action and the control target. Indeed, the system variables are split in two subsets: relevant variables, denoted by $\mathbf{w}_r$, and measured variables, $\mathbf{w}_m$. The control action is exerted only on the measured variables, and hence we deal with partial interconnection DBCs. Also, it may target either all the system variables or the relevant variables alone. It turns out that it is possible to drive to zero, in a finite number of "steps", either all or part of the system variables, by means of a partial interconnection controller, if and only if the variables that are not accessible for control are "reconstructible". On the other hand, if we search for "admissible" DBCs, the only meaningful ones in practice, we have to introduce the zero-time-controllability assumption (on the target variables, both in case they are all the system variables or only a subset of them). These properties are just the necessary and sufficient conditions for the existence of an observer-based (admissible) dead-beat controller. An observer-based DBC consists of a dead-beat observer, to estimate the target variables from the measured ones, and of a full interconnection dead-beat controller, acting on both the measured and the estimated variables.

The paper is organized as follows. In section 2, preliminary results about polynomial matrices with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ are introduced. Basic properties of behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$, and the corresponding algebraic characterizations, are discussed in section 3. All the definitions and results of these sections can be found, in extended form, in [4], and we recall them here only to make the paper reading easier. Section 4 revisits the estimation problem and the characterization of dead-beat observers (DBOs) of the relevant variables, by making use of the measured variables. The definitions and results, originally investigated in [1] for the class of systems defined on the half-plane $\mathcal{H}_0 := \{(h,k) \in \mathbb{Z} \times \mathbb{Z} : h + k \geq 0\}$, are adapted to the case of systems defined on $\mathbb{Z}_+ \times \mathbb{Z}$ (see also [9] for some recent results about the observer problem for 2D systems described by means of a state-space model). In section 5, we first extend the zero-time-controllability property to the case when we target only the relevant variables (see [2] for the 1D case). We then provide necessary and sufficient conditions for the solvability of the DBC problem by partial interconnection, both in case we want to drive to zero both the relevant variables and the measured variables, and in case we want to drive to zero only the relevant variables. Complete characterizations of DBCs and some partial parametrization results are provided, for both cases, in section 6. Finally, in section 7, we explore the possibility of achieving these targets by resorting to an observer-based DBC, that first estimates the relevant variables from the measured ones, and then applies a dead-beat control action on both the estimated and the measured ones.

2. Polynomial matrices with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ and associated operators

Let $\mathbb{R}[z_1, z_2, z_2^{-1}]$ be the ring of polynomials, with real coefficients, in the nonnegative integer powers of z_1 and in the integer powers of z_2. z_1 is associated with the time variable and z_2 with a space variable. Accordingly, we refer to such polynomials as *time-space polynomials* (for short, *TS-polynomials*) [4]. The ring of TS-polynomials is properly included in the ring of *Laurent polynomials* (*L-polynomials*) $\mathbb{R}[z_1, z_1^{-1}, z_2, z_2^{-1}]$. So, for some purposes, it is convenient to regard TS-polynomials as L-polynomials.

(Factor and zero) primeness properties for the class of TS-polynomial matrices naturally extend the analogous definitions for L-polynomial matrices [7, 20]. A full column rank TS-polynomial matrix $A(z_1, z_2)$ is *right factor prime* if in every factorization $A(z_1, z_2) = \bar{A}(z_1, z_2)\Delta(z_1, z_2)$ over the ring $\mathbb{R}[z_1, z_2, z_2^{-1}]$, with $\Delta(z_1, z_2)$ nonsingular square, the matrix $\Delta(z_1, z_2)$ is *unimodular* in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ (by this meaning that $\det \Delta(z_1, z_2) = c z_2^k$, for some $c \neq 0$ and some $k \in \mathbb{Z}$). Also, $A(z_1, z_2)$ is *right zero prime* if it admits a left inverse in $\mathbb{R}[z_1, z_2, z_2^{-1}]$, namely there exists $L(z_1, z_2)$ with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ such that (s.t.) $L(z_1, z_2)A(z_1, z_2) = I$.

A full column rank TS-polynomial matrix $A(z_1, z_2)$ is said to be *right monomic* if it is right zero prime when regarded as an L-polynomial matrix, namely it admits an L-polynomial (but not necessarily TS-polynomial) left inverse. This amounts to saying that there exists $L(z_1, z_2)$ with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ s.t. $L(z_1, z_2)A(z_1, z_2) = z_1^h I$, for some $h \in \mathbb{Z}_+$. In particular, a square TS-polynomial matrix $\Delta(z_1, z_2)$ is called *square monomic* if it is unimodular when regarded as an L-polynomial matrix, and hence $\det \Delta(z_1, z_2) = c z_1^h z_2^k$, for suitable $c \neq 0, h \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$.

Note that right zero primeness implies right factor primeness as well as right monomicity, however right factor primeness and right monomicity are not necessarily related [4]. Of course, the concepts of *left* factor/zero prime or monomic

TS-polynomial matrix can be introduced for full row rank matrices in a similar way, and enjoy analogous properties and characterizations.

Analogously to what happens with L-polynomial matrices, every TS-polynomial matrix $A(z_1, z_2)$ of rank r can always be factorized as

$$A(z_1, z_2) = L(z_1, z_2)\Delta(z_1, z_2)R(z_1, z_2), \tag{2.1}$$

for some suitable TS-polynomial matrices, with $L(z_1, z_2)$ right factor prime, $\Delta(z_1, z_2)$ $r \times r$ nonsingular square, and $R(z_1, z_2)$ left factor prime. This factorization is essentially unique, by this meaning that these three factors are uniquely determined up to (left and/or right) unimodular matrices.

The concepts of *left annihilator* and, in particular, of *minimal left annihilator* (MLA, for short) of a given TS-polynomial matrix $A(z_1, z_2)$ extend the concepts originally introduced in [15] for polynomial matrices in two indeterminates, and can be summarized as follows: if $A(z_1, z_2)$ is a TS-polynomial matrix, a TS-polynomial matrix $M(z_1, z_2)$ is a left annihilator of $A(z_1, z_2)$ if $M(z_1, z_2)A(z_1, z_2) = 0$. A left annihilator $M_m(z_1, z_2)$ of $A(z_1, z_2)$ is an MLA if it is of full row rank and for any other left annihilator $M(z_1, z_2)$ of $A(z_1, z_2)$ we have $M(z_1, z_2) = P(z_1, z_2)M_m(z_1, z_2)$ for some TS-polynomial matrix $P(z_1, z_2)$. It can be easily proved that, when the rank r of the matrix $A(z_1, z_2)$ is smaller than the number of its rows, say p, an MLA always exists, it is a $(p - r) \times p$ left factor prime matrix and is uniquely determined modulo a unimodular left factor. If the given $A(z_1, z_2)$ is of full row rank, then for consistency we define its MLA as the "void" matrix, with 0 rows and p columns [13].

The two backward shift operators, σ_1 and σ_2, along the coordinate axes of the discrete grid $\mathbb{Z} \times \mathbb{Z}$ are defined as:

$$(\sigma_1 \mathbf{w})(h, k) := \mathbf{w}(h + 1, k),\ (\sigma_2 \mathbf{w})(h, k) := \mathbf{w}(h, k + 1),$$

respectively. The forward shift operators σ_1^{-1} and σ_2^{-1} are similarly defined. Notice that σ_i, $i = 1, 2$, and σ_2^{-1} map $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$ into $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$, but this is not true for σ_1^{-1}.

If $R(z_1, z_2)$ is a $p \times \mathtt{w}$ L-polynomial matrix, we associate with it the L-polynomial matrix operator $R(\sigma_1, \sigma_2)$, acting on any 2D sequence of size $\mathtt{w}$. In the special case when $R(z_1, z_2)$ is a TS-polynomial matrix, the operator $R(\sigma_1, \sigma_2)$ maps $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$ into $(\mathbb{R}^p)^{\mathbb{Z}_+ \times \mathbb{Z}}$. If $R(z_1, z_2)$ is a TS-polynomial matrix, the associated map $R(\sigma_1, \sigma_2)$ is injective (i.e. $\ker R(\sigma_1, \sigma_2) = \{0\}$) if and only if $R(z_1, z_2)$ is right zero prime, and it is surjective if and only if $R(z_1, z_2)$ is of full row rank.

3. Basic facts about 2D behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$

A 2D *behavior* $\mathfrak{B}$ on $\mathbb{Z}_+ \times \mathbb{Z}$ is the set of solutions $\mathbf{w} = \{\mathbf{w}(h, k)\}_{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}}$ of a family of linear 2D difference equations of the following type:

$$\sum_{(i,j) \in \Sigma_R} R_{ij} \mathbf{w}(h + i, k + j) = 0, \qquad \forall\ (h, k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{3.1}$$

where the R_{ij}'s are real matrices with $\mathtt{w}$ columns (and say p rows), and the index set Σ_R is a finite subset of $\mathbb{Z}_+ \times \mathbb{Z}$. A 2D behavior $\mathfrak{B}$ described as in (3.1), is denoted by

$$\mathfrak{B} = \ker R(\sigma_1, \sigma_2), \tag{3.2}$$

where $R(z_1, z_2) = \sum_{(i,j)\in\Sigma_R} R_{ij} z_1^i z_2^j$ is a TS-polynomial matrix.

Given two TS-polynomial matrices $R_1(z_1, z_2)$ and $R_2(z_1, z_2)$, with the same number of columns $\mathtt{w}$, condition $\ker R_1(\sigma_1, \sigma_2) \subseteq \ker R_2(\sigma_1, \sigma_2)$ holds if and only if $R_2(z_1, z_2) = P(z_1, z_2)R_1(z_1, z_2)$, for some TS-polynomial matrix $P(z_1, z_2)$ of suitable size.

We now introduce autonomous behaviors.

