Dead-beat control in the behavioral approach

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Abstract

In this paper the concepts of controllability and zero-controllability of a variable $\mathbf{w}$, appearing either in a standard or in a latent variable description (as manifest variable), are introduced and characterized. By assuming this perspective, the dead-beat control (DBC) problem is posed as the problem of designing a controller, involving both $\mathbf{w}$ and the latent variable $\mathbf{c}$, such that, for the resulting controlled behavior, the variable $\mathbf{w}$ goes to zero in a finite number of steps in every trajectory. Zero-controllability of $\mathbf{w}$ turns out to be a necessary and sufficient condition for the existence of "admissible" DBC's as well as for the existence of regular DBC's. The class of minimal DBC's, namely DBC's with the least possible number of rows, is singled-out and a parametrization of such controllers is provided. Finally, a necessary and sufficient condition for the existence of DBC's that can be implemented via a feedback law, for which $\mathbf{w}$ is the input and the latent variable $\mathbf{c}$ the corresponding output, is provided.

Index Terms

Behavior, nilpotent (autonomous) behavior, controllability, zero-controllability, dead-beat controller.

I. Introduction

As clarified in some recent papers dealing with control in the behavioral setting [13], [20], the traditional perspective to control problems has deeply intertwined the idea of control itself with the concepts of input, output and feedback. This perspective, however, fails to provide the appropriate framework where to cast the controller design problem in a number of interesting cases, as, for instance, the design of passive car suspensions and of insulation equipments for noise abatement, or ships stabilization [18], [19], [20]. This is mainly due to the fact that the feedback control paradigm is based on the assumption that measurements of the output variables

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can be obtained, based on which the control input can be generated. As clearly illustrated in [20] for the special case of car suspensions, however, this assumption does not always prove to be a realistic one, and the control problem thus needs to be posed in rather different terms.

As first suggested in [17] and later explored in [18], [19], the behavioral approach provides a natural framework where control problems can be addressed in the utmost generality and without any a priori assumption regarding input/output partition and feedback connection. Indeed, in this setting, the control target is that of restricting the behavior trajectories to a subset of "good ones" and this goal is achieved by interconnection, namely by constraining either all or a subset of the system variables to obey an additional family of laws, which represent the controller laws.

Stimulated by these first contributions, there has been quite a number of papers on the control within the behavioral framework (for instance, [1], [2], [4], [6], [13]). In these papers two fundamental control set-ups have been explored: the *full interconnection* case and the *partial interconnection* case. In the former, it is assumed that all variables are the target of the control problem (for instance, the stabilization problem) and at the same time they are all available for interconnection. In the latter, the system variables are partitioned in two (or possibly more) groups, by distinguishing between to be controlled variables, typically denoted by $\mathbf{w}$, which are the object of the control specifications, and control variables, denoted by $\mathbf{c}$, which are the means through which the control target is achieved. Indeed, the controller achieves the desired result by restricting the behavior of the variables in $\mathbf{c}$ which are the only ones available for interconnection.

In this flow of research, two contributions [7], [11] address the control problem under different perspectives: in [7] the control is achieved through the control variables alone, but both $\mathbf{w}$ and $\mathbf{c}$ are target variables, in [11] the situation is just the symmetric one, since all variables are available for control but the control target is just $\mathbf{w}$. Our approach has several similarities with the one explored in [11], as we will describe in detail later.

In this paper we explore the dead-beat control (DBC) problem for discrete-time behaviors defined on $\mathbb{Z}_+$. Surprisingly enough and to the best of our knowledge, this problem has not been previously addressed, probably due to the fact that most of the contributions in the field have focused either on continuous-time behaviors or on discrete-time behaviors defined on $\mathbb{Z}$ (and hence characterized by potentially bi-infinite trajectories), for which the DBC problem has no meaning. In investigating this problem, we take a perspective that differs both from the full interconnection and from the partial interconnection set-ups, in that we assume that only the to be controlled variable $\mathbf{w}$ must be "driven to zero in a finite number of steps", but the control laws restrict the evolutions of both $\mathbf{w}$ and $\mathbf{c}$. So, there is full interconnection for control purposes, but the control target is just the variable $\mathbf{w}$.

This perspective is very close to the one assumed in [11], where the concept of *extended interconnection* was first introduced. However, while in this paper we assume that the latent variable model of the system behavior is a priori given, and hence the variable $\mathbf{c}$ and its behavior are problem data, in [11] the Authors assume as a priori information only the behavior $\mathfrak{B}$ of $\mathbf{w}$, and search for both a latent variable description of $\mathfrak{B}$ and an extended controller through which the control target (referring to $\mathbf{w}$ alone) may be achieved. In this sense, while the variable $\mathbf{c}$ in [11] plays just an instrumental role, in the present set-up it represents physical variables that can be used for control, but on which we impose no requirements. This viewpoint seems to us closer to the classical idea underlying the DBC design for state-space models, since the target is that of driving to zero (only) the state variable, but the control law involves both the state and the input.

Clearly, the partial interconnection problem represents a special case of the control problem here addressed. On the other hand, also our set-up could be restated as a partial interconnection problem, provided that the system description is fictitiously expanded by replacing the latent variable $\mathbf{c}$ with $\tilde{\mathbf{c}} = (\mathbf{c}, \mathbf{w})$. However, this approach exhibits some inconvenient, due to the increased computational complexity and the loss of a clear-cut distinction between the variable roles. On the other hand, it must be said that the DBC problem has not been previously addressed, not even in the partial interconnection case, so reducing our problem set-up to that one would not bring any benefit. We will further comment on this issue later in the paper, when dealing with the parametrization problem.

The investigation of the DBC problem under this perspective requires to introduce new concepts of controllability and of zero-controllability of the variable $\mathbf{w}$. Consistently, it will turn out that the possibility of designing DBC's with good properties (admissibility or regularity) that drive to zero in a finite number of steps all the trajectories $\mathbf{w}$, by constraining both $\mathbf{w}$ and $\mathbf{c}$, is just equivalent to the zero-controllability of $\mathbf{w}$. Special attention will be devoted to the class of minimal DBC's, by this meaning DBC's with the least number of rows, for which a parametrization will be provided. Such parametrization is quite complex, as it resorts both to polynomial and to rational parameters, under the constraint that the obtained result is polynomial. As clarified by an example, if we try to use only polynomial parameters, in general we cannot obtain a unique parametrization but families of distinct parametrizations. We will compare the parametrization here obtained with the one derived in [9] for the class of stabilizing controllers, obtained through partial interconnection.

The paper is organized as follows: at the end of this section and in section 2 background material about polynomial matrices and behaviors defined on $\mathbb{Z}_+$ is recalled. The interested reader is referred to [3], [5], [14], [12], [16] for further details. In section 3, the properties of controllability and zero-controllability of a variable $\mathbf{w}$, that represents either the entire system variable or the manifest variable in a latent variable description, are introduced, respectively, and characterized. Dead-beat controllers are the focus of section 4, where some preliminary results are given. In section 5 admissible DBC's and regular DBC's are presented and it is shown that such controllers are available if and only if $\mathbf{w}$ is zero-controllable. Minimal DBC's in turn are available only under the zero-controllability assumption, and a parametrization of such DBC's is given in section 6. Finally, in section 7, necessary and sufficient conditions for the existence of DBC's that can be obtained through a feedback connection of the controller to the original system are provided.

Notation. We consider here polynomial matrices with entries in $\mathbb{R}[z]$ and, occasionally, Laurent (L-polynomial, for short) polynomial matrices, having entries in $\mathbb{R}[z, z^{-1}]$. A polynomial matrix $H(z) \in \mathbb{R}[z]^{p\times q}$ is *right monomic* [5] if rank $H(\lambda) = q$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. This means that $H(z)$ is of full column rank and the GCD of its maximal (i.e., qth) order minors is a monomial $cz^h, c \in \mathbb{R}\setminus\{0\}, h \in \mathbb{Z}_+$. $H(z)$ is right monomic if and only if it admits a Laurent polynomial left inverse or, equivalently, the diophantine equation $L(z)H(z) = z^N I_q$, in the unknown polynomial matrix $L(z)$, is solvable for some nonnegative integer N.

$H(z) \in \mathbb{R}[z]^{p\times q}$ is *right prime* if rank $H(\lambda) = q$ for every $\lambda \in \mathbb{C}$. *Right prime* matrices are special cases of right monomic matrices. Actually, right primeness characterizations can be obtained by simply replacing in the previous equivalent conditions the word "monomial" with "unit" and the integer N by zero. Every full column rank matrix $H(z)$ can be expressed as

$H(z) = \bar{H}(z)\Delta(z)$, where $\bar{H}(z)$ is right prime and $\Delta(z)$ is nonsingular square. When so, $\Delta(z)$ is called a *greatest right divisor* of $H(z)$. *Left monomic* matrices, *left prime* matrices and *greatest left divisors* are similarly defined and characterized. Every polynomial matrix $H(z)$ factorizes over $\mathbb{R}[z]$ as $H(z) = L(z)R(z)$, where $L(z)$ is of full column rank and $R(z)$ is left prime. (* verificare che cio' sia consistente con quanto serve nelle proofs *)

The concepts of *left annihilator* and, in particular, of *minimal left annihilator* (MLA, for short) of a given polynomial matrix $H(z)$ have been originally introduced in [10] and can be summarized as follows: if $H(z)$ is a $p \times q$ polynomial matrix of rank r, a polynomial matrix $M(z)$ is a left annihilator of $H(z)$ if $M(z)H(z) = 0$. A left annihilator $\tilde{M}(z)$ of $H(z)$ is an MLA if it is of full row rank and for any other left annihilator $M(z)$ of $H(z)$ we have $M(z) = P(z)\tilde{M}(z)$ for some polynomial matrix $P(z)$. It can be easily proved that, when $r < p$, an MLA always exists, it is a $(p-r) \times p$ left prime matrix and is uniquely determined modulo a unimodular left factor. If the given $H(z)$ has full row rank, then for consistency we define [9] its MLA as the "void" matrix with 0 rows and p columns. In that case, if $K(z)$ is a given matrix with p rows, then $M(z)K(z)$ is again void.

In the following, for the sake of simplicity, the size of any vector will be denoted by means of the same typewritten letter that is used for denoting the vector itself. Accordingly, the vector $\mathbf{w}$ will have $\mathtt{w}$ entries, $\mathbf{c}$ will have $\mathtt{c}$ entries, etc.

II. Basic results about behaviors with trajectories in $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$

In this paper, all trajectories will be assumed defined on the time set $\mathbb{Z}_+$ of nonnegative integers. The left (backward) shift operator on $(\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+}$, the set of trajectories defined on $\mathbb{Z}_+$ and taking values in $\mathbb{R}^{\mathtt{v}}$, is defined as

$$\sigma : (\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+} \to (\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+} : (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \cdots) \mapsto (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots).$$

If $R(z) = \sum_{i=0}^{L} R_i z^i \in \mathbb{R}[z]^{p \times q}$ is a polynomial matrix, we associate with it the polynomial matrix operator $R(\sigma) = \sum_{i=0}^{L} R_i \sigma^i$. Results about polynomial matrix operators acting on $(\mathbb{R}^q)^{\mathbb{Z}_+}$ can be found in [16], where these results have been derived with (and compared to) those about the more common set-up of polynomial matrix operators acting on $(\mathbb{R}^q)^{\mathbb{Z}}$. Further comparisons between these two settings have been later carried on in [12] and in [14], where the few differences between the two settings have been pointed out. In this section, we only recall a few basic results. The interested reader can refer to [3] for more details. It can be proved that $R(\sigma)$ describes an injective map from $(\mathbb{R}^q)^{\mathbb{Z}_+}$ to $(\mathbb{R}^p)^{\mathbb{Z}_+}$ if and only if $R(z)$ is a right prime matrix, and a surjective map if and only if $R(z)$ is of full row rank.

In this paper, by a *behavior* $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ we mean the *linear* and *left shift invariant* set of solutions $\mathbf{w} = \{\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$ of a system of difference equations

$$R_0\mathbf{w}(t) + R_1\mathbf{w}(t+1) + \cdots + R_L\mathbf{w}(t+L) = 0,\ t \in \mathbb{Z}_+, \tag{1}$$

with $R_i \in \mathbb{R}^{p\times\mathtt{w}}$. This system is equivalently described as

$$R(\sigma)\mathbf{w} = 0, \tag{2}$$

where $R(z) := \sum_{i=0}^{L} R_i z^i$ belongs to $\mathbb{R}[z]^{p\times\mathtt{w}}$, and this leads to the short-hand notation $\mathfrak{B} = \ker\, R(\sigma)$. It has been shown in [16] that every behavior $\mathfrak{B}$ can be described as the kernel of a full row rank polynomial matrix, and that $\mathfrak{B}_1 := \ker\, R_1(\sigma) \subseteq \ker\, R_2(\sigma) =: \mathfrak{B}_2$ if and only if $R_2(z) = P(z)R_1(z)$ for some polynomial matrix $P(z)$. A behavior $\mathfrak{B}_1$ included in $\mathfrak{B}_2$ is called a *sub-behavior* of $\mathfrak{B}_2$.

A behavior $\mathfrak{B}$ is said to be *autonomous* if there exists $\delta \in \mathbb{N}$ such that if $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{B}$ and $\mathbf{w}_1|_{[0,\delta-1]} = \mathbf{w}_2|_{[0,\delta-1]}$, by this meaning that $\mathbf{w}_1(t) = \mathbf{w}_2(t)$ for $t \in [0, \delta-1]$, then $\mathbf{w}_1 = \mathbf{w}_2$. $\mathfrak{B} = \ker\, R(\sigma) \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, with $R(z) \in \mathbb{R}[z]^{p\times\mathtt{w}}$, is autonomous if and only if $R(z)$ is of full column rank $\mathtt{w}$ [15], [16]. From the previous comment about full row rank kernel representations, we deduce that every autonomous behavior can be expressed as the kernel of a nonsingular square polynomial matrix. In general, an autonomous behavior $\mathfrak{B} = \ker\, R(\sigma)$ includes finite support trajectories if and only if $\mathrm{rank}\ R(0) < \mathtt{w}$. Autonomous behaviors for which there exists $\delta \in \mathbb{N}$ such that all their trajectories have (finite) supports included in $[0, \delta - 1]$ are called *nilpotent (autonomous)* and they are kernels of polynomial matrix operators $R(\sigma)$ corresponding to right monomic matrices [15]. In particular, if $R(z)$ is nonsingular square, $\ker\, R(\sigma)$ is nilpotent if and only if $\det R(z) = c \cdot z^{\delta}$, for some $c \in \mathbb{R} \setminus \{0\}$ and some $\delta \in \mathbb{Z}_+$. If an autonomous behavior is not nilpotent, it includes at least one infinite support trajectory. It is worthwhile to remark that when dealing with behaviors defined on $\mathbb{Z}$, nilpotency cannot arise [15]. In fact, the only finite support trajectory of an autonomous behavior defined on $\mathbb{Z}$ is the zero one, and the kernel (on $\mathbb{Z}$) of a monomic matrix coincides with the zero behavior.

III. Controllability and zero-controllability properties

We here recall the definition of controllability of a behavior and introduce that of zero-controllability. The former has been investigated quite in detail. In particular, in [21] several definitions of controllability have been introduced and compared, thus showing that some of them are independent of the choice of the independent variable set (namely they hold both for $\mathbb{Z}_+$ and for $\mathbb{Z}$, and they extend to all the possible multidimensional cases), while others are not. The definition of zero-controllability here introduced is original.

Definition 1: [21] A behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ is said to be *controllable* if there exists some nonnegative integer L such that for every $N \in \mathbb{N}$ and every pair of trajectories $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{B}$, there exists $\bar{\mathbf{w}} \in \mathfrak{B}$ such that $\bar{\mathbf{w}}|_{[0,N-1]} = \mathbf{w}_1|_{[0,N-1]}$ and $\bar{\mathbf{w}}|_{[N+L,+\infty)} = \mathbf{w}_2|_{[0,+\infty)}$.

Controllable behaviors are endowed with very strong properties [21]. In particular, every controllable behavior can be described as $\mathfrak{B} = \ker\, \bar{R}(\sigma)$, for some left prime matrix $\bar{R}(z) \in \mathbb{R}[z]^{p\times \mathtt{w}}$. Given any behavior $\mathfrak{B}$, we define its *controllable part*, denoted by $\mathfrak{B}_c$, as the largest controllable sub-behavior of $\mathfrak{B}$. We have the following fundamental decomposition theorem.

Theorem 1: Given $\mathfrak{B} = \ker\, R(\sigma) \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, assume that $R(z)$ factorizes as $R(z) = L(z)\bar{R}(z)$, with $L(z) \in \mathbb{R}[z]^{p\times r}$ of full column rank and $\bar{R}(z) \in \mathbb{R}[z]^{r\times \mathtt{w}}$ left prime. Then

i) there exist some controllable behavior $\mathfrak{B}'_c$ and some autonomous behavior $\mathfrak{B}_a$ such that $\mathfrak{B} = \mathfrak{B}'_c \oplus \mathfrak{B}_a$. $\mathfrak{B}'_c$ is uniquely determined as $\mathfrak{B}_c$, the controllable part of $\mathfrak{B}$, and it can be described as $\ker \bar{R}(\sigma)$, while $\mathfrak{B}_a$ can be chosen with a certain degree of freedom.

ii) For every (full column rank) matrix $H_a \in \mathbb{R}[z]^{p_a\times \mathtt{w}}$, such that $\mathfrak{B}_a = \ker H_a(\sigma)$ appears in any such decomposition, we have that

$$\text{g.c.d.}\{\text{maximal order minors of } H_a(z)\} = \text{g.c.d.}\{\text{maximal order minors of } L(z)\}.$$

Proof: The first part of the theorem has been proved in [14]. The second part has been proved in the continuous-time case in [8]. ■

We now move to the definition and characterization of zero-controllability of a behavior.

Definition 2: A behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ is said to be *zero-controllable* if there exists some nonnegative integer L such that for every $N \in \mathbb{N}$ and every trajectory $\mathbf{w} \in \mathfrak{B}$, there exists $\bar{\mathbf{w}} \in \mathfrak{B}$ such that $\bar{\mathbf{w}}|_{[0,N-1]} = \mathbf{w}|_{[0,N-1]}$ and $\bar{\mathbf{w}}|_{[N+L,+\infty)} = 0$.

Zero-controllability can be related to the concept of set-controllability introduced in [11] (see Definition 4.1), and here recalled in the special case of behaviors defined on $\mathbb{Z}_+$.

Definition 3: Given a behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ and a sub-behavior $\mathfrak{B}' \subseteq \mathfrak{B}$, the behavior $\mathfrak{B}$ is *set-controllable to* $\mathfrak{B}'$ if there exists some nonnegative integer ρ such that for every trajectory $\mathbf{w} \in \mathfrak{B}$, there exists $\mathbf{w}' \in \mathfrak{B}'$ so that for every choice of two disjoints sets $T_1, T_2 \subset \mathbb{Z}_+$, with distance $d(T_1, T_2) := \min\{|t_1 - t_2| : t_1 \in T_1, t_2 \in T_2\} > \rho$, and every $b \in \mathbb{Z}_+$, a trajectory $\bar{\mathbf{w}}_b \in \mathfrak{B}$ can be found, satisfying:

$$\bar{\mathbf{w}}_b(t) = \begin{cases} \mathbf{w}(t), & t \in T_1; \\ \mathbf{w}'(t-b), & t \in T_2 \text{ and } t-b \in T_2. \end{cases}$$

At first sight, zero-controllability seems to be equivalent to set-controllability to the zero behavior, however, this is not the case. As shown in the following proposition, zero-controllability corresponds to the weaker property of set-controllability to a nilpotent autonomous behavior.

Proposition 1: Given a behavior $\mathfrak{B} = \ker\,(R(\sigma)) \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, with $R(z) \in \mathbb{R}[z]^{p \times \mathtt{w}}$, the following statements are equivalent ones:

i) $\mathfrak{B}$ is zero-controllable;

ii) $R(z) = L(z)\bar{R}(z)$, for some right monomic $L(z)$ and some left prime matrix $\bar{R}(z)$;

iii) $\mathfrak{B} = \mathfrak{B}_c \oplus \mathfrak{B}_a$, where $\mathfrak{B}_c$ is the controllable part of $\mathfrak{B}$ and $\mathfrak{B}_a$ is a nilpotent behavior;

iv) there exists a nilpotent behavior $\mathfrak{B}' \subseteq \mathfrak{B}$ such that $\mathfrak{B}$ is set-controllable to $\mathfrak{B}'$.