Definition 3.1. [6, 16] *Consider a 2D behavior* $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$*, with* $R(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$*. A set of variables* $\{\mathbf{w}_i : i \in \mathcal{I}\}$*,* $\mathcal{I} \subsetneq \{1, 2, \dots, \mathtt{w}\}$*, is said to be a* set of free variables *for* $\mathfrak{B}$ *if the map* $\pi_{\mathcal{I}} : \mathfrak{B} \to (\mathbb{R}^{|\mathcal{I}|})^{\mathbb{Z}_+\times\mathbb{Z}}$*, that projects every behavior trajectory onto the components indexed by* $\mathcal{I}$*, is surjective.* $\mathfrak{B}$ *is said to be* autonomous *if it has no free variables.*

Within the class of 2D autonomous behaviors, we single out the nilpotent ones. Before introducing their definition, it is convenient to introduce some notation that will be used extensively in the rest of the paper (see [4]). For any pair of nonnegative integers t_0 and t_1, with $t_0 \le t_1$, we define the *vertical strip*

$$\mathcal{S}_{t_0,t_1} := \{(h, k) \in \mathbb{Z}_+ \times \mathbb{Z} : t_0 \le h \le t_1\}.$$

When $t_0 = t_1$ we use $\mathcal{S}_{t_0}$ to denote the *vertical line* $\{(t_0, k) : k \in \mathbb{Z}\}$, when $t_0 = 0 \le t_1$ we use $\mathcal{S}_{\to t_1}$ and when $t_1 = +\infty$ we use $\mathcal{S}_{t_0\to}$. Given any trajectory $\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$ and any set $\mathcal{S}_{t_0,t_1}$, we denote the trajectory restriction to the set $\mathcal{S}_{t_0,t_1}$ by $\mathbf{w}|_{\mathcal{S}_{t_0,t_1}}$. The *support* of a trajectory $\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+\times\mathbb{Z}}$ is the set of points where the trajectory takes nonzero values, i.e. $\{(h, k) \in \mathbb{Z}_+ \times \mathbb{Z} : \mathbf{w}(h, k) \neq 0\}$.

Definition 3.2. *A 2D autonomous behavior* $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$*, with* $R(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times\mathtt{w}}$*, is said to be* nilpotent *(with respect to the half-plane* $\mathbb{Z}_+ \times \mathbb{Z}$*) if there exists* $N \in \mathbb{Z}_+$ *s.t. all the trajectories* $\mathbf{w} \in \mathfrak{B}$ *satisfy* $\mathbf{w}|_{\mathcal{S}_{N\to}} = 0$ *or, equivalently,* $\mathbf{w}(h, k) = 0,\ \forall\ (h, k) \in \mathcal{S}_{N\to}$.

Nilpotency for a 2D behavior defined on $\mathbb{Z}_+ \times \mathbb{Z}$ does not mean that each trajectory has a finite support, as in the 1D case [5], but only that its support intersects finitely many vertical lines $\mathcal{S}_t$ of $\mathbb{Z}_+ \times \mathbb{Z}$. Further insights into autonomous and nilpotent 2D behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$ can be found in [4].

Proposition 3.3. [4] *A 2D behavior* $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$*, with* $R(z_1, z_2) \in \mathbb{R}[z_1, z_2,\ z_2^{-1}]^{p\times\mathtt{w}}$*, is*

- *autonomous if and only if* $R(z_1, z_2)$ *is a full column rank matrix;*
- *nilpotent if and only if* $R(z_1, z_2)$ *is a right monomic matrix.*

4. Reconstructibility and dead-beat observers

Consider a 2D system whose behavior $\mathfrak{B}$ is described as in (3.2), for some matrix $R(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$. Assume that the system variables, grouped together in the vector $\mathbf{w}$, split into two groups: measured variables, denoted by $\mathbf{w}_m$, and variables that represent the target of our estimation problem (the "relevant" variables), denoted by $\mathbf{w}_r$. The TS-polynomial matrix $R(z_1, z_2)$ can be accordingly block-partitioned, thus leading to the following description of the 2D behavior trajectories:

$$(4.1) \qquad [R_r(\sigma_1, \sigma_2) \ \ -R_m(\sigma_1, \sigma_2)] \begin{bmatrix} \mathbf{w}_r(h,k) \\ \mathbf{w}_m(h,k) \end{bmatrix} = 0, \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

or, equivalently

$$(4.2) \qquad R_r(\sigma_1, \sigma_2)\mathbf{w}_r(h,k) = R_m(\sigma_1, \sigma_2)\mathbf{w}_m(h,k), \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

where $R_r(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_r}$ and $R_m(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_m}$. With respect to this partition of the system variables[1], the notions of observability and reconstructibility have been introduced in [1] for 2D behaviors defined on the half-plane $\mathcal{H}_0 = \{(h,k) \in \mathbb{Z}\times\mathbb{Z} : h+k \geq 0\}$. In [1] it was also assumed that additional unmeasured and not relevant variables (for instance, disturbances) were involved in the system description. The adaption to the case of behaviors described as in (4.2), and defined over the half-plane $\mathbb{Z}_+ \times \mathbb{Z}$, is immediate.

Definition 4.1. *Given a behavior $\mathfrak{B}$, described as in (4.2), we say that $\mathbf{w}_r$ is* reconstructible *from $\mathbf{w}_m$ if there exists $N \in \mathbb{Z}_+$ such that $(\mathbf{w}_r, \mathbf{w}_m)$, $(\bar{\mathbf{w}}_r, \mathbf{w}_m) \in \mathfrak{B}$ implies $\mathbf{w}_r(h,k) - \bar{\mathbf{w}}_r(h,k) = 0$, $\forall\ (h,k) \in \mathcal{S}_{N\rightarrow}$.*

A characterization of reconstructibility is provided in Proposition 4.2, below, and it is a simple adaption of the analogous result obtained in [1].

Proposition 4.2. *Given a 2D behavior $\mathfrak{B}$, described as in (4.2), $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$ if and only if $R_r(z_1, z_2)$ is right monomic.*

For a behavior $\mathfrak{B}$ described as in (4.2), a dead-beat observer (DBO) of $\mathbf{w}_r$ from $\mathbf{w}_m$ is a system that, corresponding to every trajectory $(\mathbf{w}_r, \mathbf{w}_m)$ in $\mathfrak{B}$, produces an

[1] It is worth noticing that, so long as we restrict our attention to the estimation problem alone, it is reasonable to assume that $\mathbf{w}_m$ and $\mathbf{w}_r$ correspond to disjoint sets, namely they have no common variables. However, the choice of the relevant variables may also be imposed by control purposes, as it happens when dealing with observer-based controllers, as we will see in section 7. When so, the two sets of variables are not necessarily disjoint. If this is the case, one can simply replace $\mathbf{w}_r$ with $\tilde{\mathbf{w}}_r = (\mathbf{w}_r, \mathbf{w}_{mr})$ and assume $\mathbf{w}_m = (\mathbf{w}_{m,nr}, \mathbf{w}_{mr})$, where $\mathbf{w}_r$ corresponds to the relevant (not measured) variables, $\mathbf{w}_{m,nr}$ to the measured variables that are not relevant, and $\mathbf{w}_{mr}$ are measured variables that are also relevant. In this case, we replace (4.2) with

$$\begin{bmatrix} R_r(\sigma_1,\sigma_2) & 0 \\ 0 & I \end{bmatrix} \tilde{\mathbf{w}}_r(h,k) = \begin{bmatrix} R_m(\sigma_1,\sigma_2) \\ 0 \quad I \end{bmatrix} \mathbf{w}_m(h,k).$$

estimate $\hat{\mathbf{w}}_r$ of the trajectory $\mathbf{w}_r$ (based on the measured variables $\mathbf{w}_m$ alone), that coincides with $\mathbf{w}_r$ except, possibly, on an initial strip $\mathcal{S}_{\to N-1}$. This notion, together with the additional "consistency" property of a DBO, is formalized in the following definition. For more insights into its practical meaning we refer the interested reader to [1].

Definition 4.3. *Consider a 2D behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$, described as in (4.2). The system represented by the 2D difference equation*

$$Q(\sigma_1, \sigma_2)\hat{\mathbf{w}}_r(h,k) = P(\sigma_1, \sigma_2)\mathbf{w}_m(h,k), \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{4.3}$$

with $P(z_1, z_2)$ and $Q(z_1, z_2)$ TS-polynomial matrices of suitable sizes, is said to be a dead-beat observer *(DBO) of $\mathbf{w}_r$ if the following two conditions hold:*

(a) for every $(\mathbf{w}_r, \mathbf{w}_m) \in \mathfrak{B}$ there exists $\hat{\mathbf{w}}_r$ such that $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfies (4.3);

(b) there exists $N \in \mathbb{Z}_+$ such that for every $(\mathbf{w}_r, \mathbf{w}_m)$ in $\mathfrak{B}$ and $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfying (4.3), we have $\mathbf{w}_r(h,k) - \hat{\mathbf{w}}_r(h,k) = 0$ for every $(h,k) \in \mathcal{S}_{N\to}$.

A DBO of $\mathbf{w}_r$ is consistent *if, in addition, every trajectory $(\mathbf{w}_r, \mathbf{w}_m) \in \mathfrak{B}$ satisfies (4.3).*

Theorem 4.4. *Given a 2D behavior $\mathfrak{B}$, described as in (4.2), the following facts are equivalent:*

i) there exists a consistent DBO of $\mathbf{w}_r$;

ii) there exists a DBO of $\mathbf{w}_r$;

iii) $R_r(z_1, z_2)$ is right monomic (namely, $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$).

5. Zero-time-controllability and dead-beat controllers

For 2D behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$, we have introduced the definition of zero-time-controllability [4]. For zero-time-controllable behaviors it is possible to patch any initial strip of a behavior trajectory (its portion in $\mathcal{S}_{\to N-1}$) with the zero trajectory. This means that any behavior trajectory can be driven to zero within a finite number of time instants, so that it is identically zero on some suitable half-plane $\mathcal{S}_{N+L\to}$. In this paper we extend the results obtained in [4] to the case when the system variables are partitioned into two families, and we are interested only in ensuring that one of these sets has trajectories that can be driven to zero (see also [2]).

Definition 5.1. *Given a 2D behavior $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$ with $R(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p \times \mathtt{w}}$, let I be a subset of $\{1, 2, \ldots, \mathtt{w}\}$, and let $\mathbf{w}_I$ be the subset of the system variables, consisting of all the entries of $\mathbf{w}$ indexed by I. We say that $\mathbf{w}_I$*

is zero-time-controllable *if there exists a nonnegative integer $L \in \mathbb{Z}_+$ s.t. for every $N \in \mathbb{N}$ and every $\mathbf{w} \in \mathfrak{B}$, one can find $\bar{\mathbf{w}} \in \mathfrak{B}$ s.t.*

$$\bar{\mathbf{w}}_I(h,k) = \mathbf{w}_I(h,k), \qquad \forall\ (h,k) \in \mathcal{S}_{\to N-1}, \tag{5.1}$$

$$\bar{\mathbf{w}}_I(h,k) = 0, \qquad \forall\ (h,k) \in \mathcal{S}_{N+L\to}, \tag{5.2}$$

i.e. $\bar{\mathbf{w}}_I|_{\mathcal{S}_{\to N-1}} = \mathbf{w}_I|_{\mathcal{S}_{\to N-1}}$ *and* $\sigma_1^{N+L}\bar{\mathbf{w}}_I = 0$.
When $I = \{1, 2, \ldots, \mathtt{w}\}$ we say, equivalently, that $\mathbf{w}$ or the behavior $\mathfrak{B}$ is zero-time-controllable.

Note that, when $I \subsetneq \{1, 2, \ldots, \mathtt{w}\}$, no constraint is imposed on the evolution of the remaining variables $\mathbf{w}_{\bar{I}}$, where $\bar{I}$ is the complementary set of I with respect to $\{1, 2, \ldots, \mathtt{w}\}$. In [4] we have derived algebraic characterizations of zero-time-controllable behaviors. The result can be extended to the case when we restrict our attention to a subset of the system variables, as the zero-time-controllability of $\mathbf{w}_I$ is equivalent to the zero-time-controllability (in the sense of [4]) of the projection[2] $\mathcal{P}_I\mathfrak{B}$ of $\mathfrak{B}$ on the variables $\mathbf{w}_I$ (see also [2]).

Theorem 5.2. *Given a 2D behavior, described by the difference equation:*

$$R_I(\sigma_1,\sigma_2)\mathbf{w}_I(h,k) = R_{\bar{I}}(\sigma_1,\sigma_2)\mathbf{w}_{\bar{I}}(h,k), \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{5.3}$$

where $R_I(z_1,z_2)$ and $R_{\bar{I}}(z_1,z_2)$ are TS-polynomial matrices of sizes $q \times \mathtt{w}_I$ and $q \times (\mathtt{w} - \mathtt{w}_I)$, respectively, let $M_{\bar{I}}(z_1,z_2)$ be an MLA of $R_{\bar{I}}(z_1,z_2)$. The following facts are equivalent:

i) $\mathbf{w}_I$ *is* zero-time-controllable*;*

ii) either $R_{\bar{I}}(z_1,z_2)$ is of full row rank or $H_I(z_1,z_2) := M_{\bar{I}}(z_1,z_2)R_I(z_1,z_2)$ can be expressed as

$$H_I(z_1,z_2) = L(z_1,z_2)\Delta(z_1,z_2)R(z_1,z_2),$$

with $L(z_1,z_2)$ right monomic, $\Delta(z_1,z_2)$ nonsingular square with $\det\Delta(z_1, z_2) \in \mathbb{R}[z_2,z_2^{-1}]$, and $R(z_1,z_2)$ left factor prime.

By a *controller* $\mathcal{C}$ of a given 2D behavior $\mathfrak{B} = \ker\, R(\sigma_1,\sigma_2)$, with $R(z_1,\ z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times\mathtt{w}}$, we mean a system that constrains the system trajectories, and hence is described by a difference equation of the following type

$$C(\sigma_1,\sigma_2)\mathbf{w}(h,k) = 0, \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{5.4}$$

[2] It is well-known [4, 10], that if a behavior $\mathfrak{B}$ is described by $R_I(\sigma_1,\sigma_2)\mathbf{w}_I(h,k) = R_{\bar{I}}(\sigma_1,\sigma_2)\mathbf{w}_{\bar{I}}(h,k), (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}$, then $\mathcal{P}_I\mathfrak{B} = \ker(M_{\bar{I}}(\sigma_1,\sigma_2)R_I(\sigma_1,\sigma_2))$, where $M_{\bar{I}}(z_1,z_2)$ is an MLA of $R_{\bar{I}}(z_1,z_2)$. In the special case when $R_{\bar{I}}(z_1,z_2)$ is of full row rank, its MLA is a *void* matrix and $\mathcal{P}_I\mathfrak{B} = \ker 0$, which means that $\mathcal{P}_I\mathfrak{B} = (\mathbb{R}^{\mathtt{w}_I})^{\mathbb{Z}_+\times\mathbb{Z}}$.

for a suitable TS-polynomial matrix $C(z_1, z_2)$. The overall *controlled behavior*, i.e. the behavior of the system obtained by *full interconnection* of the behavior $\mathfrak{B}$ and the controller (5.4), is described by

$$\begin{bmatrix} R(\sigma_1, \sigma_2) \\ C(\sigma_1, \sigma_2) \end{bmatrix} \mathbf{w}(h,k) = 0, \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{5.5}$$

it is denoted by $\mathcal{K}$, and it is clearly the intersection of $\mathfrak{B}$ and $\mathcal{C} := \ker C(\sigma_1, \sigma_2)$. On the other hand, if we assume that $\mathfrak{B}$ is described as in

$$R_r(\sigma_1, \sigma_2)\mathbf{w}_r(h,k) = R_m(\sigma_1, \sigma_2)\mathbf{w}_m(h,k), \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{5.6}$$

where $R_r(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_r}$, $R_m(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_m}$, and $\mathbf{w}_r$ and $\mathbf{w}_m$ represent, as in the previous sections, the relevant and the measured variables, respectively, it makes sense to consider also control by *partial interconnection*, namely to apply the control action only to the measured variables. When so, the controlled behavior $\mathcal{K}$ is described as

$$\begin{bmatrix} R_r(\sigma_1, \sigma_2) & -R_m(\sigma_1, \sigma_2) \\ 0 & -C_m(\sigma_1, \sigma_2) \end{bmatrix} \begin{bmatrix} \mathbf{w}_r(h,k) \\ \mathbf{w}_m(h,k) \end{bmatrix} = 0, \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}. \tag{5.7}$$

In either cases, whether we act by full interconnection or by partial interconnection[3], the target of the dead-beat control problem is to design, if possible, a controller s.t. either the whole controlled behavior $\mathcal{K}$ or its projection on some subset $\mathbf{w}_I$ of its variables, $\mathcal{P}_I\mathcal{K}$, is nilpotent.

Definition 5.3. *Given a 2D behavior $\mathfrak{B} = \ker\ R(\sigma_1, \sigma_2)$, with $R(z_1,\ z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, a controller $\mathcal{C}$, acting on $\mathfrak{B}$ by full or partial interconnection, is said to be a* dead-beat controller (DBC) for the system variables $\mathbf{w}_I, I \subseteq \{1, 2, \ldots, \mathtt{w}\}$, *if there exists $N \in \mathbb{Z}_+$ s.t. all the trajectories of the behavior $\mathcal{P}_I\mathcal{K} = \mathcal{P}_I(\mathfrak{B} \cap \mathcal{C})$ have supports included in the vertical strip $\mathcal{S}_{\to N-1}$.*

Remark 5.4. *The previous definition can be restated in terms of* projections. *In fact, a controller $\mathcal{C}$ is a DBC for $\mathbf{w}_I$ if and only if $\mathcal{P}_I\mathcal{K}$ is a nilpotent behavior.*

Clearly, when $I = \{1, 2, \ldots, \mathtt{w}\}$, a DBC for $\mathbf{w}_I$ becomes a DBC for the whole system variables, and we will refer to it as a DBC for $\mathfrak{B}$.

We have the following characterization of the existence of a DBC, acting on the measured variables alone.

Proposition 5.5. *Consider a 2D behavior $\mathfrak{B}$, described as in (5.6), where $R_r(z_1, z_2)$ and $R_m(z_1, z_2)$ are TS-polynomial matrices of size $p \times \mathtt{w}_r$ and $p \times \mathtt{w}_m$, respectively. The following conditions are equivalent:*

[3] In the following, when we will need to distinguish between the two of them, we will denote a controller that acts by full interconnection by $\mathcal{C}_{FI}$ and a controller that acts by partial interconnection by $\mathcal{C}_{PI}$.

i) there exists a DBC $\mathcal{C}$ for the behavior $\mathfrak{B}$, described by

$$C_m(\sigma_1, \sigma_2)\mathbf{w}_m(h,k) = 0, \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{5.8}$$

for some TS-polynomial matrix $C_m(z_1, z_2)$;

ii) there exists a DBC $\mathcal{C}$ for the variables $\mathbf{w}_r$, described as in (5.8), for some TS-polynomial matrix $C_m(z_1, z_2)$;

iii) $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$ (equivalently, $R_r(z_1, z_2)$ is right monomic).