Proof: i) $\Rightarrow$ ii) Assume that $\mathfrak{B}$ is zero-controllable, and express, without loss of generality, $R(z)$ as $R(z) = L(z)\bar{R}(z)$, with $L(z) \in \mathbb{R}[z]^{p \times r}$ of full column rank and $\bar{R}(z) \in \mathbb{R}[z]^{r \times \mathtt{w}}$ left prime. If $L(z)$ were not right monomic, a (possibly complex) $\alpha \neq 0$ and some vector $\mathbf{v} \in \mathbb{C}^r, \mathbf{v} \neq 0$, could be found such that $L(\alpha)\mathbf{v} = 0$. Then the (possibly complex valued) trajectory $\mathbf{z}(t) := \alpha^t \mathbf{v}, \forall\ t \in \mathbb{Z}_+$, satisfies $L(\sigma)\mathbf{z}(t) = 0$ for every $t \in \mathbb{Z}_+$. Express $\mathbf{z}$ as $\mathbf{z} = \mathbf{z}_r + i\mathbf{z}_i$, where $\mathbf{z}_r$ is the real part of $\mathbf{z}$ and $\mathbf{z}_i$ its imaginary part. Of course, both these sequences are real valued, belong to $\ker L(\sigma)$, and at least one of them has infinite support (since the trajectory $\mathbf{z}$ has this property). We call $\mathbf{z}_L \in (\mathbb{R}^r)^{\mathbb{Z}_+}$ such an infinite support trajectory. Since $\bar{R}(z)$ is left prime and hence it is of full row rank, it defines a surjective map on $\mathbb{Z}_+$. This implies that there exists $\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ such that $\mathbf{z}_L = \bar{R}(\sigma)\mathbf{w}$, and clearly $\mathbf{w} \in \mathfrak{B}$. Suppose now that we want to drive to zero the sequence $\mathbf{w}$ starting from $t = N + L$, meanwhile preserving it on the first N time instants, namely we want to find $\bar{\mathbf{w}} \in \mathfrak{B}$ such that $\bar{\mathbf{w}}|_{[0,N-1]} = \mathbf{w}|_{[0,N-1]}$ and $\bar{\mathbf{w}}|_{[N+L,+\infty)} = 0$. If we choose N sufficiently large, we constrain a large enough portion of $\bar{\mathbf{w}}$ to coincide with $\mathbf{w}$ and therefore we constrain, in turn, a large initial portion of the corresponding image $\bar{R}(\sigma)\bar{\mathbf{w}}$ to coincide with $\mathbf{z}_L$. But since $\bar{R}(\sigma)\bar{\mathbf{w}}$ must belong to the autonomous behavior $\ker L(\sigma)$, by constraining its initial part we essentially impose that the whole trajectory $\bar{R}(\sigma)\bar{\mathbf{w}}$ coincides with $\mathbf{z}_L$. Since $\mathbf{z}_L$ has infinite support, so does any $\bar{\mathbf{w}}$ such that $\bar{R}(\sigma)\bar{\mathbf{w}} = \mathbf{z}_L$. This implies that $\mathbf{w}$ cannot be replaced by a finite support sequence $\bar{\mathbf{w}}$ and hence zero-controllability does not hold.

ii) $\Leftrightarrow$ iii) Follows from Theorem 1, since $L(z)$ is right monomic if and only if every matrix $H_a(z)$ such that $\mathfrak{B} = \mathfrak{B}_c \oplus \ker H_a(\sigma)$ is right monomic.

iii) $\Rightarrow$ iv) Follows from Theorem 4.2 in [11].

iv) $\Rightarrow$ i) Obvious. ■

Remark 1: As far as the previous proposition is concerned, it is worthwhile noticing what the characterizations ii) and iii) become in two special cases: when $R(z)$ is either of full row rank, or of full column rank. In the former case, the matrix $L(z)$ appearing in part ii) is square monomic (possibly unimodular), and when $L(z)$ is unimodular $\mathfrak{B} = \mathfrak{B}_c$. On the other hand, when $R(z)$ is of full column rank, $\bar{R}(z)$ is square unimodular, and hence $R(z)$ is right monomic. Accordingly, the controllable part of $\mathfrak{B}$ is the zero behavior, and $\mathfrak{B} = \mathfrak{B}_a$.

To conclude the section we want to relate the concepts of controllability and of zero-controllability. Given a behavior $\mathfrak{B}$ defined on $\mathbb{Z}$, the finite support trajectories of $\mathfrak{B}$ belong to its controllable part, since all the nonzero trajectories of an autonomous behavior have necessarily infinite support. Consequently, controllability and zero-controllability turn out being equivalent properties. This is not the case, however, when working on $\mathbb{Z}_+$, and the two properties become equivalent only when we introduce an additional feature.

Differently from what happens for behaviors defined on $\mathbb{Z}$, in general behaviors on $\mathbb{Z}_+$ are only left shift-invariant. As a consequence, we can always claim that $\sigma\mathfrak{B} \subseteq \mathfrak{B}$. If also the converse is true, we call the behavior "permanent".

Definition 4: [16], [21] A behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ is *permanent* if $\sigma\mathfrak{B} = \mathfrak{B}$ (namely every trajectory $\mathbf{w} \in \mathfrak{B}$ is the shifted version $\sigma\bar{\mathbf{w}}$ of some other trajectory $\bar{\mathbf{w}} \in \mathfrak{B}$).

Proposition 2: Given a behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, the following statements are equivalent:

i) $\mathfrak{B}$ is permanent;

ii) $R(z) = L(z)\bar{R}(z)$, for some full column rank matrix $L(z)$, with $L(0)$ of full column rank, too, and some left prime matrix $\bar{R}(z)$;

iii) $\mathfrak{B} = \mathfrak{B}_a \oplus \mathfrak{B}_c$, where $\mathfrak{B}_c$ is the controllable part of $\mathfrak{B}$, and $\mathfrak{B}_a$ is an autonomous behavior devoid of finite support trajectories.

Proof: The equivalence i) $\Leftrightarrow$ ii) can be found in [16], [21]. The proof of ii) $\Leftrightarrow$ iii) can be derived, again, from Theorem 1. ■

The following results can be easily derived from the previous ones.

Proposition 3: Consider a behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$.

i) $\mathfrak{B}$ is controllable if and only if it is zero-controllable and permanent [21];

i) $\mathfrak{B}$ is controllable if and only if it is set-controllable to $\{0\}$ [11];

iii) $\mathfrak{B} = \mathfrak{B}_{ap} \oplus \mathfrak{B}_{an} \oplus \mathfrak{B}_c$, where $\mathfrak{B}_c$ is the controllable part of $\mathfrak{B}$, $\mathfrak{B}_{ap}$ is an autonomous behavior devoid of finite support trajectories and $\mathfrak{B}_{an}$ is a nilpotent behavior.

The concepts of controllability and zero-controllability can be extended to the case of behaviors described by means of latent variable models, that often are the ones naturally arising from first principles modeling. Consider the following difference equation:

$$R_w(\sigma)\mathbf{w}(t) = R_c(\sigma)\mathbf{c}(t), \qquad t \in \mathbb{Z}_+, \tag{3}$$

where $R_w(z)$ and $R_c(z)$ are polynomial matrices of sizes $p \times \mathtt{w}$ and $p \times \mathtt{c}$, respectively. We refer to $\mathbf{w}$ as to the manifest variable, and to $\mathbf{c}$ as to the latent variable. According to the standard notation for control problems in the behavioral setting (see, for instance, [1], [2], [4], [6], [11]),

$$\mathfrak{B}_{full} := \{(\mathbf{w}, \mathbf{c}) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+} \times (\mathbb{R}^{\mathtt{c}})^{\mathbb{Z}_+} \text{ satisfying } (3)\}, \tag{4}$$

is the *full system behavior*, while

$$\mathfrak{B} := \{\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+} : \exists\ (\mathbf{w}, \mathbf{c}) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+} \times (\mathbb{R}^{\mathtt{c}})^{\mathbb{Z}_+} \text{ satisfying } (3)\} \tag{5}$$

is the *manifest behavior*. $\mathfrak{B}_{full}$ represents a *latent variable description* of the behavior $\mathfrak{B}$. This latter, in turn, represents the projection of $\mathfrak{B}_{full}$ on the variable $\mathbf{w}$, i.e. $\mathfrak{B} = \mathcal{P}_w\mathfrak{B}_{full}$. It is well known (see [8]), that if $M_c(z)$ represents a minimal left annihilator of $R_c(z)$, then

$$\mathfrak{B} = \mathcal{P}_w \mathfrak{B}_{full} = \ker(M_c(\sigma) R_w(\sigma)).$$

Clearly, if $R_c(z)$ is of full row rank, $M_c(z)$ and hence $M_c(z)R_w(z)$ are void matrices with $\mathtt{w}$ columns, and hence $\mathfrak{B} = (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$.

When dealing with latent variable descriptions, it is quite natural to introduce the concepts of controllability (of zero-controllability) by referring to the variable $\mathbf{w}$ alone, and not to the pair $(\mathbf{w}, \mathbf{c})$.

Definition 5: Given a behavior $\mathfrak{B}_{full}$, the manifest variable $\mathbf{w}$ is

- *controllable* if there exists some nonnegative integer L such that for every $N \in \mathbb{N}$, every $(\mathbf{w}, \mathbf{c}) \in \mathfrak{B}_{full}$, and every $\mathbf{w}^* \in \mathfrak{B}$, one can find $(\bar{\mathbf{w}}, \bar{\mathbf{c}}) \in \mathfrak{B}_{full}$ such that

$$\bar{\mathbf{w}}|_{[0,N-1]} = \mathbf{w}|_{[0,N-1]}, \qquad \text{and} \qquad \bar{\mathbf{w}}|_{[N+L,+\infty)} = \mathbf{w}^*|_{[0,+\infty)};$$

- *zero-controllable* if there exists some nonnegative integer L such that for every $N \in \mathbb{N}$ and every $(\mathbf{w}, \mathbf{c}) \in \mathfrak{B}_{full}$, one can find $(\bar{\mathbf{w}}, \bar{\mathbf{c}}) \in \mathfrak{B}_{full}$ such that

$$\bar{\mathbf{w}}|_{[0,N-1]} = \mathbf{w}|_{[0,N-1]}, \qquad \text{and} \qquad \bar{\mathbf{w}}|_{[N+L,+\infty)} = 0.$$

The characterization of both properties can be obtained in terms of $\mathfrak{B}$, the projection of the full behavior $\mathfrak{B}_{full}$ on its manifest variables $\mathbf{w}$. Consequently, we have the following results whose proofs are straightforward and hence omitted.

Proposition 4: Given a behavior $\mathfrak{B}_{full}$, described as in (3), the following statements are equivalent:

i) the variable $\mathbf{w}$ is (zero-)controllable;

ii) the behavior $\mathfrak{B}$ is (zero-)controllable;

iii) $M_c R_w = L\bar{R}$, where $L(z)$ is right prime (right monomic), while the right factor $\bar{R}(z)$ is left prime.

Remark 2: It is worthwhile to remark that even if we have resorted to the classical concept of latent variable representation for the behavioral description (3), the vector $\mathbf{c}$ does not necessarily represent variables whose role is only instrumental in the description of the evolution of the manifest variable $\mathbf{w}$. Both $\mathbf{w}$ and $\mathbf{c}$ are system variables at the same level, however the control requirements are expressed only in terms of $\mathbf{w}$. Accordingly, the control problems will have only $\mathbf{w}$ as a target, while $\mathbf{c}$ will only be available for control purposes. For this reason, in the following, $\mathbf{w}$ will also be referred to as *to-be-controlled variable*.

As it is easily seen, (zero-)controllability of $\mathbf{w}$ in $\mathfrak{B}_{full}$ is a weaker property with respect to (zero-)controllability of the pair $(\mathbf{w}, \mathbf{c})$. Indeed, (zero-)controllability of $(\mathbf{w}, \mathbf{c})$ always implies (zero-)controllability of $\mathbf{w}$, while the converse is not true. This will allow us to solve control problems even in cases in which standard (both fully and partially interconnected) controllers do not work, as it will be clarified in the sequel.