Proof. i) $\Rightarrow$ ii) is obvious.

ii) $\Rightarrow$ iii) If ii) holds, then the behavior

$$\mathcal{K} = \ker \begin{bmatrix} R_r(\sigma_1, \sigma_2) & -R_m(\sigma_1, \sigma_2) \\ 0 & -C_m(\sigma_1, \sigma_2) \end{bmatrix}, \tag{5.9}$$

has projection $\mathcal{P}_{\mathbf{w}_r}\mathfrak{B}$ that is nilpotent. This means that if $[\, M_R(z_1, z_2) \quad M_C(z_1, z_2)\,]$ is an MLA of $\begin{bmatrix} R_m(z_1, z_2) \\ C_m(z_1, z_2) \end{bmatrix}$, then

$$[\, M_R(z_1, z_2) \quad M_C(z_1, z_2)\,] \begin{bmatrix} R_r(z_1, z_2) \\ 0 \end{bmatrix} = M_R(z_1, z_2)R_r(z_1, z_2)$$

is right monomic. But this implies that there exists a TS-polynomial matrix $L(z_1, z_2)$ and $\delta \in \mathbb{Z}_+$ such that

$$z_1^\delta I_{\mathtt{w}_r} = L(z_1, z_2)[M_R(z_1, z_2)R_r(z_1, z_2)] = [L(z_1, z_2)M_R(z_1, z_2)]R_r(z_1, z_2),$$

thus ensuring that $R_r(z_1, z_2)$ is right monomic, too.

iii) $\Rightarrow$ i) It is easily seen that, under the assumption that $R_r(z_1, z_2)$ is right monomic, the trivial controller $\mathcal{C} = \ker I_{\mathtt{w}_m}$ is a DBC for the whole behavior $\mathfrak{B}$. $\square$

Given a controller (in particular, a DBC) $\mathcal{C}$, described as in (5.4), we introduce the family of *delayed controllers* $\mathcal{C}_d, d \in \mathbb{Z}_+$, each of them described by the difference equation

$$\sigma_1^d C(\sigma_1, \sigma_2)\mathbf{w}(h,k) = 0, \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}.$$

If we denote by $\mathcal{K}_d$ the controlled behavior obtained corresponding to $\mathcal{C}_d$, then

$$\mathcal{K}_d = \ker \begin{bmatrix} R(\sigma_1, \sigma_2) \\ \sigma_1^d C(\sigma_1, \sigma_2) \end{bmatrix}. \tag{5.10}$$

Clearly, $\mathcal{C} = \mathcal{C}_0$ and $\mathcal{K} = \mathcal{K}_0$.

If $\mathcal{C}$ is a DBC for the system variables $\mathbf{w}_I$, then every $\mathcal{C}_d$ is a DBC for $\mathbf{w}_I$. This is proved in the following lemma, which extends to 2D behaviors and to partial interconnection a similar result derived for 1D systems in the case of full interconnection.

As the proof follows the same lines, *mutatis mutandis*, as the one in [2], we have chosen to provide a concise version of it.

LEMMA 5.6. *Given a 2D behavior* $\mathfrak{B} = \ker R(\sigma_1,\sigma_2)$*, with* $R(z_1, z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times \mathtt{w}}$*, let* $\mathcal{C} = \ker C(\sigma_1,\sigma_2)$*, with* $C(z_1, z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{q\times \mathtt{w}}$*, be a DBC for the system variables* $\mathbf{w}_I$*. Then, for every* $d \in \mathbb{Z}_+$*,* $\mathcal{C}_d$ *is a DBC for* $\mathbf{w}_I$*.*

Proof. It entails no loss of generality to rewrite both the behavior and the controller equations as follows, in order to separate the variables $\mathbf{w}_I$ and $\mathbf{w}_{\bar I}$:

$$\begin{aligned} R_I(\sigma_1,\sigma_2)\mathbf{w}_I(h,k) &= R_{\bar I}(\sigma_1,\sigma_2)\mathbf{w}_{\bar I}(h,k), \\ C_I(\sigma_1,\sigma_2)\mathbf{w}_I(h,k) &= C_{\bar I}(\sigma_1,\sigma_2)\mathbf{w}_{\bar I}(h,k), \qquad (h,k)\in\mathbb{Z}_+\times\mathbb{Z}. \end{aligned}$$

As $\mathcal{C}$ is a DBC for $\mathbf{w}_I$, $\mathcal{P}_I\mathcal{K}$ is nilpotent or, equivalently,

$$H_I(z_1,z_2) := \begin{bmatrix} M_R(z_1,z_2) & M_C(z_1,z_2) \end{bmatrix} \begin{bmatrix} R_I(z_1,z_2) \\ C_I(z_1,z_2) \end{bmatrix}$$

is right monomic, where $\begin{bmatrix} M_R(z_1,z_2) & M_C(z_1,z_2) \end{bmatrix}$ is an MLA of $\begin{bmatrix} R_{\bar I}(z_1,z_2) \\ C_{\bar I}(z_1,z_2) \end{bmatrix}$. Now, consider

$$\mathcal{K}_d = \mathcal{B}\cap\mathcal{C}_d = \ker \begin{bmatrix} R_I(\sigma_1,\sigma_2) & -R_{\bar I}(\sigma_1,\sigma_2) \\ \sigma_1^d C_I(\sigma_1,\sigma_2) & -\sigma_1^d C_{\bar I}(\sigma_1,\sigma_2) \end{bmatrix},$$

and let $\begin{bmatrix} \hat M_R(z_1,z_2) & \hat M_C(z_1,z_2) \end{bmatrix}$ denote an MLA of $\begin{bmatrix} R_{\bar I}(z_1,z_2) \\ z_1^d C_{\bar I}(z_1,z_2) \end{bmatrix}$. It is easily seen that $\begin{bmatrix} z_1^d M_R(z_1,z_2) & M_C(z_1,z_2) \end{bmatrix}$ is a left annihilator of that same matrix, and hence there exists a TS-polynomial matrix $P(z_1,z_2)$ such that

$$\begin{bmatrix} z_1^d M_R(z_1,z_2) & M_C(z_1,z_2) \end{bmatrix} = P(z_1,z_2)\begin{bmatrix} \hat M_R(z_1,z_2) & \hat M_C(z_1,z_2) \end{bmatrix}.$$

This ensures that

$$\mathcal{P}_I\mathcal{K}_d \subseteq \ker\Big(\sigma_1^d H_I(\sigma_1,\sigma_2)\Big).$$

As $H_I(z_1,z_2)$ is right monomic, $\mathcal{P}_I\mathcal{K}_d$ is included in the nilpotent behavior $\ker\big(\sigma_1^d H_I(\sigma_1,\sigma_2)\big)$, and hence it is nilpotent, too. This implies that $\mathcal{C}_d$ is a DBC for $\mathbf{w}_I$. □

REMARK 5.7. *Clearly,* $\mathcal{C}_d$ *and* $\mathcal{C}$ *are DBCs of the same type, meaning that if* $\mathcal{C}$ *acts on the system variables by full (partial) interconnection, so does* $\mathcal{C}_d$*.*

The concept of delayed controller allows to introduce that of *admissible DBC*, a notion first introduced in [2] for 1D behaviors, and later extended to DBCs acting by full interconnection on 2D behaviors in [4]. A controller is admissible if it can start acting on the 2D trajectory at any time, without constraining *a posteriori* its "past evolution". So, it is immediately seen that controllers that are not endowed with this property do not admit any real-time implementation.

DEFINITION 5.8. *Given a 2D behavior* $\mathfrak{B}$*, a dead-beat controller* $\mathcal{C}$ *for the system variables* $\mathbf{w}_I, I \subseteq \{1,2,\dots,\mathtt{w}\}$*, (in particular, a DBC for* $\mathfrak{B}$*) is said to be* admissible *if there exists* $L \in \mathbb{Z}_+$ *s.t. for every* $\mathbf{w} \in \mathfrak{B}$ *and every* $N \in \mathbb{N}$*, there exists* $\bar{\mathbf{w}} \in \mathcal{K}_{L+N}$*, the controlled behavior obtained corresponding to the controller* $\mathcal{C}_{L+N}$*, s.t.* $\bar{\mathbf{w}}_I(h,k)|_{\mathcal{S}_{\to N-1}} = \mathbf{w}_I(h,k)|_{\mathcal{S}_{\to N-1}}$.

In the following we will characterize the existence of DBCs and of admissible DBCs by assuming that the system variables split into relevant and measured variables, and that we resort to a partial interconnection controller $\mathcal{C} = \mathcal{C}_{PI}$, acting on the measured variables only. This means that the overall controlled behavior $\mathcal{K}$ will always be described as in (5.7). Also, we will aim at driving to zero either all the system variables ($I = \{1, 2, \dots, \mathtt{w}\}$) or just the relevant variables ($\mathbf{w}_I = \mathbf{w}_r$). In order to derive such characterizations, we will make use of the following result derived in [4].

THEOREM 5.9. *Given a 2D behavior* $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$*, with* $R(z_1,\ z_2) \in \mathbb{R}[z_1, z_2,\ z_2^{-1}]^{p\times \mathtt{w}}$*, there exists a full interconnection controller* $\mathcal{C}_{FI}$ *that acts as an admissible DBC for* $\mathfrak{B}$ *if and only if* $\mathfrak{B}$ *is zero-time-controllable. If this is the case, then every DBC for* $\mathfrak{B}$ *is admissible.*

We now extend the characterization provided by Proposition 5.5 to the case of admissible DBCs, by first addressing the case when we are interested in driving to zero only $\mathbf{w}_r$.

THEOREM 5.10. *Consider a 2D behavior* $\mathfrak{B}$*, described as in (5.6), for suitable TS-polynomial matrices* $R_r(z_1, z_2)$ *and* $R_m(z_1, z_2)$*, of size* $p \times \mathtt{w}_r$ *and* $p \times \mathtt{w}_m$*, respectively. There exists an admissible DBC* $\mathcal{C}$ *for the variables* $\mathbf{w}_r$*, described as in (5.8), for some TS-polynomial matrix (of size say* $q \times \mathtt{w}_m$*)* $C_m(z_1, z_2)$ *if and only if the following two conditions hold:*

a) $\mathbf{w}_r$ *is reconstructible from* $\mathbf{w}_m$*;*

b) the variables $\mathbf{w}_r$ *are zero-time-controllable.*

If these two conditions are satisfied, then every DBC for $\mathbf{w}_r$ *is admissible.*

Proof. [Necessity] Assume, first, that there exists an admissible DBC $\mathcal{C}$ for $\mathbf{w}_r$, described as in (5.8). This means that the overall controlled behavior $\mathcal{K}$, described as in (5.9), has projection $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}$ that is nilpotent and hence there exists $M \in \mathbb{Z}_+$ such that all trajectories in $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}$ are zero in $\mathcal{S}_{M\to}$. If we let $[\, M_R(z_1,z_2) \quad M_C(z_1,z_2)\,]$ be an MLA of $\begin{bmatrix} R_m(z_1,z_2) \\ C_m(z_1,z_2) \end{bmatrix}$, then

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K} = \ker(M_R(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2)).$$

By being $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}$ nilpotent, $M_R(z_1,z_2)R_r(z_1,z_2)$ is right monomic, and this implies (as we have already seen in the proof of Proposition 5.5) that $R_r(z_1,z_2)$ is right monomic, too, so ensuring that $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$. Now, consider

$$\mathcal{K}_d = \mathcal{B} \cap \mathcal{C}_d = \ker \begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -\sigma_1^d C_m(\sigma_1,\sigma_2) \end{bmatrix}.$$

We have already shown in the proof of Lemma 5.6 that

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_d \subseteq \ker\left(\sigma_1^d M_R(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2)\right).$$

So, the trajectories of $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_d$ have support included in $\mathcal{S}_{\rightarrow d+M-1}$, and this is true for every $d \in \mathbb{Z}_+$.