EXAMPLE 1 Consider the behavior $\mathfrak{B}_{full}$ described as in (3), with $R_w(z) = R_c(z) = z - 1$. Clearly, $\mathfrak{B}_{full} = \ker \left[R_w(\sigma) \quad -R_c(\sigma) \right]$ is neither controllable nor zero-controllable, as the full row rank matrix $\left[R_w(z) \quad -R_c(z) \right]$ is not left monomic (and hence not even left prime), however it is easily seen that since $R_c(z)$ is of full row rank, $\mathbf{w}$ is both controllable and zero-controllable.

IV. DEAD-BEAT CONTROLLERS

In this paper, by a *controller* of a given behavior (3) we mean a system (a set of difference equations) that constrains the trajectories of both the latent variable $\mathbf{c}$ and of the to-be-controlled variable $\mathbf{w}$, and hence is described by a difference equation of the following type

$$P(\sigma)\mathbf{w}(t) = Q(\sigma)\mathbf{c}(t), \qquad \forall\, t \in \mathbb{Z}_+, \tag{6}$$

for suitable polynomial matrices $P(z)$ and $Q(z)$. We define the *controller behavior* as

$$\mathcal{C} := \{(\mathbf{w}, \mathbf{c}) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+} \times (\mathbb{R}^{\mathtt{c}})^{\mathbb{Z}_+} \text{ satisfying } (6)\}.$$

The overall behavior of the system obtained by interconnecting the system (3) with the controller (6) is

$$\mathcal{K}_{full} := \mathfrak{B}_{full} \cap \mathcal{C} = \ker \begin{bmatrix} R_w(\sigma) & -R_c(\sigma) \\ P(\sigma) & -Q(\sigma) \end{bmatrix}. \tag{7}$$

The target of the control problem, however, is not the whole behavior $\mathcal{K}_{full}$, but only its projection on the to-be-controlled variable $\mathbf{w}$, and accordingly we define the *controlled behavior* as

$$\mathcal{K} := \{\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+} : \exists\, \mathbf{c} \in (\mathbb{R}^{\mathtt{c}})^{\mathbb{Z}_+} \text{ such that } (\mathbf{w}, \mathbf{c}) \in \mathfrak{B}_{full} \cap \mathcal{C}\} = \mathcal{P}_w \mathcal{K}_{full}.$$

This perspective is different both from the one adopted in the full interconnection case and from the one adopted in the partial interconnection case [18], [19], in that the controller acts both on $\mathbf{c}$ and on $\mathbf{w}$, as in the full interconnection case, but the target is only $\mathbf{w}$, as in the partial interconnection case. As commented upon in the Introduction, a similar perspective was taken in [11], where the concept of *extended interconnection* was introduced. However in [11] the starting point is only $\mathfrak{B}$, and $\mathfrak{B}_{full}$ can in turn be properly chosen in order to achieve the control target.

Within the class of controllers we are interested in those that make the resulting controlled behavior autonomous and nilpotent. We refer to them as to *dead-beat controllers*.

Definition 6: Given a behavior $\mathfrak{B}_{full}$, a controller (6) is said to be a *dead-beat controller* (DBC) for the system if in every trajectory $(\mathbf{w}, \mathbf{c}) \in \mathcal{K}_{full}$, the component $\mathbf{w}(t), t \in \mathbb{Z}_+$, has finite support, which amounts to saying that the behavior $\mathcal{K}$ is nilpotent.

The concept of DBC is rather intuitive and very much in line with the philosophy underlying the definitions introduced in the previous section: through a dead beat controller, that makes use of both the latent variable $\mathbf{c}$ and of the to be controlled variable $\mathbf{w}$, we aim at ensuring that all the trajectories of $\mathbf{w}$ go to zero in a finite number of steps. What happens of the latent variable is not relevant, and we may even accept that, in order to ensure that $\mathbf{w}$ goes to zero in a finite number of steps and remains zero, the control action related to $\mathbf{c}$ may last forever.

EXAMPLE 1 *(continued)* Consider the behavior $\mathfrak{B}_{full}$ described in Example 1 and corresponding to $R_w(z) = R_c(z) = z-1$. We have already seen that, as a whole, $\mathfrak{B}_{full}$ is neither controllable nor zero-controllable, while $\mathbf{w}$ alone is both controllable and zero-controllable. If we attempt to solve the dead-beat control problem by resorting either to (regular) full interconnected controllers or to (regular or not) partially interconnected controllers [13], the problem is unsolvable. However, dead-beat controllers in the sense of the previous definition exist. It is easy to verify that the choice $P(z) = 1$ and $Q(z) = 0$ leads to $\mathbf{w}(t) = 0,\ t \geq 0$. The prize to pay consists in the fact that $\mathbf{c}(t)$ does not vanish, in general, since $\mathbf{c}$ is constant but not necessarily zero.

A characterization of the DBC's can be easily found, and it requires a preliminary lemma.

Lemma 1: Given a behavior described as in (3), a polynomial pair $(P(z),\, Q(z))$ defines a DBC (6) for the system if and only if

$$\Gamma(z) := M(z) \begin{bmatrix} R_w(z) \\ P(z) \end{bmatrix} \tag{8}$$

is right monomic, where $M(z)$ is an MLA of $\begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix}$.

Proof: Since $\mathcal{K}_{full}$ is described as in (7), if $\begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix}$ were of full row rank then $\mathcal{K} = (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ and hence $(P(z),\, Q(z))$ could not define a DBC. So, an MLA $M(z)$ of this matrix exists and $\mathcal{K}$ can be expressed as

$$\mathcal{K} = \ker\left(M(\sigma) \begin{bmatrix} R_w(\sigma) \\ P(\sigma) \end{bmatrix} \right).$$

Therefore $\mathcal{K}$ is nilpotent if and only if $\Gamma(z)$ is right monomic. ■

Theorem 2: Given a behavior described as in (3), a pair $(P(z), Q(z))$ defines a DBC (6) for the system if and only if the equation

$$X(z, z^{-1}) \begin{bmatrix} R_w(z) & -R_c(z) \\ P(z) & -Q(z) \end{bmatrix} = \begin{bmatrix} I_{\mathtt{w}} & 0 \end{bmatrix} \tag{9}$$

has an L-polynomial solution $X(z, z^{-1})$.

Proof: If the pair defines a DBC then, by the previous lemma, we know that, given an MLA $M(z)$ of $\begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix}$, the matrix $\Gamma(z)$ given in (8) is right monomic. So, $\Gamma(z)$ admits an L-polynomial left inverse, say $L_\Gamma(z, z^{-1})$. But then

$$L_\Gamma(z, z^{-1}) M(z) \begin{bmatrix} R_w(z) & -R_c(z) \\ P(z) & -Q(z) \end{bmatrix} = \begin{bmatrix} I_{\mathtt{w}} & 0 \end{bmatrix},$$

and hence equation (9) has the L-polynomial solution $X(z, z^{-1}) = L_\Gamma(z, z^{-1}) M(z)$.
Conversely, if (9) holds, then, in particular, $X(z, z^{-1})$ is a left annihilator of $\begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix}$. Consequently,

$$X(z, z^{-1}) = F(z, z^{-1}) M(z),$$

for some L-polynomial matrix $F(z, z^{-1})$. This implies

$$F(z, z^{-1}) M(z) \begin{bmatrix} R_w(z) \\ P(z) \end{bmatrix} = I_{\mathtt{w}},$$

and therefore the polynomial matrix

$$\Gamma(z) = M(z) \begin{bmatrix} R_w(z) \\ P(z) \end{bmatrix}$$

has the left L-polynomial inverse $F(z, z^{-1})$, which ensures that $\Gamma(z)$ is right monomic. By the previous lemma, this implies that the pair $(P(z), Q(z))$ defines a DBC for the system. ■

V. ADMISSIBLE AND REGULAR DBC'S

In the previous section we have introduced the concept of DBC and showed how to characterize the pairs $(P(z), Q(z))$ that correspond to the DBC's of a given behavior (3). It is worthwhile noticing that in doing so we did not introduce any assumption on the system, and indeed *every system admits a deadbeat controller*. In fact, the goal of forcing to zero all the trajectories of $\mathcal{K}$ in a finite number of steps, is always achievable, independently of the behavior properties: it is sufficient to choose, for instance, the controller $P(z) = I_{\mathtt{w}}$ and $Q(z) = 0$, to ensure that $\mathcal{K}$ is the zero behavior.

This result may sound absurd, and indeed the contradictions deriving from the use of a controller that does not satisfy any additional requirement have been detected by Jan Willems in his first fundamental contribution about control in the behavioral setting [19]. When focusing on the specific case of DBC's, the reason for this contradiction is that it is always possible to restrict the behavior trajectories to the zero set, but this is obtained by resorting to meaningless controllers that essentially rule out any system evolution but the trivial one. For this reason it is fundamental to understand what are the features that an *admissible* DBC should reasonably endow the resulting controlled system with.

Dead beat controllers for standard state-space models provide good suggestions in this direction. Given a state-space model, if we decide to apply the DBC starting at some time $N > 0$, we expect to find among the trajectories of the controlled behavior at least one finite support trajectory that coincides with the original trajectory $\mathbf{w} \in \mathfrak{B}$ in the initial window $[0, N-1]$. When moving to the general behavior setting, what we expect is that the DBC performs its task without constraining the initial portion of the trajectories in $\mathfrak{B}$: so if it starts working at time $t = N$, it does not affect the samples of the trajectory in some initial window $[0, M-1]$ provided that $N - M$ is large enough.