Since $\mathcal{C}$ is an admissible DBC for $\mathbf{w}_r$, there exists $L \in \mathbb{Z}_+$ such that for every $N \in \mathbb{N}$ and every $(\mathbf{w}_r, \mathbf{w}_m) \in \mathfrak{B}$ a trajectory $(\bar{\mathbf{w}}_r, \bar{\mathbf{w}}_m) \in \mathcal{K}_{L+N} \subseteq \mathfrak{B}$ can be found, with $\bar{\mathbf{w}}_r = \mathbf{w}_r$ in $\mathcal{S}_{\rightarrow N-1}$. Such a trajectory $\bar{\mathbf{w}}_r$ is surely zero in $\mathcal{S}_{L+N+M\rightarrow}$. So, we have proved that there exists $L^* \in \mathbb{N}$, specifically $L^* := M + L$, such that for every $(\mathbf{w}_r, \mathbf{w}_m) \in \mathfrak{B}$ there exists a trajectory $(\bar{\mathbf{w}}_r, \bar{\mathbf{w}}_m) \in \mathcal{K}_{L+N} \subseteq \mathfrak{B}$ such that $\bar{\mathbf{w}}_r = \mathbf{w}_r$ in $\mathcal{S}_{\rightarrow N-1}$ and $\bar{\mathbf{w}}_r = 0$ in $\mathcal{S}_{N+L^*\rightarrow}$. This proves that $\mathbf{w}_r$ is zero-time-controllable.

[Sufficiency] Assume now that conditions a) and b) hold. We want to prove that an admissible DBC for $\mathbf{w}_r$ always exists. Set $C_m(z_1,z_2) = R_m(z_1,z_2)$. Correspondingly

$$\mathcal{K} = \ker\begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -R_m(\sigma_1,\sigma_2)\end{bmatrix},$$

and it is easy to verify that if $M_m(z_1,z_2)$ is an MLA of $R_m(z_1,z_2)$, then

$$\begin{bmatrix} M_m(z_1,z_2) & 0 \\ I_p & -I_p \end{bmatrix}$$

is an MLA of $\begin{bmatrix} R_m(z_1,z_2) \\ R_m(z_1,z_2)\end{bmatrix}$. Therefore

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K} = \ker\left(\begin{bmatrix} M_m(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2) \\ R_r(\sigma_1,\sigma_2)\end{bmatrix}\right) = \ker R_r(\sigma_1,\sigma_2).$$

By assumption a), $R_r(z_1,z_2)$ is right monomic and hence $C_m(z_1,z_2) = R_m(z_1,z_2)$ defines a DBC for $\mathbf{w}_r$.
We now need to prove that this DBC is admissible. By the zero-time-controllability property, there exists a nonnegative integer L such that for every $N \in \mathbb{N}$ and every $(\mathbf{w}_r, \mathbf{w}_m) \in \mathfrak{B}$, one can find $(\bar{\mathbf{w}}_r, \bar{\mathbf{w}}_m) \in \mathfrak{B}$ such that

$$\begin{aligned} \bar{\mathbf{w}}_r(h,k) &= \mathbf{w}_r(h,k), & \forall\ (h,k) \in \mathcal{S}_{\rightarrow N-1} \\ \bar{\mathbf{w}}_r(h,k) &= 0, & \forall\ (h,k) \in \mathcal{S}_{N+L\rightarrow}. \end{aligned} \tag{5.11}$$

We want to show that the condition appearing in the definition of admissible DBC is verified for this choice of L. To this end we have to show that for every $N \in \mathbb{N}$ and every $(\mathbf{w}_r, \mathbf{w}_m) \in \mathfrak{B}$ a trajectory $(\bar{\mathbf{w}}_r, \bar{\mathbf{w}}_m) \in \mathcal{K}_{L+N} \subseteq \mathfrak{B}$ can be found, with $\bar{\mathbf{w}}_r = \mathbf{w}_r$ in $\mathcal{S}_{\rightarrow N-1}$. Now, consider

$$\mathcal{K}_{L+N} = \mathcal{B} \cap \mathcal{C}_{L+N} = \ker\begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -\sigma_1^{L+N}R_m(\sigma_1,\sigma_2)\end{bmatrix}.$$

It is easily seen that

$$\begin{bmatrix} M_m(z_1,z_2) & 0 \\ z_1^{L+N} I & -I \end{bmatrix}$$

is an MLA of $\begin{bmatrix} R_m(z_1,z_2) \\ z_1^{L+N} R_m(z_1,z_2) \end{bmatrix}$. This implies that

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_{L+N} = \ker\left(\begin{bmatrix} M_m(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2) \\ \sigma_1^{L+N} R_r(\sigma_1,\sigma_2) \end{bmatrix}\right).$$

So, it is easy to see that the same trajectory $\bar{\mathbf{w}}_r$ that satisfies (5.11), and whose existence is ensured by the zero-time-controllability property, is necessarily a trajectory of both $\ker(M_m(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2))$ and $\ker(\sigma_1^{L+N} R_r(\sigma_1,\sigma_2))$. Therefore $\bar{\mathbf{w}}_r \in \mathcal{P}_{\mathbf{w}_r}\mathcal{K}_{L+N}$ and this makes the definition of admissible DBC satisfied. This completes the proof of sufficiency.

By repeating the same reasoning we just used, we can show that if a) and b) hold, and $C_m(z_1,z_2)$ defines a DBC, then an MLA of $\begin{bmatrix} R_m(z_1,z_2) \\ C_m(z_1,z_2) \end{bmatrix}$ can always be expressed as

$$\begin{bmatrix} M_m(z_1,z_2) & 0 \\ M_{R_m}(z_1,z_2) & M_{C_m}(z_1,z_2) \end{bmatrix},$$

while a left annihilator of $\begin{bmatrix} R_m(z_1,z_2) \\ z_1^{N+L} C_m(z_1,z_2) \end{bmatrix}$ is

$$\begin{bmatrix} M_m(z_1,z_2) & 0 \\ z_1^{N+L} M_{R_m}(z_1,z_2) & M_{C_m}(z_1,z_2) \end{bmatrix}$$

and this ensures that

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_{L+N} \subseteq \ker\left(\begin{bmatrix} M_m(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2) \\ \sigma_1^{L+N} M_{R_m}(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2) \end{bmatrix}\right).$$

So, zero-time-controllability of $\mathbf{w}_r$ ensures the admissibility of any DBC for $\mathbf{w}_r$. $\square$

The following proposition addresses the existence of an admissible DBC for the behavior $\mathfrak{B}$. Its proof follows similar reasonings to those used in the previous proof, and hence it is omitted.

Proposition 5.11. *Consider a 2D behavior $\mathfrak{B}$, described as in (5.6), for suitable TS-polynomial matrices $R_r(z_1,z_2)$ and $R_m(z_1,z_2)$, of size $p \times \mathtt{w}_r$ and $p \times \mathtt{w}_m$, respectively. There exists an admissible DBC $\mathcal{C}$ for the behavior $\mathfrak{B}$, described as in (5.8), if and only if the following two conditions hold:*

a) $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$;

b) the behavior $\mathfrak{B}$ is zero-time-controllable.

If these two conditions are satisfied, then every DBC for $\mathfrak{B}$ is admissible.

6. Characterization of the DBCs

The aim of this section is to provide a characterization of the TS-polynomial matrices $C_m(z_1, z_2)$ that describe the (admissible) DBCs that drive to zero either all the system variables or just the relevant variables $\mathbf{w}_r$ of a 2D behavior described as in (5.6). As in the previous section, we address, first, the case of DBCs that drive to zero both $\mathbf{w}_r$ and $\mathbf{w}_m$.

Proposition 6.1. *Consider a 2D behavior $\mathfrak{B}$ described as in (5.6), with $R_r(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_r}$ and $R_m(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_m}$. Let $M_r(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{(p-\mathrm{rank}(R_r))\times p}$ be an MLA of $R_r(z_1, z_2)$, and suppose that the following two conditions hold:*

a) $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$;

b) $\mathfrak{B}$ is zero-time-controllable.

The TS-polynomial $C_m(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{q\times \mathtt{w}_m}$ defines a (necessarily admissible) DBC for $\mathfrak{B}$ if and only if the matrix

$$H_m(z_1, z_2) := \begin{bmatrix} M_r(z_1, z_2)R_m(z_1, z_2) \\ C_m(z_1, z_2) \end{bmatrix} \tag{6.1}$$

is right monomic.

Proof. [Necessity] If $C_m(z_1, z_2)$ defines an admissible DBC for $\mathfrak{B}$, then the controlled behavior, $\mathcal{K}$, described as in (5.9), is nilpotent. But this clearly implies that also its projection on $\mathbf{w}_m$ is. Since an MLA for

$$\begin{bmatrix} R_r(z_1, z_2) \\ 0 \end{bmatrix} \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{(p+q)\times \mathtt{w}_r}$$

is just

$$\begin{bmatrix} M_r(z_1, z_2) & 0 \\ 0 & I_q \end{bmatrix},$$

this means that

$$\mathcal{P}_{\mathbf{w}_m}\mathcal{K} = \ker \begin{bmatrix} M_r(\sigma_1, \sigma_2)R_m(\sigma_1, \sigma_2) \\ C_m(\sigma_1, \sigma_2) \end{bmatrix} = \ker H_m(\sigma_1, \sigma_2),$$

is nilpotent. This is equivalent to the fact that $H_m(z_1, z_2)$ is right monomic.