By assuming this perspective, we want to introduce the concept of *admissible DBC*. To this end, we need some mathematical preliminaries. Consider the difference equation (7), describing the behavior $\mathcal{K}_{full}$ of the overall system. As we enlightened within the proof of Lemma 1, if $\mathcal{C}$ defines a DBC for system (3), the matrix $\begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix}$ cannot be of full row rank. If so,

$\mathcal{K} = (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, and hence it could not be a nilpotent behavior. An MLA $M(z)$ of $\begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix}$ can always be described as follows:

$$M(z) = \begin{bmatrix} M_c(z) & 0 \\ M_1(z) & M_2(z) \end{bmatrix},$$

where $M_c(z)$ is an MLA of $R_c(z)$. Accordingly, the behavior $\mathcal{K}$ can be equivalently described as

$$\begin{aligned} \mathcal{K} &= \mathcal{P}_w\ \mathcal{K}_{full} = \ker\left(\begin{bmatrix} M_c(\sigma) & 0 \\ M_1(\sigma) & M_2(\sigma) \end{bmatrix} \begin{bmatrix} R_w(\sigma) \\ P(\sigma) \end{bmatrix} \right) \\ &= \ker \begin{bmatrix} M_c(\sigma)R_w(\sigma) \\ M_1(\sigma)R_w(\sigma) + M_2(\sigma)P(\sigma) \end{bmatrix} \subseteq \ker\left[\, M_c(\sigma)R_w(\sigma)\,\right] = \mathfrak{B}. \end{aligned} \tag{10}$$

Starting from $\mathcal{C}$, we introduce the *delayed controllers* $\mathcal{C}_i, i \in \mathbb{Z}_+$, described by the difference equation

$$\sigma^i P(\sigma)\mathbf{w}(t) = \sigma^i Q(\sigma)\mathbf{c}(t), \qquad t \in \mathbb{Z}_+. \tag{11}$$

If we denote by $\mathcal{K}_i$ the controlled behavior obtained corresponding to $\mathcal{C}_i$, we can describe it as

$$\mathcal{K}_i = \ker\left(\begin{bmatrix} M_c(\sigma) & 0 \\ M_1^{(i)}(\sigma) & M_2^{(i)}(\sigma) \end{bmatrix} \begin{bmatrix} R_w(\sigma) \\ \sigma^i P(\sigma) \end{bmatrix} \right) = \ker \begin{bmatrix} M_c(\sigma)R_w(\sigma) \\ M_1^{(i)}(\sigma)R_w(\sigma) + \sigma^i M_2^{(i)}(\sigma)P(\sigma) \end{bmatrix}, \tag{12}$$

where

$$M_i(z) = \begin{bmatrix} M_c(z) & 0 \\ M_1^{(i)}(z) & M_2^{(i)}(z) \end{bmatrix}$$

is an MLA of $\begin{bmatrix} R_c(z) \\ z^i Q(z) \end{bmatrix}$. Clearly, $\mathcal{C} = \mathcal{C}_0$, $\mathcal{K} = \mathcal{K}_0$ and $M(z) = M_0(z)$.

The controller $\mathcal{C}_i$ acts on the trajectories $(\mathbf{w}, \mathbf{c})$ of $\mathfrak{B}_{full}$ as the original controller $\mathcal{C}$, but instead of performing the control action from $t = 0$ onward, it starts at $t = i$. Note, however, that this does not mean that the controlled trajectories are unconstrained at the time instants preceding $t = i$. We want to show that if $\mathcal{C}$ is a DBC for the given system, then every $\mathcal{C}_i$ is.

Lemma 2: Given a behavior described as in (3), if the controller $\mathcal{C}$ described by the difference equation (6) is a DBC for the system, then such is any controller $\mathcal{C}_i, i \in \mathbb{Z}_+$, described as in (11).

Proof: If $\mathcal{C}$ is a DBC for the system, then the corresponding controlled behavior $\mathcal{K}$, described as in (10), is nilpotent and hence

$$\begin{bmatrix} M_c(z)R_w(z) \\ M_1(z)R_w(z) + M_2(z)P(z) \end{bmatrix} = \begin{bmatrix} M_c(z) & 0 \\ M_1(z) & M_2(z) \end{bmatrix} \begin{bmatrix} R_w(z) \\ P(z) \end{bmatrix}$$

is right monomic. We also notice that an MLA $[\, M_1^{(i)}(z) \quad M_2^{(i)}(z) \,]$ of $\begin{bmatrix} R_c(z) \\ z^i Q(z) \end{bmatrix}$, can be obtained from the polynomial matrix $[\, z^i M_1(z) \quad M_2(z) \,]$ by simply extracting its greatest left divisor $\Delta_i(z)$, namely:

$$\Delta_i(z)\,[\, M_1^{(i)}(z) \quad M_2^{(i)}(z) \,] = [\, z^i M_1(z) \quad M_2(z) \,],$$

with $\Delta_i(z)$ nonsingular square and $[\, M_1^{(i)}(z) \quad M_2^{(i)}(z) \,]$ left prime. Accordingly, $\mathcal{K}_i$, the controlled behavior corresponding to $\mathcal{C}_i$, can be expressed as

$$\mathcal{K}_i = \ker \left(\begin{bmatrix} M_c(\sigma) & 0 \\ M_1^{(i)}(\sigma) & M_2^{(i)}(\sigma) \end{bmatrix} \begin{bmatrix} R_w(\sigma) \\ \sigma^i P(\sigma) \end{bmatrix} \right)$$

and it is immediately seen that

$$\mathcal{K}_i \;\subseteq\; \ker \left(\begin{bmatrix} M_c(\sigma) & 0 \\ \sigma^i M_1(\sigma) & M_2(\sigma) \end{bmatrix} \begin{bmatrix} R_w(\sigma) \\ \sigma^i P(\sigma) \end{bmatrix} \right) = \ker \begin{bmatrix} M_c(\sigma) R_w(\sigma) \\ \sigma^i (M_1(\sigma) R_w(\sigma) + M_2(\sigma) P(\sigma)) \end{bmatrix}.$$

Since the matrix

$$\begin{bmatrix} M_c(z) R_w(z) \\ z^i (M_1(z) R_w(z) + M_2(z) Q(z)) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & z^i I \end{bmatrix} \left(\begin{bmatrix} M_c(z) & 0 \\ M_1(z) & M_2(z) \end{bmatrix} \begin{bmatrix} R_w(z) \\ P(z) \end{bmatrix} \right)$$

is right monomic, $\mathcal{K}_i$ is a nilpotent behavior and hence $\mathcal{C}_i$ is a DBC. ■

The result of the previous lemma allows us to introduce the following definition of admissible DBC.

Definition 7: Given a behavior $\mathfrak{B}_{full}$, a dead-beat controller $\mathcal{C}$ described as in (6) is said to be *admissible* if there exists $L \in \mathbb{Z}_+$ such that for every $\mathbf{w} \in \mathfrak{B} = \mathcal{P}_w \mathfrak{B}_{full}$ and every $N \in \mathbb{N}$, there exists $\bar{\mathbf{w}} \in \mathcal{K}_{L+N}$, the nilpotent behavior obtained corresponding to the controller $\mathcal{C}_{L+N}$, such that $\bar{\mathbf{w}}(t)|_{[0,N-1]} = \mathbf{w}(t)|_{[0,N-1]}$.

We are in a position to relate zero-controllability of $\mathbf{w}$ to the existence of an admissible DBC.

Theorem 3: A behavior $\mathfrak{B}_{full}$ described as in (3) admits an admissible DBC if and only if $\mathbf{w}$ is zero-controllable. If this is the case, then every DBC is admissible.

Proof: Assume, first, that the system admits an admissible DBC $\mathcal{C}$, described by the matrix pair $(P(z), Q(z))$, and, hence $\mathcal{K}$, described as in (10), is a nilpotent behavior. This implies that there exists $M \in \mathbb{Z}_+$ such that all trajectories in $\mathcal{K}$ are zero for $t \geq M$. On the other hand, we have shown that for every $i \in \mathbb{Z}_+$

$$\begin{aligned}\mathcal{K}_i &\subseteq \ker \begin{bmatrix} M_c(\sigma)R_w(\sigma) \\ \sigma^i(M_1(\sigma)R_w(\sigma) + M_2(\sigma)Q(\sigma)) \end{bmatrix} \\ &\subseteq \ker \left(\sigma^i \begin{bmatrix} M_c(\sigma)R_w(\sigma) \\ M_1(\sigma)R_w(\sigma) + M_2(\sigma)Q(\sigma) \end{bmatrix} \right),\end{aligned}$$

and the behavior on the right hand-side is nilpotent, with trajectories which are identically zero at least for $t \geq i+M$. So, also the trajectories of $\mathcal{K}_i$ have finite support included in $[0, i+M-1]$, and this is true for every $i \in \mathbb{Z}_+$.

Since $\mathcal{C}$ is an admissible DBC, there exists $L \in \mathbb{Z}_+$ such that for every $N \in \mathbb{N}$ and every $\mathbf{w} \in \mathfrak{B}$ a trajectory $\bar{\mathbf{w}} \in \mathcal{K}_{L+N} \subseteq \mathfrak{B}$ can be found, coinciding with $\mathbf{w}$ in $[0, N-1]$. Such a trajectory $\bar{\mathbf{w}}$ is surely zero for $t \geq L+N+M$. So, we have proved that there exists $L^* \in \mathbb{N}$, specifically $L^* := M + L$, such that for every $\mathbf{w} \in \mathfrak{B}$ there exists a trajectory $\bar{\mathbf{w}} \in \mathcal{K}_{L+N} \subseteq \mathfrak{B}$ coinciding with $\mathbf{w}$ in $[0, N-1]$ and zero for $t \geq N + L^*$. This proves that $\mathbf{w}$ is zero-controllable.

We have already pointed out that a DBC always exists, independently of the behavior properties. We want to show that when $\mathbf{w}$ is zero-controllable, every DBC is admissible (this, obviously, implies that there exists an admissible one). Let $(P(z), Q(z))$ be the pair of polynomial matrices that describes a DBC. We have only to verify that it is admissible. By the zero-controllability property, there exists a nonnegative integer L such that for every $N \in \mathbb{N}$ and every $(\mathbf{w}, \mathbf{c}) \in \mathfrak{B}_{full}$, one can find $(\bar{\mathbf{w}}, \bar{\mathbf{c}}) \in \mathfrak{B}_{full}$ such that

$$\bar{\mathbf{w}}|_{[0,N-1]} = \mathbf{w}|_{[0,N-1]}, \qquad \text{and} \qquad \bar{\mathbf{w}}|_{[N+L,+\infty)} = 0. \tag{13}$$

We want to show that this same nonnegative integer L makes the definition of admissible DBC satisfied. To this end we have to show that for every $N \in \mathbb{N}$ and every $\mathbf{w} \in \mathfrak{B}$, there exists $\bar{\mathbf{w}} \in \mathcal{K}_{L+N}$ coinciding with $\mathbf{w}$ in $[0, N-1]$. By resorting to the same reasonings we previously used, we can claim that

$$\begin{aligned}\mathcal{K}_{L+N} &= \ker \left(\begin{bmatrix} M_c(\sigma) & 0 \\ M_1^{(L+N)}(\sigma) & M_2^{(L+N)}(\sigma) \end{bmatrix} \begin{bmatrix} R_w(\sigma) \\ \sigma^{N+L}P(\sigma) \end{bmatrix} \right) \\ &= \ker \left(\begin{bmatrix} I & 0 \\ M_1^{(L+N)}(\sigma) & M_2^{(L+N)}(\sigma) \end{bmatrix} \begin{bmatrix} M_c(\sigma)R_w(\sigma) \\ \sigma^{N+L}P(\sigma) \end{bmatrix} \right).\end{aligned}$$

So, it is easy to see that the same trajectory $\bar{\mathbf{w}} \in \mathfrak{B}$ that satisfies (13), and whose existence is ensured by the zero-controllability property, is necessarily a trajectory of both $\mathfrak{B} = \ker(M_c(\sigma)R_w(\sigma))$ and $\ker(\sigma^{L+N}P(\sigma))$. Therefore $\bar{\mathbf{w}} \in \mathcal{K}_{L+N}$ and this makes the definition of admissible DBC satisfied. ■

The following example illustrates why a delayed DBC unavoidably constraints the whole controlled trajectory, and hence admissible DBC's do not exist for behaviors for which $\mathbf{w}$ is not zero-controllable.

EXAMPLE 2 Consider the behavior described by the following difference equation:

$$\begin{bmatrix} \sigma - 1 & 0 \\ 0 & \sigma - 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} c(t), \ t \in \mathbb{Z}_+.$$

For this behavior, $\mathbf{w} = [\, w_1 \quad w_2 \,]^T$ is not zero-controllable. Let $\mathcal{C}$ be any DBC for this system. Despite each delayed controller $\mathcal{C}_i$ ($i \in \mathbb{Z}_+$) seems to start its control action only at $t = i$, it actually imposes the constraint $w_2(t) = 0$ for any $t \geq 0$. This is due to the fact that the plant equations constrain $w_2(t)$ to assume a constant value, and the DBC action ensures that $w_2(t) = 0$ after a finite number of steps. Therefore no DBC can be admissible, since it constrains the second component of the trajectory $\mathbf{w}(t)$ to be zero-valued independently of the time when the DBC starts to act. In other words, there is no possibility of driving to zero a trajectory $\mathbf{w}(t)$ unless $w_2(0) = 0$.

In the literature about behaviors, particular attention has been devoted to the so-called *regular interconnections* [1], [2], [19]. The connection of a plant and a controller is a regular one if the controller laws are not redundant with respect to the system laws.

Definition 8: Given a behavior $\mathfrak{B}_{full}$ described as in (3), a DBC $\mathcal{C}$ described as in (6) is *regular* if

$$\text{rank} \begin{bmatrix} R_w(z) & -R_c(z) \\ P(z) & -Q(z) \end{bmatrix} = \text{rank} \,[\, R_w(z) \quad -R_c(z) \,] + \text{rank} \,[\, P(z) \quad -Q(z) \,] \,.$$

By making use of the characterization of set-controllable behaviors given in [11], Theorem 5.7, in terms of extended regular interconnections, which are just regular controllers described as in (6), we will show that a regular DBC exists if and only if $\mathbf{w}$ is zero-controllable. In addition, for the class of behaviors described as in (3) and characterized by certain values of $\mathtt{w}, \mathtt{c}$ and rank $R_c(z)$, the DBC's are described by polynomial matrices $[\,P(z) \quad -Q(z)\,]$ whose number of rows is lower bounded by the minimal value $\mathtt{w} - p + r$. Such *minimal DBC's* can always be obtained under the zero-controllability assumption and only in that case. So, as a general result, the zero-controllability of $\mathbf{w}$ is the property that allows to design DBC's that are effective and not redundant. For all the behaviors that do not exhibit this property, control to zero can be obtained only at the expenses of redundancy.

Theorem 4: Consider a behavior described as in (3), and assume without loss of generality that $[\,R_w(z) \quad -R_c(z)\,] \in \mathbb{R}[z]^{p\times(\mathtt{w}+\mathtt{c})}$ is of full row rank. Let r be the rank of $R_c(z)$. The following statements are equivalent:

i) $\mathbf{w}$ is zero-controllable;
ii) there exists a DBC that corresponds to a polynomial matrix $[\,P(z) \quad -Q(z)\,]$ with $\mathtt{w}-p+r$ rows;
iii) there exists a regular DBC.

Proof: i) $\Rightarrow$ ii) Suppose that $\mathbf{w}$ is zero-controllable. If $R_c(z)$ is of full row rank, then $p = r$ and a DBC that corresponds to a polynomial matrix $[\,P(z) \quad -Q(z)\,]$ with $\mathtt{w}$ rows is simply $P(z) = I_{\mathtt{w}}$ and $Q(z) = 0$. If $R_c(z)$ is not of full row rank, we let

$$U(z) := \begin{bmatrix} S_c(z) \\ M_c(z) \end{bmatrix}$$

be a unimodular matrix such that

$$U(z)R_c(z) = \begin{bmatrix} B_c(z) \\ 0 \end{bmatrix},$$

with $B_c(z) := S_c(z)R_c(z)$ of full row rank. Accordingly we set

$$U(z)R_w(z) = \begin{bmatrix} B_w(z) \\ M_c(z)R_w(z) \end{bmatrix}.$$

By the full row rank assumption on $[\,R_w(z) \quad -R_c(z)\,]$, the matrix $M_c(z)R_w(z)$ is of full row rank, and we can factorize it as $M_c(z)R_w(z) = L(z)\bar{R}(z)$, with $L(z)$ nonsingular square and $\bar{R}(z) \in \mathbb{R}[z]^{(p-r)\times\mathtt{w}}$ left prime. In the special case when $L(z)\bar{R}(z)$ is nonsingular square, $\bar{R}(z)$ is a unimodular factor. By assumption i), $L(z)$ is square monomic and hence $\det L(z) = z^k$ for some $k \in \mathbb{Z}_+$.

If $\bar{R}(z)$ is unimodular, then $\mathtt{w} - p + r = 0$ and, indeed, by choosing as $P(z)$ and $Q(z)$ the void matrices we obtain a DBC, since $\mathcal{K} = \mathfrak{B}$ is nilpotent. If $\bar{R}(z)$ is not unimodular, let $\bar{C}(z) \in \mathbb{R}^{(\mathtt{w}-p+r)\times \mathtt{w}}$ be a completion of $\bar{R}(z)$ to a unimodular matrix. Then by choosing $P(z) = \bar{C}(z)$ and $Q(z) = 0$, we obtain the DBC we were searching for.

ii) $\Rightarrow$ iii) Let $P(z)$ and $Q(z)$ be polynomial matrices such that $[\, P(z) \quad -Q(z)\,]$ defines a DBC with $\mathtt{w} - p + r$ rows for the system (3). We want to prove that $P(z)$ and $Q(z)$ define a regular DBC. To this end, let $U(z)$ be a unimodular matrix such that

$$U(z) \begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix} = \begin{bmatrix} \tilde{R}_c(z) \\ \hline 0 \end{bmatrix},$$

where $\tilde{R}_c(z)$ is of full row rank $\tilde{r}$. Clearly, as $\mathrm{rank}\tilde{R}_c(z) \geq \mathrm{rank} R_c(z)$, $\tilde{r} \geq r$. Accordingly,

$$U(z) \begin{bmatrix} R_w(z) \\ P(z) \end{bmatrix} = \begin{bmatrix} \tilde{R}_w(z) \\ \hline \tilde{P}(z) \end{bmatrix},$$

where $\tilde{P}(z) \in \mathbb{R}[z]^{(\mathtt{w}+r-\tilde{r})\times \mathtt{w}}$. Since $\mathcal{K} = \ker\, \tilde{P}(\sigma)$, in order for $\mathcal{K}$ to be a nilpotent behavior, it must be $\mathrm{rank}\, \tilde{P}(z) = \mathtt{w}$. But this implies $\mathtt{w} + r - \tilde{r} \geq \mathtt{w}$, namely $r - \tilde{r} \geq 0$. On the other hand, we know that $r \leq \tilde{r}$, so it must be $r = \tilde{r}$. This means that $\tilde{P}(z)$ is nonsingular square. Finally,

$$\begin{aligned} \mathrm{rank} \begin{bmatrix} R_w(z) & -R_c(z) \\ P(z) & -Q(z) \end{bmatrix} &= \mathrm{rank} \begin{bmatrix} \tilde{R}_w(z) & -\tilde{R}_c(z) \\ \tilde{P}(z) & -0 \end{bmatrix} \\ &= \mathrm{rank}\tilde{P}(z) + \mathrm{rank}\tilde{R}_c(z) = \mathtt{w} + r = p + (\mathtt{w} - p + r) \\ &= \mathrm{rank}\, [\, R_w(z) \quad -R_c(z)\,] + \mathrm{rank}\, [\, P(z) \quad -Q(z)\,]\,. \end{aligned}$$

This completes the proof.

iii) $\Rightarrow$ i) By Theorem 5.7 in [11], if there exists a regular controller described as in (6) such that $\mathcal{K} = \mathfrak{B}'$ is nilpotent, then $\mathfrak{B}$ is set-controllable to the nilpotent behavior $\mathfrak{B}' \subset \mathfrak{B}$, but as shown in Proposition 1, this amounts to saying that $\mathfrak{B}$ is zero-controllable. ■

Remark 3: It is worthwhile noticing that, as clarified within the proof, $\mathtt{w} - p + r$ is the minimal number of rows a DBC may exhibit. Indeed, this is the minimal number of rows that a controller $\mathcal{C}$ needs in order to make $\mathcal{K} = \mathcal{P}_w \mathfrak{B}_{full}$ autonomous. So, it is not surprising that if such a DBC can be found then it is necessarily a regular one, namely it does not introduce any kind of redundancy. However, regular DBC's are not necessarily minimal, as shown by the following example.

EXAMPLE 3 Consider the behavior described by the following difference equation:

$$(\sigma - 1)w(t) = [\,1 \quad 1\,]\,\mathbf{c}(t),\ t \in \mathbb{Z}_+,$$

where $\mathtt{w} = p = r = 1$. $\mathbf{w}$ is controllable and hence zero-controllable. The DBC described by the following equations

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} w(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{c}(t),\ t \in \mathbb{Z}_+,$$

is clearly regular, however it is not minimal as it has $2 > \mathtt{w} - p + r = 1$ rows.

On the other hand, when $\mathbf{w}$ is zero-controllable, all DBC's are admissible, but obviously not all admissible DBC's are regular, as shown by the following example.

EXAMPLE 4 Consider the behavior described by the following difference equation:

$$\begin{bmatrix} \sigma - 1 \\ \sigma + 1 \end{bmatrix} w(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{c}(t),\ t \in \mathbb{Z}_+.$$

$\mathbf{w}$ is zero-controllable and the DBC described by the following equations

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} w(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{c}(t),\ t \in \mathbb{Z}_+,$$

is (admissible but) not regular.

These two examples clarified that the concepts of admissible, regular and minimal DBC's are distinct ones, even if zero-controllability of $\mathbf{w}$ is a necessary and sufficient condition for the existence of DBC's of any of these classes. The following picture describes how such classes are related.

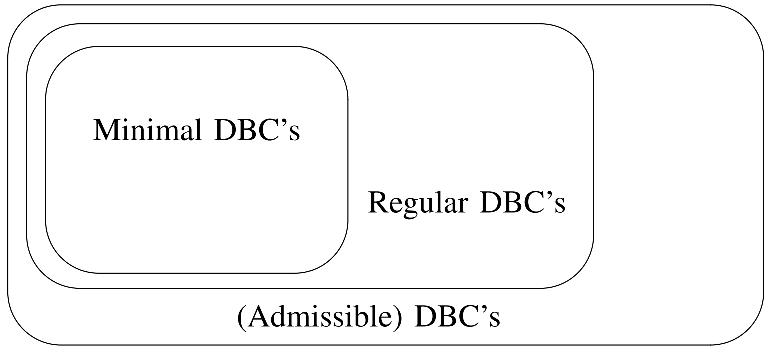

Figure 1: Classes of DBC's for systems with $\mathbf{w}$ zero-controllable.