[Sufficiency] By reversing the last part of the previous reasoning, we can claim that if $H_m(z_1, z_2)$ is right monomic, then $\mathcal{P}_{\mathbf{w}_m}\mathcal{K}$ is nilpotent, and hence there exists $M \in \mathbb{Z}_+$ such that $\mathbf{w}_m \in \mathcal{P}_{\mathbf{w}_m}\mathcal{K}$ implies $\mathbf{w}_m(h, k) = 0$ in $\mathcal{S}_{M\to}$. Since every $(\mathbf{w}_r, \mathbf{w}_m) \in \mathcal{K}$ satisfies (5.6), it follows that

$$R_r(\sigma_1, \sigma_2)\mathbf{w}_r(h, k) = 0, \qquad \forall\ (h, k) \in \mathcal{S}_{M\to},$$

and if $L_r(z_1, z_2)$ satisfies

$$L_r(z_1, z_2)R_r(z_1, z_2) = z_1^\delta I_{\mathtt{w}_r},$$

for some $\delta \in \mathbb{Z}_+$, then

$$\sigma_1^\delta \mathbf{w}_r(h, k) = 0, \qquad \forall\ (h, k) \in \mathcal{S}_{M\to},$$

which amounts to saying that

$$\mathbf{w}_r(h, k) = 0, \qquad \forall\ (h, k) \in \mathcal{S}_{M+\delta\to}.$$

So, $C_m(z_1, z_2)$ defines a DBC for $\mathfrak{B}$, and since $\mathfrak{B}$ is zero-time-controllable, such DBC is also admissible. □

Remark 6.2. *A complete description of all TS-polynomial matrices $C_m(z_1,\ z_2)$ that make the matrix $H_m(z_1, z_2)$ right monomic is a little involved and probably not worth the effort. We want to provide, however, a parametrization of a large class of such matrices: all matrices described in the following way*

$$C_m(z_1, z_2) = z_1^\nu I - P(z_1, z_2)M_r(z_1, z_2)R_m(z_1, z_2), \qquad \begin{array}{l} \nu \in \mathbb{Z}_+, \\ P(z_1, z_2)\ \text{TS}-\text{polynomial}, \end{array}$$

make $H_m(z_1, z_2)$ right monomic. Indeed, from

$$\begin{bmatrix} P(z_1, z_2) & I \end{bmatrix} \begin{bmatrix} M_r(z_1, z_2)R_m(z_1, z_2) \\ z_1^\nu I - P(z_1, z_2)M_r(z_1, z_2)R_m(z_1, z_2) \end{bmatrix} = z_1^\nu I,$$

it is immediately seen that $H_m(z_1, z_2)$ admits an L-polynomial inverse, and hence it is right monomic.

We now characterize all the matrices $C_m(z_1, z_2)$ that describe (admissible) DBCs that drive to zero the variables $\mathbf{w}_r$ alone.

Proposition 6.3. *Consider a 2D behavior $\mathfrak{B}$ described as in (5.6), with $R_r(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_r}$ and $R_m(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_m}$. Let $M_r(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{(p-\mathrm{rank}(R_r))\times p}$ be an MLA of $R_r(z_1, z_2)$, and let $L_r(z_1,\ z_2)$ be a TS-polynomial matrix of size $\mathtt{w}_r \times p$ such that*

$$L_r(z_1, z_2)R_r(z_1, z_2) = z_1^\delta I_{\mathtt{w}_r},$$

for some $\delta \in \mathbb{Z}_+$. Suppose that the following two conditions hold:

a) $\mathbf{w}_r$ *is reconstructible from* $\mathbf{w}_m$*;*

b) $\mathbf{w}_r$ *is zero-time-controllable.*

The TS-polynomial matrix $C_m(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{q\times \mathtt{w}_m}$ defines a (necessarily admissible) DBC for $\mathbf{w}_r$ if and only if there exists $\nu \in \mathbb{Z}_+$ such that

$$\ker H_m(\sigma_1,\sigma_2) \subseteq \ker(z_1^{\nu} L_r(\sigma_1,\sigma_2)R_m(\sigma_1,\sigma_2)), \tag{6.2}$$

where $H_m(z_1,z_2)$ is defined as in (6.1).

Proof. [Necessity] If $C_m(z_1,z_2)$ defines an admissible DBC for $\mathbf{w}_r$, then the projection of

$$\mathcal{K} = \ker \begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -C_m(\sigma_1,\sigma_2) \end{bmatrix}$$

over $\mathbf{w}_r$ is nilpotent, which means that there exists $M \in \mathbb{Z}_+$ such that $\mathbf{w}_r \in \mathcal{P}_{\mathbf{w}_r}\mathcal{K}$ implies $\mathbf{w}_r(h,k) = 0$ in $\mathcal{S}_{M\to}$. On the other hand, as we showed in the proof of the previous proposition, $\mathcal{P}_{\mathbf{w}_m}\mathcal{K} = \ker H_m(\sigma_1,\sigma_2)$. We want to prove that (6.2) holds.

For every $(\mathbf{w}_r,\mathbf{w}_m) \in \mathcal{K}$, condition

$$R_r(\sigma_1,\sigma_2)\mathbf{w}_r(h,k) = R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

implies

$$\sigma_1^{\delta}\mathbf{w}_r = L_r(\sigma_1,\sigma_2)R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

and hence

$$0 = L_r(\sigma_1,\sigma_2)R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \qquad \forall\ (h,k) \in \mathcal{S}_{M+\delta\to}.$$

This proves that $\mathbf{w}_m \in \ker(\sigma_1^{M+\delta}L_r(\sigma_1,\sigma_2)R_m(\sigma_1,\sigma_2))$.

[Sufficiency] Consider the controlled behavior $\mathcal{K}$, obtained corresponding to $C_m(z_1, z_2)$. We have already proved that $\mathcal{P}_{\mathbf{w}_m}\mathcal{K} = \ker H_m(\sigma_1,\ \sigma_2)$. So, if (6.2) holds,

$$\mathcal{P}_{\mathbf{w}_m}\mathcal{K} \subseteq \ker(\sigma_1^{\nu}L_r(\sigma_1,\sigma_2)R_m(\sigma_1,\sigma_2)),$$

for some $\nu \in \mathbb{Z}_+$. Therefore, for every $(\mathbf{w}_r,\mathbf{w}_m) \in \mathcal{K}$, condition

$$R_r(\sigma_1,\sigma_2)\mathbf{w}_r(h,k) = R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

implies

$$\begin{aligned} \sigma_1^{\delta+\nu}\mathbf{w}_r(h,k) &= \sigma_1^{\nu}L_r(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2)\mathbf{w}_r(h,k) \\ &= \sigma_1^{\nu}L_r(\sigma_1,\sigma_2)R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k) \\ &= 0, \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}. \end{aligned}$$

Therefore $\mathbf{w}_r(h,k) = 0$ in $\mathcal{S}_{\delta+\nu\to}$. This proves that $C_m(z_1,z_2)$ defines a DBC for $\mathbf{w}_r$, and since $\mathbf{w}_r$ is zero-time-controllable, such a DBC is also admissible. □

Remark 6.4. *Clearly, the characterization given in Proposition 6.3 is weaker than the one given in Proposition 6.1. Indeed, this latter can be rewritten as: "there* *exists $\nu \in \mathbb{Z}_+$ such that* $\ker H_m(\sigma_1,\sigma_2) \subseteq \ker(\sigma_1^\nu I_{\mathtt{w}_m})$*," and when this condition is verified a fortiori condition (6.2) is.*

Remark 6.5. *As in the previous case, we can provide a parametrization of a class of matrices $C_m(z_1,z_2)$ such that the corresponding $H_m(z_1,z_2)$ satisfies condition (6.2). We want to show that all the matrices described in the following way*

$$C_m(z_1,z_2) = z_1^\nu L_r(z_1,z_2)R_m(z_1,z_2) - P(z_1,z_2)M_r(z_1,z_2)R_m(z_1,z_2),$$
$$\nu \in \mathbb{Z}_+,$$
$$P(z_1,z_2)\ \text{TS} - \text{polynomial},$$

make (6.2) satisfied. Indeed, from

$$\begin{aligned}[\,P(z_1,z_2) \quad I\,] &\begin{bmatrix} M_r(z_1,z_2)R_m(z_1,z_2) \\ z_1^\nu L_r(z_1,z_2)R_m(z_1,z_2) - P(z_1,z_2)M_r(z_1,z_2)R_m(z_1,z_2)\end{bmatrix} \\ &= z_1^\nu L_r(z_1,z_2)R_m(z_1,z_2),\end{aligned}$$

it is immediately seen that (6.2) holds.

7. Observer-based DBCs

In the previous sections we have explored the possibility of driving either all or part of the system variables to zero, by resorting to a partial interconnection controller acting on the measured variables alone. We have seen that the existence of admissible DBCs that drive to zero either $(\mathbf{w}_r,\mathbf{w}_m)$ or $\mathbf{w}_r$ is equivalent to the reconstructibility of $\mathbf{w}_r$ together with the zero-time-controllability either of the whole behavior $\mathfrak{B}$ or of $\mathbf{w}_r$. These two properties have proved to be equivalent, respectively, to the existence of a dead-beat observer and of an admissible dead-beat controller acting on all the variables (namely a full interconnection controller), this latter targeting either $(\mathbf{w}_r,\mathbf{w}_m)$ or $\mathbf{w}_r$.

So, the idea naturally arises that a DBC by partial interconnection could always be realized as an "observer-based DBC", by this meaning that we can first design a dead-beat observer that provides an estimate $\hat{\mathbf{w}}_r$ of the unaccessible but relevant variables $\mathbf{w}_r$, and then design a full interconnection DBC, acting on the pair $(\hat{\mathbf{w}}_r,\mathbf{w}_m)$, and driving to zero either $(\mathbf{w}_r,\mathbf{w}_m)$ or $\mathbf{w}_r$. Also, in this case we will address the "admissibility" issue, by this meaning that we will impose that the (shifted versions of the) DBCs allow to preserve the initial portion either of the pair of trajectories $(\mathbf{w}_r,\mathbf{w}_m)$ or of the trajectory $\mathbf{w}_r$ alone.