VI. Minimal DBC parametrization

Given a behavior described as in (3), with $\mathbf{w}$ zero-controllable and $[\,R_w(z) \quad -R_c(z)\,] \in \mathbb{R}[z]^{p\times(\mathtt{w}+\mathtt{c})}$ of full row rank matrix, we want to provide a parametrization of all minimal DBC's of the system, namely all DBC's with the minimal number of rows and hence associated with some polynomial matrix

$$[\,P(z) \quad -Q(z)\,] \in \mathbb{R}[z]^{(\mathtt{w}-p+r)\times(\mathtt{w}+\mathtt{c})},$$

where r denotes the rank of $R_c(z)$. By Theorem 3, each such DBC will be admissible. By Theorem 4 (and following remark), it will also be regular and the matrix $[\,P(z) \quad -Q(z)\,]$ will necessarily be of full row rank.

In order to solve this problem, we proceed as in the proof of Theorem 4. We distinguish the case $r < p$ from the case $r = p$.

- Case 1: If $r < p$, we let

$$U(z) := \begin{bmatrix} S_c(z) \\ M_c(z) \end{bmatrix}$$

be a unimodular matrix such that

$$U(z)R_c(z) = \begin{bmatrix} B_c(z) \\ 0 \end{bmatrix},$$

with $B_c(z) \in \mathbb{R}[z]^{r\times \mathtt{c}}$ of full row rank. Note that $M_c(z)$ is an MLA of $R_c(z)$. Accordingly we set

$$U(z)R_w(z) = \begin{bmatrix} B_w(z) \\ M_c(z)R_w(z) \end{bmatrix},$$

and we factorize $M_c(z)R_w(z)$ as $\Delta_M(z)\bar{R}(z)$, with $\Delta_M(z)$ $(p-r)\times(p-r)$ square monomic and $\bar{R}(z)\in\mathbb{R}[z]^{(p-r)\times \mathtt{w}}$ left prime. Clearly, $\mathfrak{B}=\ker(M_c(\sigma)R_w(\sigma))$. Accordingly, $\mathcal{K}_{full}$ can be described as the kernel of the following polynomial matrix

$$\begin{bmatrix} B_w(z) & -B_c(z) \\ \Delta_M(z)\bar{R}(z) & 0 \\ P(z) & -Q(z) \end{bmatrix} \in \mathbb{R}[z]^{(\mathtt{w}+r)\times(\mathtt{w}+\mathtt{c})}. \tag{14}$$

Let $V(z)$ be a unimodular matrix such that

$$V(z)\begin{bmatrix} B_c(z) \\ Q(z) \end{bmatrix} = \begin{bmatrix} \tilde{R}_c(z) \\ 0 \end{bmatrix},$$

with $\tilde{R}_c(z)$ of full row rank. As shown in the proof of Theorem 4, $\mathrm{rank}\ \tilde{R}_c(z)=\mathrm{rank}\ B_c(z)=\mathrm{rank}\ R_c(z)=r$. This implies that all the rows of $Q(z)$ must be linearly dependent (on $\mathbb{R}(z)$) on the rows of $B_c(z)$:

$$Q(z)=W(z)B_c(z), \qquad \exists\ W(z)\in\mathbb{R}^{(\mathtt{w}-p+r)\times r}(z). \tag{15}$$

Consequently, $\mathcal{K}_{full}$ can be described as the kernel of

$$\begin{bmatrix} B_w(z) & -B_c(z) \\ \Delta_M(z)\bar{R}(z) & 0 \\ P(z) & -W(z)B_c(z) \end{bmatrix} \in \mathbb{R}[z]^{(\mathtt{w}+r)\times(\mathtt{w}+\mathtt{c})},$$

under the constraint that $W(z)B_c(z)$ is polynomial. Let $D_W(z)^{-1}N_W(z)$ be a left coprime matrix fraction description (MFD) of $W(z)$. This amounts to saying that $[\,N_W(z)\ \ -D_W(z)\,]$ is an MLA of $\begin{bmatrix} B_c(z) \\ W(z)B_c(z) \end{bmatrix}$. Accordingly

$$\mathcal{K}=\ker\left(\begin{bmatrix} \Delta_M(\sigma)\bar{R}(\sigma) \\ N_W(\sigma)B_w(\sigma)-D_W(\sigma)P(\sigma) \end{bmatrix}\right).$$

$\mathcal{K}$ is nilpotent if and only if

$$H_{TOT}(z)=\begin{bmatrix} \Delta_M(z)\bar{R}(z) \\ N_W(z)B_w(z)-D_W(z)P(z) \end{bmatrix}$$

is square monomic. If we represent (without loss of generality) $N_W(z)B_w(z)-D_W(z)P(z)$ as

$$N_W(z)B_w(z)-D_W(z)P(z)=[\,T_R(z)\quad T_C(z)\,]\begin{bmatrix} \bar{R}(z) \\ \bar{C}(z) \end{bmatrix},$$

where $\bar{C}(z)$ is a completion of $\bar{R}(z)$ to a unimodular matrix and $T_R(z), T_C(z)$ are polynomial matrices, then $H_{TOT}(z)$ is square monomic if and only if $T_C(z)$ is square monomic. So, to summarize, in the case $r < p$, the matrices of the DBC's with minimal number of rows are all the polynomial matrices that can be obtained as

$$\begin{array}{rcl} Q(z) & = & W(z)B_c(z) \\ P(z) & = & W(z)B_w(z) - D_W(z)^{-1}\left[\, T_R(z) \quad T_C(z)\,\right] \begin{bmatrix} \bar{R}(z) \\ \bar{C}(z) \end{bmatrix}, \end{array} \tag{16}$$

where $W(z)$ is a rational matrix, $D_W^{-1}(z)N_W(z)$ is a left coprime MFD of $W(z)$, $T_R(z)$ and $T_C(z)$ are polynomial matrices, and $T_C(z)$ is square monomic.

• Case 2: if $R_c(z)$ is of full row rank (namely $r = p$), then $S_c(z) = I_p$, $B_c(z) = R_c(z)$, $B_w(z) = R_w(z)$ and the matrix $M_c(z)R_w(z)$ does not appear. So, we can apply the same reasoning as in Case 1 and obtain that $\mathcal{K}_{full}$ can be described as the kernel of

$$\begin{bmatrix} R_w(z) & -R_c(z) \\ P(z) & -W(z)R_c(z) \end{bmatrix} \in \mathbb{R}[z]^{(\mathtt{w}+r)\times(\mathtt{w}+\mathtt{c})},$$

where $W(z)$ is a rational matrix such that $W(z)R_c(z)$ is polynomial. If we denote, again, by $D_W(z)^{-1}N_W(z)$ a left coprime MFD of $W(z)$, then

$$\mathcal{K} = \ker\left(N_W(\sigma)R_w(\sigma) - D_W(\sigma)P(\sigma)\right),$$

and this is a nilpotent behavior if and only if the square matrix

$$H_{TOT}(z) = N_W(z)R_w(z) - D_W(z)P(z)$$

is monomic. This brings us to the parametrization:

$$\begin{array}{rcl} Q(z) & = & W(z)R_c(z) \\ P(z) & = & W(z)R_w(z) - D_W(z)^{-1}T_C(z), \end{array} \tag{17}$$

where $W(z)$ is a rational function, $D_W^{-1}(z)N_W(z)$ is a left coprime MFD of $W(z)$, and $T_C(z)$ is square monomic.

EXAMPLE 5 Consider a behavior described as in (3), with

$$R_w(z) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; R_c(z) = \begin{bmatrix} 1+z & 0 \\ 0 & 1+z \end{bmatrix}.$$

It is easily seen that $R_c(z)$ (and hence, a fortiori, $[\, R_w(z) \quad -R_c(z)\,]$) is of full row rank. So, w is zero-controllable and we are in Case 2 previously discussed (namely $p = r = 2$). We express the polynomial matrix $Q(z)$ as

$$Q(z) = [\, q_1(z) \quad q_2(z)\,] = W(z)R_c(z) = (z+1)\,[\, w_1(z) \quad w_2(z)\,],$$

where $w_1(z)$ and $w_2(z)$ are rational functions. Clearly, it must be

$$W(z) = [\, w_1(z) \quad w_2(z) \,] = [\, \tfrac{q_1(z)}{z+1} \quad \tfrac{q_2(z)}{z+1} \,] \,.$$

Two cases possibly arise: (a) $W(z)$ is polynomial, namely both $q_1(z)$ and $q_2(z)$ are multiple of $z+1$; (b) $W(z)$ is not polynomial, which amounts to saying that at least one between $q_1(z)$ and $q_2(z)$ is not a multiple of $z+1$.

Case (a): If $W(z)$ is polynomial, a left coprime MFD of $W(z)$ is $D_W(z)^{-1}N_W(z)$ with $D_W(z) = 1$ and $N_W(z) = W(z) = [\, w_1(z) \quad w_2(z) \,]$. Accordingly

$$P(z) = W(z)R_w(z) - T_C(z) = [\, w_1(z) \quad w_2(z) \,] \begin{bmatrix} 1 \\ 1 \end{bmatrix} - T_C(z) = w_1(z) + w_2(z) - T_C(z),$$

where $T_C(z)$ is monomic. So, a parametrization in terms of the polynomials $w_1(z)$, $w_2(z)$ and $T_C(z)$, this latter constrained to be monomic, is:

$$\begin{aligned} Q(z) &= [\, (z+1)w_1(z) \quad (z+1)w_2(z) \,] \\ P(z) &= w_1(z) + w_2(z) - T_C(z). \end{aligned} \tag{18}$$

Case (b): In this case $D_W(z) = z+1$ and $N_W(z) = [\, q_1(z) \quad q_2(z) \,]$. Accordingly

$$\begin{aligned} P(z) &= W(z)R_w(z) - D_W(z)^{-1}T_C(z) \\ &= (z+1)^{-1} [\, q_1(z) \quad q_2(z) \,] \begin{bmatrix} 1 \\ 1 \end{bmatrix} - (z+1)^{-1}T_C(z) = \frac{q_1(z) + q_2(z) - T_C(z)}{z+1}, \end{aligned}$$

where $T_C(z)$ is monomic. Clearly, the only way for $P(z)$ to be polynomial is by imposing that $(z+1) \mid q_1(z) + q_2(z) - T_C(z)$. This implies $q_1(z) + q_2(z) - T_C(z) = (z+1)q_3(z)$. So, by expressing $q_2(z)$ as

$$q_2(z) = (z+1)q_3(z) - q_1(z) + T_C(z),$$

we can obtain a parametrization in terms of the polynomials $q_1(z), q_3(z)$ and $T_C(z)$, this latter constrained to be monomic:

$$\begin{aligned} Q(z) &= [\, q_1(z) \quad (z+1)q_3(z) - q_1(z) + T_C(z) \,] \\ P(z) &= q_3(z). \end{aligned} \tag{19}$$

Remark 4: Note that the pairs (Q, P) obtained through (18) are all different from the pairs described in (19), as it is easily seen by equating their expressions. So, (18) and (19) represent two disjoint DBC's families, which parametrize all the minimal complexity DBC's. Therefore this example clearly shows that the parametrization problem is a very complicate one, as a double parametrization is required even if the example is very simple, being D_W just a scalar polynomial.