Consider a 2D behavior described as in

$$(7.1) \qquad R_r(\sigma_1,\sigma_2)\mathbf{w}_r(h,k) = R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

where $R_r(z_1,z_2)$ and $R_m(z_1,z_2)$ are TS-polynomial matrices of size $p\times\mathtt{w}_r$ and $p\times\mathtt{w}_m$, respectively. Suppose that $\hat{\mathbf{w}}_r$ is the estimate of $\mathbf{w}_r$ provided by a (possibly consistent) DBO, based on the knowledge of the measured variables $\mathbf{w}_m$, and described as follows:

$$Q(\sigma_1,\sigma_2)\hat{\mathbf{w}}_r(h,k) = P(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \qquad (h,k)\in\mathbb{Z}_+\times\mathbb{Z}, \tag{7.2}$$

where the TS-polynomial matrices $Q(z_1,z_2)$ and $P(z_1,z_2)$ have size $d\times\mathtt{w}_r$ and $d\times\mathtt{w}_m$, respectively. Finally, we introduce a full interconnection (w.r.t. the pair $(\hat{\mathbf{w}}_r,\mathbf{w}_m)$) controller

$$C_r(\sigma_1,\sigma_2)\hat{\mathbf{w}}_r(h,k) = C_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \qquad (h,k)\in\mathbb{Z}_+\times\mathbb{Z}, \tag{7.3}$$

where the TS-polynomial matrices $C_r(z_1,z_2)$ and $C_m(z_1,z_2)$ have size $q\times\mathtt{w}_r$ and $q\times\mathtt{w}_m$, respectively. The overall system is the one depicted in Figure 1

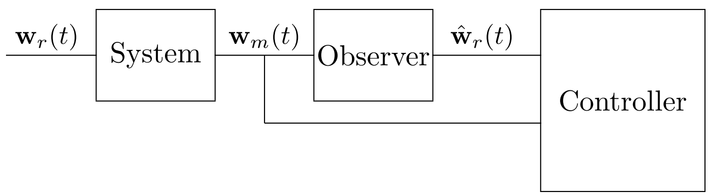

Fig.1 Observer-based DBC structure.

and its behavior $\mathcal{K}_{tot}$ is described by the following difference equation:

$$\begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) & 0 \\ 0 & -P(\sigma_1,\sigma_2) & Q(\sigma_1,\sigma_2) \\ 0 & -C_m(\sigma_1,\sigma_2) & C_r(\sigma_1,\sigma_2) \end{bmatrix} \begin{bmatrix} \mathbf{w}_r(h,k) \\ \mathbf{w}_m(h,k) \\ \hat{\mathbf{w}}_r(h,k) \end{bmatrix} = 0, \quad (h,k)\in\mathbb{Z}_+\times\mathbb{Z}. \tag{7.4}$$

If we set

$$H_{tot}(z_1,z_2) := \begin{bmatrix} R_r(z_1,z_2) & -R_m(z_1,z_2) & 0 \\ 0 & -P(z_1,z_2) & Q(z_1,z_2) \\ 0 & -C_m(z_1,z_2) & C_r(z_1,z_2) \end{bmatrix},$$

clearly $\mathcal{K}_{tot} = \ker H_{tot}(\sigma_1,\sigma_2)$.

The estimated variables $\hat{\mathbf{w}}_r$ represent additional variables, that need to be eliminated in order to reduce ourselves to the partial interconnection problem considered in the previous sections. Indeed, we are interested in the behavior $\mathcal{K} := \mathcal{P}_{(\mathbf{w}_r,\mathbf{w}_m)}\mathcal{K}_{tot}$, which can be described as follows. If $[\,M_Q(z_1,z_2) \quad M_{C_r}(z_1,z_2)\,]$ is an MLA of $\begin{bmatrix} Q(z_1,z_2) \\ C_r(z_1,z_2) \end{bmatrix}$, then

$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & M_Q(z_1,z_2) & M_{C_r}(z_1,z_2) \end{bmatrix}$$

is an MLA of $\begin{bmatrix} 0_{p\times\mathtt{w}_r} \\ Q(z_1,z_2) \\ C_r(z_1,z_2) \end{bmatrix}$. This implies that

$$\mathcal{K} = \ker \begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -(M_Q(\sigma_1,\sigma_2)P(\sigma_1,\sigma_2) + M_{C_r}(\sigma_1,\sigma_2)C_m(\sigma_1,\sigma_2)) \end{bmatrix}. \tag{7.5}$$

So, in order to investigate the existence and admissibility of observer-based DBCs, we need to address the existence and admissibility of partial interconnection DBCs taking the special form

$$(7.6)\ [M_Q(\sigma_1,\sigma_2)P(\sigma_1,\sigma_2)+M_{C_r}(\sigma_1,\sigma_2)C_m(\sigma_1,\sigma_2)]\mathbf{w}_m(h,k)=0,\ (h,k)\in\mathbb{Z}_+\times\mathbb{Z},$$

where the TS-polynomial matrices appearing in (7.6) have the interpretation previously given. It is worth noticing that having reduced the observer-based DBC problem to the problem of designing DBCs, by partial interconnection, taking the form

$$\mathcal{C}=\ker\left[\,0\quad -(M_Q(\sigma_1,\sigma_2)P(\sigma_1,\sigma_2)+M_{C_r}(\sigma_1,\sigma_2)C_m(\sigma_1,\sigma_2))\,\right],$$

by the results derived in the previous sections, it immediately follows that the admissibility of an observer-based DBC will always be related to the zero-time-controllability property (either of the whole behavior $\mathfrak{B}$ or of the variables $\mathbf{w}_r$, alone).

Now we are in a position to show that the same conditions that allow to solve the two DBC problems (by partial interconnection) allow to solve the analogous observer-based DBC problems.

Proposition 7.1. *Given a 2D behavior $\mathfrak{B}$, described as in (7.1), the following conditions are equivalent:*

i) there exists an observer-based [admissible] *DBC for the behavior $\mathfrak{B}$, described as in (7.2)-(7.3);*

ii) $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$ [and $\mathfrak{B}$ is zero-time-controllable].

Proof. i) $\Rightarrow$ ii) Suppose that there exists an observer-based [admissible] DBC for $\mathfrak{B}$. Since $\mathcal{K}=\mathcal{P}_{(\mathbf{w}_r,\mathbf{w}_m)}\mathcal{K}_{tot}$ is nilpotent, the matrix

$$\begin{bmatrix} R_r(z_1,z_2) & -R_m(z_1,z_2) \\ 0 & -(M_Q(z_1,z_2)P(z_1,z_2)+M_{C_r}(z_1,z_2)C_m(z_1,z_2)) \end{bmatrix}$$

is right monomic, and this ensures that $R_r(z_1,z_2)$ is right monomic, too. Finally, as previously remarked, if the observer-based DBC is admissible, this means that the controller

$$\mathcal{C}=\ker\left[\,0\quad -(M_Q(\sigma_1,\sigma_2)P(\sigma_1,\sigma_2)+M_{C_r}(\sigma_1,\sigma_2)C_m(\sigma_1,\sigma_2))\,\right]$$

is an admissible DBC for the behavior $\mathfrak{B}$, and hence $\mathfrak{B}$ is zero-time-controllable.

ii) $\Rightarrow$ i) If $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$, $R_r(z_1,z_2)$ is right monomic, and by choosing $(Q(z_1,z_2),P(z_1,z_2))=(R_r(z_1,z_2),R_m(z_1,z_2))$ we obtain a (consistent) DBO. We aim at showing that by choosing $C_r(z_1,z_2)=0$ and $C_m(z_1,z_2)=I_{\mathtt{w}_m}$ we obtain an observer-based [admissible] DBC for the behavior $\mathfrak{B}$. It is easy to verify that in this case

$$M_Q(z_1,z_2)=\begin{bmatrix} M_r(z_1,z_2) \\ 0 \end{bmatrix} \quad \text{and} \quad M_{C_r}(z_1,z_2)=\begin{bmatrix} 0 \\ I_{\mathtt{w}_m} \end{bmatrix},$$

and this leads to rewrite (7.6) in the following form:

$$\begin{bmatrix} M_r(\sigma_1,\sigma_2)P(\sigma_1,\sigma_2) \\ I_{\mathtt{w}_m} \end{bmatrix} \mathbf{w}_m(h,k) = 0, \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}.$$

So, being both $R_r(z_1,z_2)$ and $\begin{bmatrix} M_r(z_1,z_2)P(z_1,z_2) \\ I \end{bmatrix}$ right monomic matrices, it easily follows that

$$\mathcal{K} = \ker \begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -M_r(\sigma_1,\sigma_2)P(\sigma_1,\sigma_2) \\ 0 & -I \end{bmatrix},$$

is a nilpotent behavior. Finally, under the additional assumption that $\mathfrak{B}$ is zero-time-controllable, the observer-based DBC is necessarily an admissible one. □

Remark 7.2. *It is easy to see that, under assumption ii) of the previous proposition, for every choice of a DBO and a DBC we obtain an observer-based [admissible] DBC. In fact, from the description of $\mathcal{K}_{tot}$ it follows*

$$\begin{aligned} &R_r(\sigma_1,\sigma_2)\mathbf{w}_r(h,k) = R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \\ &C_r(\sigma_1,\sigma_2)\hat{\mathbf{w}}_r(h,k) = C_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \end{aligned}$$

that can be rewritten as

$$\begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ C_r(\sigma_1,\sigma_2) & -C_m(\sigma_1,\sigma_2) \end{bmatrix} \begin{bmatrix} \mathbf{w}_r(h,k) \\ \mathbf{w}_m(h,k) \end{bmatrix} = \begin{bmatrix} 0 \\ C_r(\sigma_1,\sigma_2) \end{bmatrix} (\mathbf{w}_r(h,k) - \hat{\mathbf{w}}_r(h,k)). \tag{7.7}$$

Since the first matrix appearing in the previous equation is right monomic, a TS-polynomial matrix $L(z_1,z_2)$ exists such that

$$\begin{aligned} \sigma_1^\delta \begin{bmatrix} \mathbf{w}_r(h,k) \\ \mathbf{w}_m(h,k) \end{bmatrix} &= L(\sigma_1,\sigma_2) \begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ C_r(\sigma_1,\sigma_2) & -C_m(\sigma_1,\sigma_2) \end{bmatrix} \begin{bmatrix} \mathbf{w}_r(h,k) \\ \mathbf{w}_m(h,k) \end{bmatrix} \\ &= L(\sigma_1,\sigma_2) \begin{bmatrix} 0 \\ C_r(\sigma_1,\sigma_2) \end{bmatrix} (\mathbf{w}_r(h,k) - \hat{\mathbf{w}}_r(h,k)) \end{aligned}$$

for some $\delta \in \mathbb{Z}_+$. As the signal $\mathbf{w}_r(h,k) - \hat{\mathbf{w}}_r(h,k)$ is zero on some half-plane $\mathcal{S}_{M\rightarrow}$, the right hand side of the previous identity is a trajectory with support included in a vertical strip. So, both $\mathbf{w}_r$ and $\mathbf{w}_m$ have the same property, thus showing that $\mathcal{K}$ is a nilpotent behavior. This ensures that (7.6) defines a DBC for $\mathfrak{B}$. Moreover, in case of zero-time-controllability, this DBC is necessarily an admissible one.