Remark 5: By referring to the previous parametrizations, it is worthwhile noticing that a (in general, proper) subset of all pairs $(P(z), Q(z))$ that correspond to minimal DBC's can be easily obtained from (16) and (17), by simply constraining $W(z)$ to be a polynomial matrix and therefore $D_W(z)$ to be equal to $I_{\mathtt{w}-p+r}$.

Remark 6: As anticipated in the Introduction, it would possible to restate the DBC problem addressed in this paper as a DBC problem achieved through partial interconnection. This would require to fictitiously replace the variable $\mathbf{c}$ with the variable $\tilde{\mathbf{c}} = (\mathbf{c}, \mathbf{w})$, that introduces a copy of the to-be-controlled variable. By exploiting this trick, we could adapt the parametrization of the "minimal" stabilizing controller (through partial interconnection) provided in [9] in order to achieve a parametrization of the minimal DBC's. This solution, even if feasible, would bring to a parametrization (of course, equivalent but) different from ours; and in several instances this method for obtaining a parametrization of the minimal DBC's would lead to much more involved calculations with respect to those required by our method[1]. Nonetheless, it must be remarked that even the parametrization in [9] resorts to multiple polynomial parameters, and in several cases the set of all minimal DBC's would be the union of disjoint sets of parametrized DBC's.

The problem of finding a complete parametrization of all the (minimal or not) DBC's of a given plant remains however an open one, which needs further investigation.

VII. Input/output structure of DBC's

The concepts of zero-controllability and of DBC investigated in the previous sections represent a generalization of the perspective and set-up commonly adopted for state-space models: there is a variable to be controlled to zero (the to be controlled variable $\mathbf{w}$) and the control action is

[1] Unfortunately, to support our claim by means of examples we should provide a detailed description of the parametrization obtained in [9], a task that we believe is out of the purposes of this paper.

obtained by linking the values of $\mathbf{w}$ and of the latent variable $\mathbf{c}$. Accordingly, one may want to investigate under what conditions it is possible to obtain a DBC that naturally implements a feedback law, by this meaning that for such a DBC $\mathbf{c}$ represents the controller output and $\mathbf{w}$ the controller input. This amounts to searching for conditions ensuring that a polynomial pair $(P(z), Q(z))$ can be found, with $Q(z)$ full row rank or, in particular, nonsingular square, such that the corresponding system (6) is a DBC for the system (3). We have the following result.

Proposition 5: Given a system described as in (3), with $\mathbf{w}$ zero-controllable, the following statements are equivalent:

i) the following condition holds:

$$\mathtt{w} \leq \operatorname{rank} \left[R_w(z) \quad -R_c(z) \right]; \tag{20}$$

ii) there exists a DBC described as in (6) with $\left[P(z) \quad -Q(z) \right] \in \mathbb{R}[z]^{\mathtt{c} \times (\mathtt{w}+\mathtt{c})}$, and $Q(z)$ nonsingular square;

iii) there exists a DBC described as in (6) with $\left[P(z) \quad -Q(z) \right] \in \mathbb{R}[z]^{k \times (\mathtt{w}+\mathtt{c})}$, and $Q(z)$ of full row rank.

Proof: Assume w.l.o.g. that $\left[R_w(z) \quad -R_c(z) \right] \in \mathbb{R}[z]^{p \times (\mathtt{w}+\mathtt{c})}$ is a full row rank matrix, so that condition (20) becomes $\mathtt{w} \leq p$. Also, we let r denote the rank of $R_c(z)$ and we assume, as in the previous sections, that the behavior $\mathfrak{B}_{full}$ is described as the kernel of the following polynomial matrix:

$$\begin{bmatrix} B_w(z) & -B_c(z) \\ \Delta_M(z)\bar{R}(z) & 0 \end{bmatrix} \in \mathbb{R}[z]^{p \times (\mathtt{w}+\mathtt{c})},$$

where $B_c(z)$ is of full row rank, and the matrices $\Delta_M(z)$ and $\bar{R}(z)$, if they exist (provided that $R_c(z)$ is not of full row rank, namely that $r < p$) are square monomic and left prime, respectively. It entails no loss of generality assuming $B_c(z) = \Delta_c(z)\bar{R}_c(z)$, where $\Delta_c(z)$ is nonsingular square and $\bar{R}_c(z)$ is left prime. Also, we denote by $\bar{C}(z) \in \mathbb{R}[z]^{(\mathtt{w}-p+r) \times \mathtt{w}}$ a completion of $\bar{R}(z)$ to a unimodular matrix and by $\bar{C}_c(z) \in \mathbb{R}[z]^{(\mathtt{c}-r) \times \mathtt{c}}$ a completion of $\bar{R}_c(z)$ to a unimodular matrix (they can possibly be void matrices or unimodular matrices in case the matrices we are completing are already unimodular or, on the contrary, void matrices, respectively).

i) $\Rightarrow$ ii) If $p \geq \mathtt{w}$, then $r \geq \mathtt{w} - p + r$. We want to show that

$$\left[P(z) \quad -Q(z) \right] = \begin{bmatrix} B_w(z) - \begin{bmatrix} I_{\mathtt{w}-p+r} \\ 0 \end{bmatrix} \bar{C}(z) & -\Delta_c(z)\bar{R}_c(z) \\ 0 & -\bar{C}_c(z) \end{bmatrix} \begin{matrix} \} \ r \\ \} \ \mathtt{c} - r \end{matrix}$$

is a DBC we are searching for. Surely $Q(z)$ is nonsingular square since

$$Q(z) = \begin{bmatrix} \Delta_c(z) & 0 \\ 0 & I_{\mathtt{c}-r} \end{bmatrix} \begin{bmatrix} \bar{R}_c(z) \\ \bar{C}_c(z) \end{bmatrix}.$$

On the other hand, it is easily seen that an MLA of

$$\begin{bmatrix} \Delta_c(z)\bar{R}_c(z) \\ 0 \\ Q(z) \end{bmatrix} = \begin{bmatrix} \Delta_c(z)\bar{R}_c(z) \\ 0 \\ \Delta_c(z)\bar{R}_c(z) \\ \bar{C}_c(z) \end{bmatrix}$$

is given by

$$\begin{bmatrix} 0 & I_{p-r} & 0 & 0 \\ I_r & 0 & -I_r & 0 \end{bmatrix},$$

and therefore $\mathcal{K}_{full}$ is described as the kernel of the following polynomial matrix

$$\begin{bmatrix} 0 & I_{p-r} & 0 & 0 \\ I_r & 0 & -I_r & 0 \end{bmatrix} \begin{bmatrix} B_w(z) \\ \Delta_M(z)\bar{R}(z) \\ B_w(z) - \begin{bmatrix} I_{\mathtt{w}-p+r} \\ 0 \end{bmatrix} \bar{C}(z) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Delta_M(z)\bar{R}(z) \\ \begin{bmatrix} I_{\mathtt{w}-p+r} \\ 0 \end{bmatrix} \bar{C}(z) \end{bmatrix} = \begin{bmatrix} \Delta_M(z) & 0 \\ 0 & I_{\mathtt{w}-p+r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{R}(z) \\ \bar{C}(z) \end{bmatrix},$$

which is clearly right monomic, thus proving that the given controller is a DBC.

ii) $\Rightarrow$ iii) Obvious.

iii) $\Rightarrow$ i) Suppose that a DBC (6) with $Q(z)$ $k \times \mathtt{c}$ full row rank exists. Then, by Theorem 2, an L-polynomial matrix exists such that

$$X(z, z^{-1}) \begin{bmatrix} R_w(z) & -R_c(z) \\ P(z) & -Q(z) \end{bmatrix} = [\, I_{\mathtt{w}} \quad 0 \,].$$

Since $X(z, z^{-1})$ is a left inverse of $\begin{bmatrix} R_w(z) \\ P(z) \end{bmatrix}$ its rank is $\mathtt{w}$. On the other hand, being a left annihilator of $\begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix}$ its rank must be not greater than the rank of an MLA for the same matrix. So,

$$\mathtt{w} \leq (p+k) - \text{rank} \begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix} \leq (p+k) - k = p,$$

where we have used the fact that $\text{rank} \begin{bmatrix} R_c(z) \\ Q(z) \end{bmatrix} \geq \text{rank}\ Q(z) = k$. ∎

As previously recalled, a DBC with $Q(z)$ either of full row rank or nonsingular square corresponds to the possibility of implementing the control action through a feedback connection, with $\mathbf{w}$ as an input (a maximal input in the nonsingular square case) and $\mathbf{c}$ as an output (possibly including some free variables, in turn). It is worthwhile to remark a few aspects:

- The previous proposition states that such DBC's exist if and only if the cardinality of $\mathbf{w}$ is not greater than the number of (independent) equations of the plant. If condition (20) does not hold, the controller has to impose direct constraints on the to be controlled variables in order to achieve the task of driving them to zero in a finite number of steps. This means that it has to constrain a priori the trajectories of $\mathbf{w}$ of the original behavior in order to guarantee that the control action is successful. Indeed, it is easy to see that when $Q(z)$ is not of full row rank, we can always obtain for the DBC an equivalent description of the following type:

$$\begin{aligned} P_1(\sigma)\mathbf{w}(t) &= Q_1(\sigma)\mathbf{c}(t), \\ P_2(\sigma)\mathbf{w}(t) &= 0, \qquad\qquad t \in \mathbb{Z}_+, \end{aligned}$$

  with $Q_1(z)$ of full row rank and $P_2(z) \neq 0$. The former equation represents a feedback control action, while the latter represents a constraint directly imposed on the to-be-controlled variables.
- (* lo teniamo o e' pericoloso?? *) The previous proposition bears some similarities with analogous results in [13] (see Section 7). It must be remarked, however, that the two settings are rather different since in [13] the partial interconnection case is considered, and the problem of splitting the control variable $\mathbf{c}$ in the form $(\mathbf{c}_1, \mathbf{c}_2)$, with $\mathbf{c}_2$ (maximally) free for the controller, is considered. Clearly, also in that case a constraint similar to (20) has been obtained.
- One may wonder why considering also the case when $Q(z)$ is of full row rank instead of just the case when $Q(z)$ is nonsingular square. The reason is that, in general, if we accept that $Q(z)$ is only of full row rank, we can obtain DBC's of lower complexity. In particular, if we impose that $Q(z)$ is nonsingular square we may end up considering DBC's that are not regular and hence introduce some redundancy with respect to the original system laws.

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