We now address observer-based DBCs targeting only the relevant variables.

Proposition 7.3. *Given a 2D behavior $\mathfrak{B}$, described as in (7.1), the following conditions are equivalent:*

i) there exists an observer-based [admissible] *DBC for the variables $\mathbf{w}_r$, described as in (7.2)-(7.3);*

ii) $\mathbf{w}_r$ *is reconstructible from* $\mathbf{w}_m$ [and $\mathbf{w}_r$ is zero-time-controllable].

Proof. i) $\Rightarrow$ ii) Suppose that there exists an observer-based [admissible] DBC for $\mathbf{w}_r$. Clearly, $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_{tot} = \mathcal{P}_{\mathbf{w}_r}\mathcal{K}$, and, by assumption, $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}$ is nilpotent. From (7.5), it follows that

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K} = \ker(A_1(\sigma_1, \sigma_2)R_r(\sigma_1, \sigma_2))$$

where $[\, A_1(z_1, z_2) \quad A_2(z_1, z_2)\,]$ is an MLA of

$$\begin{bmatrix} R_m(z_1, z_2) \\ M_Q(\sigma_1, \sigma_2)P(z_1, z_2) + M_{C_r}(z_1, z_2)C_m(z_1, z_2) \end{bmatrix}.$$

As $A_1(z_1, z_2)R_r(z_1, z_2)$ is right monomic, then $R_r(z_1, z_2)$ is monomic, too. Finally, if the observer-based DBC for $\mathbf{w}_r$ is admissible, this means that the controller

$$\mathcal{C} = \ker [\, 0 \quad -(M_Q(\sigma_1, \sigma_2)P(\sigma_1, \sigma_2) + M_{C_r}(\sigma_1, \sigma_2)C_m(\sigma_1, \sigma_2))\,]$$

is an admissible DBC for $\mathbf{w}_r$, and hence $\mathbf{w}_r$ is zero-time-controllable.

ii) $\Rightarrow$ i) By choosing $(Q(z_1, z_2), P(z_1, z_2)) = (R_r(z_1, z_2), R_m(z_1, z_2))$, $C_r(z_1, z_2) = 0$ and $C_m(z_1, z_2) = I_{\mathtt{w}_m}$ as in the previous Proposition 7.1, we get an observer-based DBC for $\mathfrak{B}$, and hence *a fortiori* for $\mathbf{w}_r$. Under the additional zero-time-controllability assumption on $\mathbf{w}_r$,

$$\mathcal{C} = \ker [\, 0 \quad -(M_Q(\sigma_1, \sigma_2)P(\sigma_1, \sigma_2) + M_{C_r}(\sigma_1, \sigma_2)C_m(\sigma_1, \sigma_2))\,]$$

is an admissible DBC for $\mathbf{w}_r$, and hence the observer-based DBC is admissible. □

Remark 7.4. *A remark analogous to Remark 7.2 holds true also for observer-based DBCs targeting only* $\mathbf{w}_r$. *Starting from equation (7.7), and by premultiplying it by an MLA of* $\begin{bmatrix} R_m(z_1, z_2) \\ C_m(z_1, z_2) \end{bmatrix}$, *we obtain the following equation*

$$A(\sigma_1, \sigma_2)\mathbf{w}_r(h, k) = B(\sigma_1, \sigma_2)(\mathbf{w}_r(h, k) - \hat{\mathbf{w}}_r(h, k)), \qquad (h, k) \in \mathbb{Z}_+ \times \mathbb{Z},$$

for suitable TS-polynomial matrices $A(z_1, z_2)$ *and* $B(z_1, z_2)$, *with* $A(z_1, z_2)$ *right monomic, since we are now dealing with a DBC for* $\mathbf{w}_r$ *alone. By the same reasoning adopted in the previous remark, it follows that* $\mathbf{w}_r$ *belongs to a nilpotent behavior, thus showing that (7.6) defines a DBC for* $\mathbf{w}_r$. *Moreover, in case of zero-time-controllability of* $\mathbf{w}_r$, *this DBC is necessarily an admissible one.*

8. Conclusions

In this paper we have addressed the dead-beat control problem, by partial interconnection, of 2D behaviors, defined on the grid $\mathbb{Z}_+ \times \mathbb{Z}$ and having the time as independent variable. The existence of such a DBC, driving to zero (in a finite number of time instants) either all or part of the system variables, is equivalent to the reconstructibility of the variables that are not accessible for control. On the other hand, if we constrain our search to "admissible" DBCs, we have to introduce the additional assumption that either $\mathfrak{B}$ or the relevant variables are zero-time-controllable. Such characterizations prove to be also the necessary and sufficient conditions for solving the same problem by resorting to an observer-based (admissible) dead-beat controller.

REFERENCES

[1] M. Bisiacco and M.E. Valcher. Dead-beat estimation problems for 2D behaviors. *Mult. Sys. Sign. Process.*, 19:287–306, 2008.

[2] M. Bisiacco and M.E. Valcher. Dead-beat control in the behavioral approach. *IEEE Trans. Aut. Contr.*, 57, no.9:2163–2175, 2012.

[3] M. Bisiacco and M.E. Valcher. Dead-beat control of two-dimensional behaviors. *Proceedings of the 51st IEEE Conf. on Decision and Control*, pp. 3195-3202, Maui, Hawaii, 2012.

[4] M. Bisiacco and M.E. Valcher. Zero-time-controllability and dead-beat control of two-dimensional behaviors. *SIAM J. on Control and Optimization*, 51, no. 1: 195-220, 2013.

[5] M. Bisiacco, M.E. Valcher, and J.C. Willems. A behavioral approach to estimation and dead-beat observer design with applications to state-space models. *IEEE Trans. Aut. Contr.*, 51, no.11:1787–1797, 2006.

[6] E. Fornasini, P. Rocha, and S. Zampieri. State space realization of 2-D finite-dimensional behaviours. *SIAM J. Contr. Optimiz.*, 31:1502–1517, 1993.

[7] E. Fornasini and M.E. Valcher. nD polynomial matrices with applications to multidimensional signal analysis. *Multidim. Sys. Sign. Proc.*, 8:387–407, 1997.

[8] D. Napp, P. Rapisarda, and P. Rocha. Time-relevant stability of 2D systems. *Automatica*, 47, no. 11:2373–2382, 2011.

[9] L. Ntogramatzidis and M. Cantoni. Detectability subspaces and observer synthesis for two-dimensional systems. *Multidimensional Systems and Signal Processing*, 23: 79-96, 2012.

[10] U. Oberst. Multidimensional constant linear systems. *Acta Applicandae Mathematicae*, 20:1–175, 1990.

[11] U. Oberst and M. Scheicher. The asymptotic stability of stable and time- autonomous discrete multidimensional behaviors. *Multidimensional Systems and Signal Processing*, submitted, 2012.

[12] U. Oberst and M. Scheicher. Time-autonomy and time-controllability of discrete multidimensional behaviors. *Int. J. of Control*, 85, no.8:990–1009, 2012.

[13] C. Praagman, H.L. Trentelman, and R. Zavala Yoe. On the parametrization of all regularly implementing and stabilizing controllers. *SIAM J. Contr. Optim.*, 45, no.6:2035–2053, 2007.

[14] P. Rapisarda and P. Rocha. Lyapunov functions for time-relevant 2D systems, with application to first-orthant stable systems. *Automatica*, 48, no. 9:1998–2006, 2012.

[15] P. Rocha. *Structure and Representation of 2-D Systems.* PhD thesis, University of Groningen, The Netherlands, 1990.

[16] P. Rocha and J. Wood. Trajectory control and interconnections of 1D and nD systems. *SIAM J. Control Optim.*, 40, no.1:107–134, 2001.

[17] A.J. Sasane. Time-autonomy and time-controllability of 2-D behaviours that are tempered in the spatial direction. *Multidimensional Systems and Signal Processing*, 15:97–116, 2004.

[18] A.J. Sasane and T. Cotroneo. Conditions for time-controllability of behaviours. *International Journal of Control*, 75:61–67, 2002.

[19] A.J. Sasane, E.G.F. Thomas, and J.C. Willems. Time-autonomy versus time-controllability. *Systems and Control Letters*, 45:145–153, 2002.

[20] D.C. Youla and G. Gnavi. Notes on n-dimensional system theory. *IEEE Trans. Circ. Sys.*, CAS-26:105–111, 1979.

[21] E. Zerz. Primeness of multivariate polynomial matrices. *Systems & Control Letters*, 29, no. 3:139–145, 1996.

[22] E. Zerz and L. Wagner. Finite multidimensional behaviors. *Multidimensional Systems and Signal Processing*, 24: 5-15, 2012.